7
Quantum gravity

7.1 Introduction

There have been many different attempts to provide a quantum description of gravitational phenomena. Although there is at present no immediate experimental evidence of quantum effects of the gravitational field, it is expected on general grounds that at sufficiently high energies quantum effects may be relevant. The fact that quantum field theories in general involve virtual processes of arbitrarily high energies may suggest that an understanding of quantum gravity may be needed to provide a complete picture of quantum fields. Ultraviolet divergences arise as a consequence of an idealization in which one expects the field theory in question to be applicable up to arbitrarily high energies. It is generally accepted that for high energies gravitational corrections could play a role. On the other hand, classical general relativity predicts in very general settings the appearance of singularities in which energies, fields and densities become intense enough to suggest the need for quantum gravitational corrections.

In spite of the many efforts invested over the years in trying to apply the rules of quantum mechanics to the gravitational field, most attempts have remained largely incomplete due to conceptual and technical difficulties. There are good reasons why the merger of quantum mechanics and gravity as we understand them at present is a difficult enterprise. We now present a brief and incomplete list of the issues involved. The reader should realize that every one of these problems is to some extent currently being actively investigated by several groups and some of these difficulties could eventually be overcome.

• It is not clear which theory of gravity to start from at a classical level. The fact that general relativity is the simplest viable theory does not necessarily mean it is appropriate for quantization. Some people argue that a successful theory of gravity should also incorporate all other

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interactions in nature in a unified fashion.

• The rules of quantum mechanics, as we know them at present, may not be applicable to systems without a defined notion of time, as is the case for generally covariant theories of gravity.

• It is not clear that a continuous description of spacetime and fields will be enough to provide a framework to quantize gravity. It may be that the description provided by general relativity is an "effective" macroscopic theory with an underlying, more fundamental theory. As in the case of the Fermi model, quantizing the effective theory can be misleading.

• There is a tendency to incorporate into quantum descriptions of the gravitational field issues related to the quantization of the universe as a whole ("quantum cosmology"). As a consequence it is not clear what the measurement process exactly is and how to define observers and measurable quantities for the theory.

As well as these more fundamental problems, several attempts to quantize the gravitational field have encountered more specific difficulties. Again, we present just a brief list and many of these difficulties are currently being studied by several researchers.

• Attempts based on perturbation theory, in which one starts with a fixed background metric and quantizes deviations from it have led to non-renormalizable theories. This has sometimes been perceived as a pathology of the classical theory of gravity chosen, and has motivated the study of quantizations of theories other than Einstein's, most notably higher order theories, supersymmetric theories and theories based on strings. Another point of view is to notice that these attempts ignore the rich non-linear, geometric and topological nature of general relativity. This suggests that from the beginning they offered little hope of dealing appropriately with the fundamental difficulties listed above. It is therefore not entirely surprising that they encounter difficulties at some point.

• In recent years there has been great interest in considering string theories as the fundamental theory of particles and gravity. Apart from possibly being able to unify all interactions, string theory was expected to be perturbatively much better behaved than regular field theories based on point particles. In spite of this better behavior, which makes each term in the perturbation expansion finite, the series diverges rather badly. Again one could view this as a failure of perturbative techniques and it is still possible that a non-perturbative theory of strings could yield the correct quantum theory of gravity.

• The use of path integral quantization techniques has been advocated for gravity since it is naturally covariant and allows us to consider in a dynamical fashion the geometric and topological nature of gravity. With the exception of some mini-superspace examples, several technical difficulties have prevented the application of these techniques to gravity.

Among them is the lack of understanding of the appropriate gauge invariant measure of integration in the path integral, the unboundedness of the Einstein action and the inapplicability of Wick rotation techniques without a notion of time.

• Canonical techniques have been applied to quantum gravity for quite some time. All the discussion in this book will be focused on this kind of approach and we will discuss in detail some of the difficulties that appear. Among the difficulties is the choice of a natural time variable in the theory, the construction of gauge invariant observables, the imposition of an appropriate Hilbert space structure compatible with regularized constraints enforcing gauge invariance and the fact that the spacetime topology is fixed.

• Other more radical approaches try to start from theories that are fundamentally different from general relativity or other field theories, usually with some degree of discreteness, and try to recover the usual theories in some limit. The main difficulty is that they are faced with the burden of checking that all desirable features of the usual field theories are reproduced and that no unexpected behaviors are introduced.

In this book we will concentrate on a very specific approach to quantum gravity: we will apply canonical quantization techniques to general relativity.

The use of canonical quantization techniques is suggested by the results on Yang–Mills theories that we introduced in chapter 5. As we saw, one can obtain considerable progress in the canonical formulation using loop variables. Although there has been recent progress on the use of loop techniques at a covariant level, most of the emphasis up to now has been on canonical approaches. The choice of general relativity (in four spacetime dimensions) as the theory of gravity to be quantized is based on the fact that it is the simplest purely geometric theory available and it should serve well as a testbed for quantization techniques, even if it ultimately is superseded by another theory.

The canonical approach to quantum general relativity had been considered extensively in the past and had several complications. As we will see, general relativity is a constrained system and the constraint equations turned out to be unmanageable at a quantum level. The situation changed a few years ago with the introduction of a new set of variables that has allowed a significant amount of progress. In particular, the new variables cast general relativity in a form that is similar to that of a Yang–Mills theory and is therefore quite suited to the techniques we have developed in this book.

The outline of this chapter is as follows. In the first section we recall the traditional Hamiltonian formulation of general relativity. In the next section we work out the new canonical formulation. In the last section we use the new Hamiltonian formulation to quantize canonically the theory as if it were a Yang–Mills theory, obtaining a connection representation. This will be the starting point for the development of the loop representation in the next chapter.

7.2 The traditional Hamiltonian formulation

7.2.1 Lagrangian formalism

General relativity is a theory of gravity in which the gravitational interaction is accounted for by a deformation of spacetime. The fundamental variable for the theory is the spacetime metric g_{ab}. The action for the theory is given by

$$S = \int d^4x\sqrt{-g}R(g_{ab}) + \int d^4x\sqrt{-g}\mathcal{L}(\text{matter}), \tag{7.1}$$

where g is the determinant of g_{ab}, $R(g_{ab})$ is the curvature scalar and we have also included a term to take into account possible couplings to matter, although we will largely concentrate on the vacuum case. The equations of motion are obtained by varying the action with respect to the spacetime metric,

$$\frac{\delta S}{\delta g^{ab}} = 0 \quad : \quad R_{ab} - \tfrac{1}{2}Rg_{ab} = \frac{\delta S_{\text{matter}}}{\delta g^{ab}}, \tag{7.2}$$

and are the well known Einstein equations. The action is invariant under diffeomorphisms of the spacetime manifold (which can also be viewed as invariance under coordinate transformations). We will see that this symmetry is intimately tied into the structure of the Einstein equations.

7.2.2 The split into space and time

The standard Hamiltonian formulation for general relativity was developed by Arnowitt, Deser and Misner (ADM) [124]. To cast the theory in a canonical form, we need to split spacetime into space and time. Without a notion of time, there is no notion of evolution and therefore no Hamiltonian in the traditional sense. This may seem odd at first; one of the main points of general relativity is to cast space and time on the same footing and this approach seems to separate them again. We will see that the issue is more subtle. Although the canonical formalism manifestly breaks the spacetime covariance of the theory by singling out a particular time direction, in the end the formalism itself will tell us that it really did not matter which direction of time we took to begin with. The covariance is restored by certain relations that appear in the canonical formulation

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and the time picked is a "fiducial" one for construction purposes only. This is a similar situation to that which one faces when one formulates the theory of a relativistic particle canonically. We will see more details of this immediately.

We consider a spacetime ${}^4M$ with metric g_{ab} that has topology ${}^3\Sigma \times R$ where ${}^3\Sigma$ is a space-like surface with respect to g_{ab}. We will assume ${}^3\Sigma$ is a Cauchy surface, i.e., a surface such that the light cones emanating from it span all the spacetime to the future of ${}^3\Sigma$. Associated with the foliation is a time-like, future directed, vector t^a and a function on spacetime t such that its level surfaces coincide with the leaves of the foliation ${}^3\Sigma_t$ and such that $t^a\partial_a t = 1$. This vector field can be interpreted as describing the "flow of time" among the leaves of the foliation, but it should be realized that it has been introduced fiducially and cannot be connected with the measurements of any clock until we have a metric appropriately determined by the Einstein equations. We introduce a unit vector field n^a normal to the foliation. In combination with the spacetime metric this defines a unique, positive-definite spatial metric on the three-dimensional slice,

$$q_{ab} := g_{ab} + n_a n_b. \tag{7.3}$$

Notice that since we have a spacetime metric all indices are raised and lowered with it. The vector field t^a can be decomposed in components normal and tangential to ${}^3\Sigma$ as

$$t^a = Nn^a + N^a, \tag{7.4}$$

where the scalar N is known as the "lapse" and N^a is a vector on ${}^3\Sigma$ and is usually referred to as the "shift" vector. The decomposition can be seen in figure 7.1. It is clear that the quantities N and N^a contain information about the particular foliation rather than information intrinsic to spacetime.

From the information contained in q_{ab}, N^a and N one can reconstruct the spacetime metric,

$$g^{ab} = q^{ab} - n^a n^b, \tag{7.5}$$

where n^a can be easily constructed from N and N^a and q^{ab} is the inverse of q_{ab} in the tangent space to ${}^3\Sigma_t$ (see Wald [123]). In fact, one can explicitly choose coordinates (t, x^i) such that the metric reads

$$ds^2 = -N^2dt^2 + q_{ij}(dx^i + N^i dt)(dx^j + N^j dt), \tag{7.6}$$

where q_{ij} and N^i are the coordinate components of q_{ab} and N^a. We therefore see that the lapse has the interpretation of the "time time"

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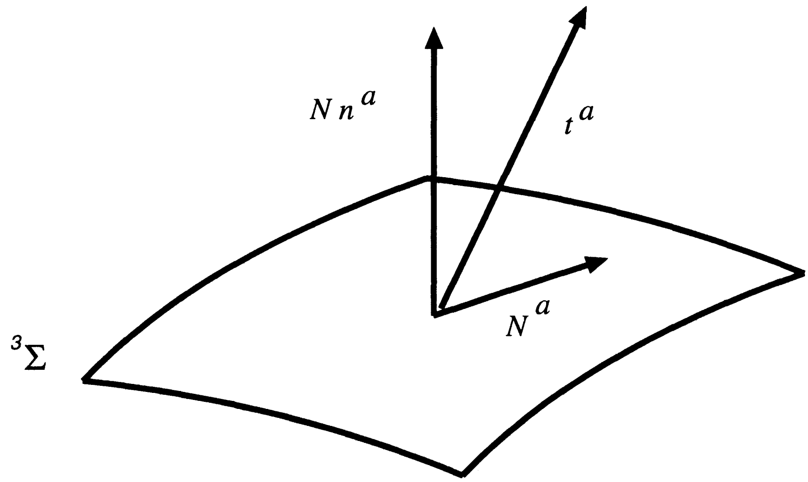

Fig. 7.1. The foliation introduced for the canonical formulation of general relativity.

component of the metric and the shift represents off-diagonal "time space" components.

An important quantity in the canonical description is the extrinsic curvature of the surface $^3\Sigma$. This is defined by

$$K_{ab} := q_a^c q_b^d \nabla_c n_d, \tag{7.7}$$

where ∇ is the torsion-free derivative compatible with g_{ab}. The extrinsic curvature measures the rate of change of the spatial metric along the congruence defined by n^a and therefore gives an idea of the "bending" of the spatial surfaces in spacetime. One can easily check that

$$K_{ab} = \tfrac{1}{2}\mathcal{L}_{\vec{n}} q_{ab}, \tag{7.8}$$

and also that

$$\dot{q}_{ab} := \mathcal{L}_{\vec{t}} q_{ab} = 2NK_{ab} + \mathcal{L}_{\vec{N}}\, q_{ab}. \tag{7.9}$$

That is, the extrinsic curvature allows us to give a measure of the variation of the three-dimensional metric with respect to the fiducial time introduced by the foliation, i.e., K_{ab} essentially contains the information about the "time derivative" of q_{ab}.

We have introduced up to now a series of quantities defined on the spatial surface in terms of which we can reconstruct the spacetime metric and its time derivatives. We now proceed to rewrite the Einstein action in terms of these variables (see reference [123]),

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$$S = \int dt L, \tag{7.10}$$

$$L = \int d^3x N\sqrt{q}(^3R + K_{ab}K^{ab} - K^2), \tag{7.11}$$

where q is the determinant of q_{ab} in a basis adapted to $^3\Sigma$ such that $\sqrt{-g} = N\sqrt{q}$, ^3R is the (intrinsic) curvature of the spatial metric and $K := K_{ab}q^{ab}$. To achieve this particular form of the action surface terms have to be added appropriately. In this book we will always deal with compact three-surfaces (like those that arise in some cosmologies) and will therefore ignore these issues. If one wants to consider non-compact spatial slices (as is needed in asymptotically flat spacetimes like those that describe stars and black holes) one can achieve the same form of the action by imposing appropriate boundary conditions at infinity. This can be done in a straightforward manner (see reference [123]).

We now have the action of general relativity in a reasonable form to allow a canonical formulation. We have it expressed in terms of variables that are functions of "space" and that "evolve in time". This is the usual setup for constructing canonical formulations.

We pick as the canonical variable the three-metric q_{ab} and compute its conjugate momentum,

$$\tilde{\pi}_{ab} := \frac{\delta L}{\dot{q}_{ab}} = \sqrt{q}(K^{ab} - Kq^{ab}) \tag{7.12}$$

and we see that the conjugate momentum to the metric is essentially given by the extrinsic curvature ("time derivative").

The variables N and N^a have vanishing conjugate momenta, since the action (7.11) does not contain time derivatives of them. This implies the canonical formulation will have constraints.

We can now perform the Legendre transform and obtain the Hamiltonian of the theory

$$H(\tilde{\pi}, q) = \int d^3x(\tilde{\pi}^{ab}q_{ab} - \tilde{\mathcal{L}}), \tag{7.13}$$

where $\tilde{\mathcal{L}}$ is the Lagrangian density ($L = \int d^3x\tilde{\mathcal{L}}$). Replacing $\dot{q}$ in terms of $\tilde{\pi}$ one gets

$$H(\tilde{\pi}, q) = \int d^3x(N(-\sqrt{q}R + (\sqrt{q})^{-1}(\tilde{\pi}^{ab}\tilde{\pi}_{ab} - \tfrac{1}{2}\tilde{\tilde{\pi}}^2)) - 2N^bD_a\tilde{\pi}^a_b, \tag{7.14}$$

where $\tilde{\pi} = \tilde{\pi}^{ab}q_{ab}$ (and squared has double density weight) and D_a is the torsion-free covariant derivative compatible with q_{ab}.

The variables q_{ab} and $\tilde{\pi}^{ab}$ have the straightforward simplectic structure of conjugate pairs,

$$\{q_{ab}(x), \tilde{\pi}^{cd}(y)\} = \delta_a^c \delta_b^d \delta(x-y). \tag{7.15}$$

7.2.3 Constraints

Having cast the theory in a Hamiltonian form, let us step back a minute and analyze the formalism that we have built. We started from a four-dimensional metric g^{ab} and we now have in its place the three-dimensional q^{ab} and the "lapse" and "shift" functions N and N^a. We defined a conjugate momentum for q_{ab}. However, notice that nowhere in the formalism does a time derivative of the lapse or shift appear. That means their conjugate momenta are zero. That is, our theory has constraints. In fact, if we rewrite the action using the expression for the Hamiltonian given above, we get

$$S = \int dt \int d^3x \, ((\tilde{\pi}_{ab}\dot{q}^{ab} + \underset{\sim}{N}(-qR + (\tilde{\pi}^{ab}\tilde{\pi}_{ab} - \tfrac{1}{2}\tilde{\tilde{\pi}}^2)) - 2N^b D_a \tilde{\pi}_b^a), \tag{7.16}$$

where the inverse-densitized lapse $\underset{\sim}{N}$ is defined as $(\sqrt{q})^{-1}N$. If we vary the action with respect to $\underset{\sim}{N}$ and N^b in order to get their respective equations of motion, we get four expressions, functions of $\tilde{\pi}$ and q which should vanish identically, and are usually called $\tilde{C}^a$ and $\tilde{\tilde{\mathcal{H}}}$,

$$\tilde{C}_a(\pi, q) = 2D_b \tilde{\pi}_a^b, \tag{7.17}$$

$$\tilde{\tilde{\mathcal{H}}}(\pi, q) = -\tilde{\tilde{q}}R + (\tilde{\pi}^{ab}\tilde{\pi}_{ab} - \tfrac{1}{2}\tilde{\pi}^2). \tag{7.18}$$

For calculational simplicity, these equations are usually "smoothed out" with arbitrary test fields on the three-manifold, $C(\vec{N}) = \int d^3x N^a \tilde{C}_a$, $\mathcal{H}(\underset{\sim}{N}) = \int d^3x \underset{\sim}{N} \tilde{\tilde{\mathcal{H}}}$.

These equations are "instantaneous" laws, i.e., they must be satisfied *on each hypersurface*. They tell us that if we want to prescribe data for a gravitational field, not every pair of $\tilde{\pi}$ and q will do; equations (7.17), (7.18) should be satisfied. The counting of degrees of freedom is done in the following way: we have a 12-dimensional phase space. In that space we have four constraints and we can fix four gauge conditions. We are therefore left with a four-dimensional constraint-free phase space, which gives two degrees of freedom. (General relativity being a field theory the previous counting holds per each point of the spatial surface.)

These equations have the same character as the Gauss law has for electromagnetism, which tells us that not any vector field would necessarily work as an electric field, it must have vanishing divergence in vacuum. As is well known, the Gauss law appears as a consequence of the $U(1)$ invariance of the Maxwell equations. An analogous situation appears here.

To understand this, consider the Poisson bracket of any quantity with the constraint $C(\vec{N})$. It is straightforward to check that (exercise)

$$\{f(\tilde{\pi},q),C(\vec{N})\}=\mathcal{L}_{\vec{N}}f(\tilde{\pi},q). \tag{7.19}$$

Therefore we see that the constraint $C(\vec{N})$ "Lie drags" the function $f(\pi,q)$ along the vector $\vec{N}$. Technically, it is the infinitesimal generator of diffeomorphisms of the three-manifold in phase space. As the Gauss law (in the canonical formulation of Maxwell's theory) is the infinitesimal generator of $U(1)$ gauge transformations, the constraint here is the infinitesimal generator of spatial diffeomorphisms. This clearly shows why we have this constraint in the theory: it is the canonical representation of the fact that the theory is invariant under spatial diffeomorphisms. The constraint $C(\underset{\sim}{N})$ is analogously associated with the invariance under spacetime diffeomorphisms of general relativity, it is related to the time reparametrization invariance of the theory.

We can now work out the equations of motion of the theory by either varying the action with respect to q^{ab} and $\tilde{\pi}_{ab}$ or taking the Poisson bracket of these quantities with the Hamiltonian constraint.

The above system of constraints is first class (for the definition of this see chapter 3). Computing the Poisson algebra one gets

$$\{C(\vec{N}),C(\vec{M})\}=C(\mathcal{L}_{\vec{M}}\vec{N}), \tag{7.20}$$

$$\{C(\vec{N}),\mathcal{H}(\underset{\sim}{M})\}=\mathcal{H}(\mathcal{L}_{\vec{N}}\underset{\sim}{M}), \tag{7.21}$$

$$\{\mathcal{H}(\underset{\sim}{N}),\mathcal{H}(\underset{\sim}{M})\}=C(\vec{K}), \tag{7.22}$$

where the vector $\vec{K}$ is defined by $K^a=qq^{ab}(\underset{\sim}{N}\partial_b\underset{\sim}{M}-\underset{\sim}{M}\partial_a\underset{\sim}{N})$. The reader should notice, however, that the algebra is not a true Lie algebra, since one of the structure constants (the one defined by the last equation) is not a constant but depends on the fields q^{ab} (through the definition of the vector $\vec{K}$). At a quantum mechanical level this will imply that the fields should appear to the left of the constraint in the appropriate factor ordering to ensure consistency.

7.2.4 Quantization

Having cast the theory in a canonical form, we can now proceed to a canonical quantization, following the general quantization scheme outlined in chapter 3. One picks as canonical algebra the pair q_{ab} and $\tilde{\pi}^{ab}$, and represents them as quantum operators acting on a set of wavefunctionals $\Psi[q]$ in the obvious fashion: $\hat{q}_{ab}$ as a multiplicative operator and $\hat{\pi}^{ab}=-i\delta/\delta q_{ab}$. One wants the wavefunctions to be invariant under the symmetries of the theory. As we saw the symmetries are represented in

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this language as constraints. The requirement that the wavefunctions be annihilated by the constraints (promoted to operatorial equations) implements the symmetries at the quantum level. The wavefunctionals that are annihilated by the constraints are the physical states of the theory. Notice that we do not yet have a Hilbert space. One needs to introduce an inner product on the space of physical states in order to compute expectation values and make physical predictions. Only at this point does one have an actual Hilbert space. How to find this inner product is not prescribed by standard canonical quantization (we will discuss this in the next section). Under this inner product the physical states should be normalizable. The expectation values, by the way, only make sense for quantities that are invariant under the symmetries of the theory (quantities that classically have vanishing Poisson brackets with all the constraints). We call them physical observables. For the gravitational case *none* is known for compact spacetimes (we will return to this issue later). The observables of the theory should be self-adjoint operators with respect to the inner product in order to yield real expectation values.

It is at the level of the constraints that we run into trouble. We have to promote the constraints we discussed in the last subsection to quantum operators. This in itself is a troublesome issue, since general relativity being a field theory, issues of regularization and factor ordering appear. One can, — at least formally — find factor orderings in which the diffeomorphism constraint becomes the infinitesimal generator of diffeomorphisms on the wavefunctions. Therefore the requirement that a wavefunction be annihilated by it just translates itself in the fact that the wavefunction has to be invariant under diffeomorphisms. This is not difficult to accomplish (formally!). One simply requires that the wavefunctions be not actually functionals of the three-metric q^{ab}, but of the "three-geometry" (i.e., meaning the properties of the three-metric invariant under diffeomorphisms). Thus, what we are saying is just a restatement of the fact that the functional should be invariant under diffeomorphisms. One can come up with several examples of functionals that meet this requirement. The real trouble appears when we want the wavefunctions to be annihilated by the Hamiltonian constraint. This constraint does not have a simple geometrical interpretation in terms of three-dimensional quantities (remember that the idea that it represents "time evolution" does not help here, since we are always talking about equations that hold *on the three-surface* without any explicit reference to time). Therefore we are just forced to proceed directly: to promote the constraint to a wave equation, use some factor ordering (hopefully with some physical motivation), pick some regularization and try to solve the resulting equation (the Wheeler–DeWitt equation). It turns out that this task has never been accomplished in general (it has been in simplified mini-superspace

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examples). One of the difficulties encountered in this direction is the fact that the constraint is a non-polynomial function of the basic variables (remember it involves the scalar curvature, a non-polynomial function of the three-metric).

Therefore the program of canonical quantization stalls here. It could well be that the constraints do not admit a consistent factor ordering and the quantum theory may not exist. Having been unable to find the physical states of the theory we are in a bad position to introduce an inner product (since we do not know on what space of functionals to act) and actually make physical predictions. This issue is compounded by the fact that we do not know any observables for the system, which puts us in a more clueless situation with respect to the inner product. This state of affairs had already been reached in the work of DeWitt in the 1960s [125] and little improvement has been made until recently. We will see in the next section that the use of a new set of variables improves the situation with respect to the Hamiltonian constraint, giving hope of maybe allowing us to attack the problem of the inner product. Moreover, we will see that the new formulation allows a natural contact with the main ideas of this book.

7.3 The new Hamiltonian formulation

As we saw in the previous section, the traditional canonical approach to quantum general relativity faces serious obstructions at a very early stage. On the other hand, as we saw, the canonical quantization of Yang–Mills theories has been more successful. For many years efforts were directed towards casting general relativity in such a way that it resembled a Yang–Mills theory more with the hope that quantization techniques and ideas developed for the latter would become applicable to general relativity. This led to several attempts that started from a gauge theory approach with the aim of deriving a theory of gravity based on gauging a particular symmetry group. This, in general, led to new theories of gravity that involved higher order terms in the Hilbert action [126]. There is another possible approach: to keep the Einstein equations for the gravitational theory but reinterpret them as statements about a connection instead of a metric. The simplest way to achieve such a reformulation is to consider the Palatini variational principle. In this, one varies the metric and the spacetime connection as independent variables. One retrieves the Christoffel definition of the connection as one of the field equations. Attempts to formulate gravity in terms of connections in this way go back to Einstein and Dirac in the 1940s. In order to have a formalism as close as possible to a usual Yang–Mills formulation, one could take

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the Palatini principle based on tetrads and $SO(3,1)$ connections. This route was studied in some detail by Kijowski [127]. Unfortunately, the canonical theory based on such connections has second class constraints (in 3+1 dimensions). When one eliminates these, non-polynomialities are introduced and one is led back to the traditional Hamiltonian formulation [61]. It is remarkable that in 2+1 dimensions one actually can formulate the theory in terms of connections, although historically this was realized later and through a different construction. We will review the 2+1 case later.

In 3+1 dimensions, the only successful attempt to obtain a canonical theory in terms of a connection that yields first class constraints is that due to Ashtekar [51]. It is based on the use of self-dual connections. Not only do the constraints remain first class but they are relatively simple polynomial functions. The price to be paid is that the self-dual connections are complex. In the next subsections we will develop this formalism. The treatment will follow closely the book by Ashtekar [2], we direct the reader to it for extensive details.

7.3.1 Tetradic general relativity

To introduce the new variables, we first need to introduce the notion of tetrads. A tetrad is a vector basis in terms of which the metric of spacetime looks locally flat,

$$g_{ab} = e_a^I e_b^J \eta_{IJ}, \tag{7.23}$$

where $\eta_{IJ} = \mathrm{diag}(-1,1,1,1)$ is the Minkowski metric, and equation (7.23) simply expresses that g_{ab}, when written in terms of the basis e_a^I, is locally flat. If spacetime were truly flat, one could perform such a transformation globally, integrating the basis vectors into a coordinate transformation $e_a^I = \partial x^I/\partial x'^a$. In a curved spacetime these equations cannot be integrated and the transformation to a flat space only works locally, the flat space in question being the "tangent space". From equation (7.23) it is immediate to see that given a tetrad, one can reconstruct the metric of spacetime. One can also see that although g_{ab} has only ten independent components, the e_a^I have sixteen. This is due to the fact that equation (7.23) is invariant under Lorentz transformations on the indices $I, J\ldots$. That is, these indices behave as if existing in flat space. In summary, tetrads have all the information needed to reconstruct the metric of spacetime but there are extra degrees of freedom in them, and this will have a reflection in the canonical formalism.

7.3.2 The Palatini action

We now write the Einstein action in terms of tetrads. We introduce a covariant derivative via $D_a K_I = \partial_a K_I + \omega_{aI}{}^J K_J$. Here $\omega_{aI}{}^J$ is a Lorentz connection (its associated covariant derivative annihilates the Minkowski metric). We define a curvature by $\Omega_{ab}{}^{IJ} = \partial_{[a}\omega_{b]}{}^{IJ} + [\omega_a, \omega_b]^{IJ}$, where $[\,,]$ is the commutator in the Lorentz Lie algebra. The Ricci scalar of this curvature can be expressed as $e_I^a e_J^b \Omega_{ab}^{IJ}$ (indices I, J are raised and lowered with the Minkowski metric). The Einstein action can be written as

$$S(e,\omega) = \int d^4x \; e \; e_I^a e_J^b \Omega_{ab}^{IJ}, \tag{7.24}$$

where e is the determinant of the tetrad (equal to $\sqrt{-g}$).

We will now derive the Einstein equations by varying this action with respect to e and ω as independent quantities. To take the metric and connection as independent variables in the action principle was first considered by Palatini [128].

As a shortcut to performing the calculation (this derivation is taken from reference [2]), we introduce a (torsion-free) connection compatible with the tetrad via $\nabla_a e_I^b = 0$. The difference between the two connections we have introduced is a field $C_{aI}{}^J$, defined by $C_{aI}{}^J V_J = (D_a - \nabla_a)V_I$. We can compute the difference between the curvatures (R_{ab}^{IJ} is the curvature of ∇_a), $\Omega_{ab}{}^{IJ} - R_{ab}{}^{IJ} = \nabla_{[a}C_{b]}{}^{IJ} + C_{[a}{}^{IM}C_{b]M}{}^J$. The reason for performing this intermediate calculation is that it is easier to compute the variation by reexpressing the action in terms of ∇ and $C_a{}^{IJ}$ and then noting that the variation with respect to $\omega_a{}^{IJ}$ is the same as the variation with respect to C_a^{IJ}. The action therefore is

$$S = \int d^4x \; e \; e_I^a e_J^b (R_{ab}{}^{IJ} + \nabla_{[a}C_{b]}{}^{IJ} + C_{[a}{}^{IM}C_{b]M}{}^J). \tag{7.25}$$

The variation of this action with respect to $C_a{}^{IJ}$ is easy to compute: the first term simply does not contain $C_a{}^{IJ}$ so it does not contribute. The second term is a total divergence (notice that ∇ is defined so that it annihilates the tetrad), the last term yields $e_M^{[a} e_N^{b]} \delta_{[I}^M \delta_{J]}^K C_{bK}{}^N$. It is easy to check that the prefactor in this expression is non-degenerate and therefore the vanishing of this expression is equivalent to the vanishing of $C_{bK}{}^N$. So this equation basically tells us that ∇ coincides with D when acting on objects with only internal indices. Thus the connection D is completely determined by the tetrad and Ω coincides with R (some authors refer to this fact as the vanishing of the torsion of the connection). We now compute the second equation, straightforwardly varying with respect to the tetrad. We get (after substituting $\Omega_{ab}{}^{IJ}$ by $R_{ab}{}^{IJ}$ as given by the

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previous equation of motion)

$$e^c_I R_{cb}{}^{IJ} - \tfrac{1}{2} R_{cd}{}^{MN} e^c_M e^d_N e^J_b = 0, \tag{7.26}$$

which, after multiplication by e_{Ja} just tells us that the Einstein tensor $R_{ab} - \frac{1}{2} R g_{ab}$ of the metric defined by the tetrads vanishes. We have therefore proved that the Palatini variation of the action in tetradic form yields the usual Einstein equations.

There is a difference between the first order (Palatini) tetradic form of the theory and the usual one. One sees that a solution to the Einstein equations we presented above is simply $e^b_J = 0$. This solution would correspond to a vanishing metric and is therefore forbidden in the traditional formulation since quantities, such as the Ricci or Riemann tensor are not defined for a vanishing metric. However, the first order action and equation of motion are well defined for vanishing triads. We therefore see that strictly speaking the first order tetradic formulation is a "generalization" of general relativity that contains the traditional theory in the case of non-degenerate triads. We will see this subtlety playing a role in subsequent chapters. It should be noticed that the potential of allowing vanishing metrics in general relativity offers new possibilities for some old questions, since one could envisage the formalism "going through", say, the formation of singularities. It also allows for topology change [129].

Is there any advantage in this formulation over the traditional one? The answer is no. If one performs a canonical decomposition of the first order tetradic action, one finds that the momentum canonically conjugate to the connection is quadratic in the tetrads. The factorizability of the momenta leads to new constraints in the theory that turn out to be second class. If one eliminates them through the Dirac procedure one returns to the traditional formulation [61].

7.3.3 The self-dual action

Up to now the treatment has been totally traditional. We will now take a conceptual step that allows the introduction of the Ashtekar variables. We will reconstruct the tetradic formalism of the previous subsection but we will introduce a change. Instead of considering the connection $\omega_a{}^{IJ}$ we will consider its self-dual part with respect to the internal indices and we will call it $A_a{}^{IJ}$, i.e., $iA_a{}^{IJ} = \frac{1}{2}\epsilon_{MN}{}^{IJ} A_a{}^{MN}$. Now, to really be able to do this, the connection must be complex (or one should work in an Euclidean signature). Therefore for the time being we will consider *complex general relativity* and we will then specify appropriately how to recover the traditional real theory. The connection now takes values in the (complex) self-dual subalgebra of the Lie algebra of the Lorentz group.

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We will propose as action,

$$S(e,A) = \int d^4x \; e \; e^a_J e^b_K F_{ab}{}^{JK}, \tag{7.27}$$

where $F_{ab}{}^{JK}$ is the curvature of the self-dual connection and it can be checked that it corresponds to the self-dual part of the curvature of the usual connection.

We can now repeat the calculations of the previous subsection for the self-dual case. When one varies the self-dual action with respect to the connection $A_a{}^{IJ}$ one obtains that this connection is the self-dual part of a torsion-free connection that annihilates the triad (if one repeated step by step the previous subsection argument, the self-dual part of $C_a{}^{IJ}$ would vanish). The variation with respect to the tetrad follows along very similar lines except that $\Omega_{ab}{}^{IJ}$ is everywhere replaced by $F_{ab}{}^{IJ}$. The final equation one arrives at again tells us that the Ricci tensor vanishes. Remarkably, the self-dual action leads to the (complex) Einstein equations. This essentially can be explained by the fact that the two actions differ by terms that on-shell are a pure divergence. This implies that the imaginary part of the equations of motion identically vanishes. If one works it out explicitly one finds that this corresponds to the Bianchi identities.

7.3.4 The new canonical variables

As we said before, if one takes the Palatini action principle in terms of tetrads and performs a canonical decomposition, second class constraints appear and one is led back to the traditional formulation. A quite different thing happens if one decomposes the self-dual action. Let us therefore proceed to do the 3+1 split. As we did before, we introduce a vector $t^a = Nn^a + N^a$. Taking the action

$$S(e,A) = \int d^4x \; e \; e^a_I e^b_J F_{ab}{}^{IJ} \tag{7.28}$$

and defining the vector fields $E^a_I = q^a_b e^b_I$ (where $q^a_b = \delta^a_b + n^a n_b$ is the projector on the three-surface), which are orthogonal to n^a, we have

$$S(e,A) = \int d^4x \; (e \; E^a_I E^b_J F_{ab}{}^{IJ} - 2 \; e \; E^a_I e^d_J n_d n^b F_{ab}{}^{IJ}). \tag{7.29}$$

We now define $\tilde{E}^a_I = \sqrt{q} E^a_I$, which is a density on the three-manifold. The determinant of the triad can be written as $e = N\sqrt{q}$. We also introduce the vector in the "internal space" induced by n^a, defined by $n_I = e^d_I n_d$. With these definitions, and exploiting the self-duality of $F_{ab}{}^{IJ}$ to write $F_{ab}{}^{IJ} = -i\frac{1}{2}\epsilon^{IJ}{}_{MN}F_{ab}{}^{MN}$, we get

$$S(e,A) = \int d^4x \; (-\tfrac{i}{2}\underset{\sim}{N}\tilde{E}^a_I\tilde{E}^b_J\epsilon^{IJ}{}_{MN}F_{ab}{}^{MN} - 2Nn^b\tilde{E}^a_I n_J F_{ab}{}^{IJ}). \tag{7.30}$$

The action is now written in canonical form and the conjugate variables can be read off directly. The configuration variable is the self-dual connection A_a . The conjugate momentum is the self-dual part of $-i\tilde{E}^a_J \epsilon^J_{MN}$,

$$\tilde{\pi}^a_{MN} = \tilde{E}^a_{[M} n_{N]} - \frac{i}{2} \tilde{E}^a_I \epsilon^I_{MN}. \tag{7.31}$$

Now, in terms of the canonical variables the Lagrangian takes the form

$$\int_\Sigma d^3x \mathrm{Tr}(-\tilde{\pi}^a \mathcal{L}_t A_a + N^a \tilde{\pi}^b F_{ab} - (A \cdot t) D_a \tilde{\pi}^a - \underset{\sim}{N} \tilde{\pi}^a \tilde{\pi}^b F_{ab}), \tag{7.32}$$

where all references to the internal vector n^I have disappeared. The projection of the spacetime connection on the time-like direction $(A \cdot t)$ is arbitrary and acts as a Lagrange multiplier.

Since n_I is not a dynamical variable it can be gauge fixed. We fix $n^I = (1, 0, 0, 0)$ and therefore $\epsilon^{IJKL} n_L = \epsilon^{IJK0}$. Since A^{IJ}_a and $\tilde{\pi}^a_{IJ}$ are self-dual, they can be determined by their $0I$ components. We may therefore define

$$A^i_a = i A^{0I}_a, \quad \tilde{E}^a_i = \tilde{\pi}^a_{0I}, \tag{7.33}$$

where internal indices i, j refer to the $SO(3)$ Lie algebra. In fact, as is well known the self-dual Lorentz Lie algebra is isomorphic to the (complexified) $SO(3)$ algebra

The new variables satisfy the Poisson bracket relations

$$\{A^i_a(x), \tilde{E}^b_j(y)\} = +i\delta^b_a \delta^i_j \delta^3(x - y). \tag{7.34}$$

The constraints may be read off from the Lagrangian (7.32) and take the form

$$\tilde{\mathcal{G}}^i = D_a \tilde{E}^{ai}, \tag{7.35}$$
$$\tilde{C}_a = \tilde{E}^b_i F^i_{ab}, \tag{7.36}$$
$$\tilde{\tilde{\mathcal{H}}} = \epsilon^{ij}_k \tilde{E}^a_i \tilde{E}^b_j F^k_{ab}, \tag{7.37}$$

and the Hamiltonian is again a linear combination of the constraints.

The last four equations correspond to the usual diffeomorphism and Hamiltonian constraints of canonical general relativity. The first three equations are extra constraints that stem from our use of triads as fundamental variables. These equations, which have exactly the same form as a Gauss law of an $SU(2)$ Yang–Mills theory, are the generators of infinitesimal $SU(2)$ transformations. They tell us that the formalism is invariant under triad rotations, as it should be.

Notice that a dramatic simplification of the constraint equations has occurred. In particular the Hamiltonian constraint is a polynomial function of the canonical variables, of quadratic order in each variable. Moreover, the canonical variables, and the phase space of the theory are exactly those of a (complex) $SU(2)$ Yang–Mills theory. The reduced phase space is actually a subspace of the reduced phase space of a (complex) Yang–Mills theory (the phase space modulo the Gauss law), since general relativity has four more constraints that further reduce its phase space. This resemblance of the formalism to that of a Yang–Mills theory will be the starting point of all the results we will introduce in the rest of the book.

In terms of the new variables, the structure of the constraints is simple enough for the reader to be able to compute the constraint algebra without great effort (this computation can also be carried out with the traditional variables and the results are the same). We only summarize the results here. To express them in a simpler form (and to avoid confusing manipulations of distributions while performing the computations), it is again convenient to smooth out the constraints with arbitrary test fields and to perform some recombinations. We denote

$$\mathcal{G}(N_i) = \int d^3x N_i (\mathcal{D}_a \tilde{E}^a)^i, \tag{7.38}$$

$$C(\vec{N}) = \int d^3x N^b \tilde{E}^a_i F^i_{ab} - \mathcal{G}(N^a A^i_a), \tag{7.39}$$

$$\mathcal{H}(\underset{\sim}{N}) = \int d^3x \underset{\sim}{N} \epsilon^{ij}{}_k \tilde{E}^a_i \tilde{E}^b_j F^k_{ab}, \tag{7.40}$$

and as before the notation is unambiguous. The constraint algebra then reads

$$\{\mathcal{G}(N_i), \mathcal{G}(N_j)\} = \mathcal{G}([N_i, N_j]), \tag{7.41}$$

$$\{C(\vec{N}), C(\vec{M})\} = C(\mathcal{L}_{\vec{M}} \vec{N}), \tag{7.42}$$

$$\{C(\vec{N}), \mathcal{G}(N_i)\} = \mathcal{G}(\mathcal{L}_{\vec{N}} N_i), \tag{7.43}$$

$$\{C(\vec{N}), \mathcal{H}(\underset{\sim}{M})\} = \mathcal{H}(\mathcal{L}_{\vec{N}} \underset{\sim}{M}), \tag{7.44}$$

$$\{\mathcal{G}(N_i), \mathcal{H}(\underset{\sim}{N})\} = 0, \tag{7.45}$$

$$\{\mathcal{H}(\underset{\sim}{N}), \mathcal{H}(\underset{\sim}{M})\} = C(\vec{K}) - \mathcal{G}(A^i_a K^a), \tag{7.46}$$

where the vector $\vec{K}$ is defined by $K^a = 2\tilde{E}^a_i \tilde{E}^b_i (\underset{\sim}{N} \partial_a \underset{\sim}{M} - \underset{\sim}{M} \partial_a \underset{\sim}{N})$. Here we clearly see that the constraints are first class. The reader should notice, however, that the algebra is not a true Lie algebra, since one of the structure constants (the one defined by the last equation) is not a constant but depends on the fields $\tilde{E}^a_i$ (through the definition of the vector $\vec{K}$).

The new variables are simply related to the traditional Hamiltonian variables:

$$A^i_a = \Gamma^i_a - iK^i_a, \qquad qq^{ab} = \tilde{E}^a_i \tilde{E}^b_i, \tag{7.47}$$

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where $K_a^i = K_{ab}E^{bi}$ and Γ_a^i is the spin connection compatible with the triad.

The evolution equations for the canonical variables are obtained taking the Poisson brackets of the variables with the Hamiltonian,

$$\dot{A}_a^i = -i\epsilon^{ijk}\underset{\sim}{N}\tilde{E}_j^b F_{abk} - N^b F_{ab}^i, \tag{7.48}$$

$$\dot{\tilde{E}}_i^a = i\epsilon_i^{jk} D_b(\underset{\sim}{N}\tilde{E}_j^a \tilde{E}_k^b) - 2D_b(N^{[a}\tilde{E}^{b]i}). \tag{7.49}$$

A similar simplification to that introduced in the constraints is evident in the equations of motion.

As we mentioned above, because of the self-duality used in the definition of the canonical variables, these are in general complex. The situation is totally analogous to that introduced when we discussed the harmonic oscillator and Maxwell theory in the Bargmann representation in section 4.5. If we want to recover the classical theory we must take a "section" of the phase space that corresponds to the dynamics of real relativity. This can be done. One gives data on the initial surface that correspond to a real spacetime and the evolution equations will keep these data real through the evolution. Now, strictly speaking, this procedure is not really canonical, since we are imposing these conditions by hand at the end. That does not mean it is not useful*. In fact, one can eliminate the reality conditions and have a canonical theory. However, much of the beauty of the new formulation is lost, in particular the structure of the resulting constraints is basically that of the traditional formalism.

The issue of the reality conditions acquires a different dimension at the quantum level. A point of view that is strongly advocated, and may turn out to be correct, is the following. Start by considering the complex theory and apply the usual steps towards canonical quantization After the space of physical states has been found, when one looks for an inner product, the reality conditions are used in order to choose an inner product that implements them. That is, the reality conditions can be a guideline to finding the appropriate inner product of the theory. One simply requires that the quantities that have to be real according to the reality conditions of the classical theory become self-adjoint operators under the chosen inner product. This solves two difficulties at once, since it allows us to recover the real quantum theory and the appropriate inner product at the same time. This point of view is strictly speaking a deviation from standard Dirac quantization, and works successfully for several model problems [130]. The success or failure in quantum gravity of this approach

* A non-trivial example where it can be worked to the end is the Bianchi II cosmology [132].

is yet to be tested and is one of the most intriguing and attractive features of the formalism. (For a critical viewpoint, see reference [131].)

In terms of the basic variables, the reality conditions are

$$(\tilde{E}_i^a \tilde{E}^{bi})^* = \tilde{E}_i^a \tilde{E}^{bi}, \tag{7.50}$$

$$(\epsilon^{ijk} \tilde{E}_i^{(a} D_c(\tilde{E}_k^{b)} \tilde{E}_j^c))^* = (\epsilon^{ijk} \tilde{E}_i^{(a} D_c(\tilde{E}_k^{b)} \tilde{E}_j^c)). \tag{7.51}$$

This particular form of the reality conditions may be useful to select real initial data for classical evolutions. However, if one wants to impose the conditions as adjointness relations of operators with respect to a quantum inner product, it is clear that one would need to recast the conditions in terms of physical observables, since these are the only quantities defined in the space of physical states. In particular equations (7.50),(7.51) are not well defined in that space.

Up to now we have discussed the theory in vacuum. There is no difficulty in incorporating matter fields in the new variable formulation. The constraints can be made polynomial in a natural fashion for coupling to scalar fields, Yang–Mills fields, and fermions. It is remarkable that Dirac fermions can be introduced only coupled to the self-dual part of the connection. A complete discussion can be found in references [133, 2].

It is immediate to include a cosmological constant in the framework. In the Einstein action the cosmological constant appears as $\int d^4x\sqrt{-g}\Lambda$. This action can be immediately canonically decomposed as

$$S_\Lambda = \int dt \int d^3x \underset{\sim}{N} q \Lambda, \tag{7.52}$$

and this can be written in terms of the new variables noting that the determinant of the three-metric is given by

$$q = \frac{1}{6} \underset{\sim}{\eta}_{abc} e^{ijk} \tilde{E}_i^a \tilde{E}_j^b \tilde{E}_k^c. \tag{7.53}$$

The only change introduced in the canonical theory is that the Hamiltonian constraint gains an extra term,

$$\mathcal{H}(\underset{\sim}{N}) = \int d^3x \underset{\sim}{N} \epsilon^{ij}{}_k \tilde{E}_i^a \tilde{E}_j^b F_{ab}^k + \frac{\Lambda}{6} \int d^3x \underset{\sim}{N} \underset{\sim}{\eta}_{abc} e^{ijk} \tilde{E}_i^a \tilde{E}_j^b \tilde{E}_k^c. \tag{7.54}$$

And again, is a polynomial expression. There is no modification to the other constraints, since the entire term in the action is proportional to $\underset{\sim}{N}$.

7.4 Quantum gravity in terms of connections

7.4.1 Formulation

The casting of general relativity as a theory of a connection has important implications at the quantum mechanical level. One can now proceed to quantize the theory exactly like we did in chapter 5, picking a polarization in which wavefunctions are functionals of a connection

$$\Psi[A]. \tag{7.55}$$

The Gauss law will immediately require that these be gauge invariant functions, i.e., functionals in the space of connections modulo gauge transformations. Notice that this is a significant departure from the traditional picture where one considered functionals of a three-metric, or if one imposed the diffeomorphism constraint, of a three-geometry.

As in the Yang–Mills case a representation for the Poisson algebra of the canonical variables considered can be simply achieved by representing the connection as a multiplicative operator and the triad as a functional derivative:

$$\hat{A}_a^i \Psi(A) = A_a^i \Psi(A), \tag{7.56}$$

$$\hat{\tilde{E}}_i^a \Psi(A) = \frac{\delta}{\delta A_a^i} \Psi(A). \tag{7.57}$$

It should be emphasized that a difference with the Yang–Mills case arises since the connection is complex. The wavefunctions considered are holomorphic functions of the connection and the functional derivative treats as independent the connection and its complex conjugate.

We would now like to use this choice in the representation of the canonical algebra to promote the constraint equations to operatorial equations. Since the constraint equations involve operator products, a regularization is needed. This is a fundamental point. Most of the issues one faces when promoting the constraints to wave equations do not have a unique answer unless one has a precise regularization. There is not a complete regularized picture of the theory at present. We will introduce some of the issues in this chapter and will return to them in chapters 8 and 11 as we develop the quantum theory and some of its consequences.

Ignoring for the time being the regularization issue, one can promote the constraints formally to operator equations if one picks a factor ordering. Two factor orderings have been explored: with the triads either to the right or the left of the connections.

7.4.2 Triads to the right and the Wilson loop

If one orders the triads to the right, the constraints become

$$\hat{\tilde{\mathcal{G}}}^i = D_a \frac{\delta}{\delta A_a^i}, \tag{7.58}$$

$$\hat{\tilde{\mathcal{C}}}_a = F_{ab}^i \frac{\delta}{\delta A_b^i}, \tag{7.59}$$

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$$\hat{\tilde{\tilde{\mathcal{H}}}} = \epsilon^{ijk} F^i_{ab} \frac{\delta}{\delta A^j_a} \frac{\delta}{\delta A^k_b}. \tag{7.60}$$

This ordering was first considered by Jacobson and Smolin [134] because the Gauss law and the diffeomorphism constraint formally (without a regularization) generate gauge transformations and diffeomorphisms on the wavefuctions.

There is a potential problem when one considers the algebra of constraints. Remember that it is not a true algebra, but as we discussed, the commutator of two Hamiltonians has a structure "constant" that depends on one of the canonical variables, the triad. This means that in this ordering such a "constant" would have to appear to the right of the resulting commutator, which is not expected. In fact, an explicit calculation of the formal commutator shows the triads appear to the right. Therefore, it is not immediate that acting on a solution the commutator of two Hamiltonians vanishes and it has to be checked explicitly.

The simplest solution to the constraints in this representation is

$$\Psi[A] = \text{constant}. \tag{7.61}$$

This state is annihilated by all the constraints formally and it is easy to check that it is also annihilated with simple point-splitting regularizations. This state is less trivial than one may imagine. It has been explored in the context of Bianchi models and it has a quite non-trivial form if transformed into the traditional variables [135].

Jacobson and Smolin set out to find less obvious solutions to the constraint equations in this formalism. If one starts by considering the Gauss law, one would like the wavefunctionals to be invariant under $SU(2)$ gauge transformations. An example of such functionals is the Wilson loop,

$$W(A,\gamma) = \text{Tr}\left(\text{Pexp} \oint_\gamma dy^a A_a(y)\right). \tag{7.62}$$

In fact, as we have seen *any* gauge invariant function of a connection can be expressed as a combination of Wilson loops. In view of this, one can consider Wilson loops as an infinite family of wavefunctions in the connection representation parametrized by a loop $\Psi_\gamma(A) = W(\gamma, A)$ that forms an (overcomplete) basis of solutions to the quantum Gauss law constraint.

What happens to the diffeomorphism constraint? Evidently Wilson loops are not solutions. When a diffeomorphism acts on a Wilson loop, it gives as a result a Wilson loop with the loop displaced by the diffeomorphism performed. Therefore they are not annihilated by the diffeomorphism constraint and cannot become candidates for physical states of quantum gravity. In spite of that, they are worth exploring a bit more.

Remember they form an overcomplete basis in terms of which any physical state should be expandable (since any physical state has to be gauge invariant). We will therefore explore what happens when we act with the Hamiltonian constraint on them. To perform this calculation we only need the formula for the action of a triad on a holonomy along an open path $\gamma_o^{o'}$,

$$\hat{\tilde{E}}_i^a(x)U(\gamma_o^{o'}) = \frac{\delta}{\delta A_a^i(x)}U(\gamma_o^{o'}) = \oint_\gamma dy^a \delta^3(x-y)U(\gamma_o^y)\tau^i U(\gamma_y^{o'}), \quad (7.63)$$

where τ^i are $-i\sqrt{2}/2$ times the Pauli matrices.

The reason why we are considering an open path is to avoid ambiguities when we act with the second derivative. The expression for the action on the Wilson loop we are interested in is obtained in the limit in which o and o' coincide. We now act with a second triad,

$$\frac{\delta}{\delta A_a^i(x)}\frac{\delta}{\delta A_b^j(x)}U(\gamma_o^{o'}) =$$
$$\oint_\gamma dy^b \oint_{\gamma_o^y} dz^a \delta(x-y)\delta(x-z)U(\gamma_o^z)\tau^i U(\gamma_z^y)\tau^j U(\gamma_y^{o'})$$
$$+ \oint_\gamma dy^b \oint_{\gamma_y^{o'}} dz^a \delta(x-y)\delta(x-z)U(\gamma_o^y)\tau^j U(\gamma_y^z)\tau^i U(\gamma_z^{o'}). \quad (7.64)$$

We now take the trace and obtain the action of the Hamiltonian,

$$H(x)\Psi_\gamma[A] =$$
$$F_{ab}^k(x)\epsilon_{ijk}\left[\oint_\gamma dy^b \oint_{\gamma_o^y} dz^a \delta(x-y)\delta(x-z)\mathrm{Tr}(\tau^i U(\gamma_z^y)\tau^j U(\gamma_{y\,o}^{\,z}))\right.$$
$$\left.+ \oint_\gamma dy^b \oint_{\gamma_y^{o'}} dz^a \delta(x-y)\delta(x-z)\mathrm{Tr}(\tau^j U(\gamma_y^z)\tau^i U(\gamma_{z\,o}^{\,y}))\right], \quad (7.65)$$

where the notation $U(\gamma_{y\,o}^{\,z})$ denotes the portion of the loop going from y to z through the basepoint o.

If the loop has no kinks or intersections, the portion γ_z^y shrinks to a point due to the presence of the Dirac delta functions and the action of the Hamiltonian can be written as

$$\hat{\mathcal{H}}(x)\Psi_\gamma[A] =$$
$$F_{ab}^k(x)\epsilon_{ijk}\left[\oint_\gamma dy^b \oint_\gamma dz^a \delta(x-y)\delta(x-z)\mathrm{Tr}(\tau^i\tau^j U(\gamma_{y\,o}^{\,z}))\right.$$
$$+ \oint_\gamma dy^b \oint_\gamma dz^a \delta(x-y)\delta(x-z)\mathrm{Tr}(\tau^j\tau^i U(\gamma_{z\,o}^{\,y})) =$$
$$\left.\oint_\gamma dy^b \oint_\gamma dz^a \delta(x-y)\delta(x-z)\mathrm{Tr}(\delta^{ij}U(\gamma_{z\,o}^{\,y}))\right], \quad (7.66)$$

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where we have extended the second integral along the whole loop, since no additional contributions are added due to the fact that the loop is smooth.

Notice that we have a quantity $F^i_{ab}\epsilon_{ijk}$ which is antisymmetric in both a, b and i, j contracted with an expression that is symmetric in both a, b and i, j. Therefore, the expression vanishes! We have just proved that a Wilson loop formed with the Ashtekar connection is a (formal) solution of the Hamiltonian constraint of quantum gravity. This is a remarkable fact. Notice that up to this discovery *no* solution of this constraint was known in a general case (without making mini-superspace approximations). Historically, this discovery fostered the interest for loops in this context and led to the use of the loop representation.

A key to this result was the consideration of smooth non-intersecting loops. If the loops have intersections or kinks, the proof we presented above does not work. Moreover, it should be stressed that the result is *formal.* The expressions considered involve one-dimensional integrals of three-dimensional Dirac delta functions. In a particular coordinate system they are proportional to $\delta^2(0)$. Therefore we are canceling divergent terms.

To see if this result holds beyond the formal level, a regularization is needed. Two different regularizations were considered by Jacobson and Smolin [134]. The first one is based on "flux tubes", a process in which the loops are thickened out. The main drawback of this method is that it is not gauge invariant. Under this regularization, smooth loops solve the constraint with suitable prescriptions for limiting procedures. The second regularization method is based on a point-splitting of the two functional derivatives of the Hamiltonian constraint. Although point-splitting in general breaks gauge invariance (since point-split quantities exist at different points of the manifold and transform with different transformation matrices) one can restore gauge invariance connecting the point-split quantities with holonomies along paths connecting the split points. Unfortunately, under this procedure smooth Wilson loops *fail* to satisfy the constraint. An anomaly appears that is proportional to terms that depend on the curvature of the loops ("acceleration terms") and is non-vanishing. We will see that the role of the acceleration terms is different in the loop representation and there is a sense in which smooth loops correspond to solutions of the constraints. We will return to these and other regularization issues later.

Even ignoring the regularization issues of the Hamiltonian constraint, there are two main drawbacks to these solutions: they do not solve the diffeomorphism constraint and they fail to solve the Hamiltonian constraint if the loops have intersections.

Why care about loops with intersections? Why not just restrict ourselves to smooth loops? The problem appears when we try to get some sort of understanding of what these wavefunctionals are. The first question that comes to mind is what is the metric for such a state. This, in principle, is a meaningless question, since the metric is not an observable in the sense of Dirac, but let us ask it anyway to see where it leads. The metric acting on one of these states gives

$$\hat{\tilde{\tilde{q}}}^{ab}(x)\Psi_\gamma(A) = \frac{\delta}{\delta A^i_a}\frac{\delta}{\delta A^i_b}\Psi_\gamma(A) = X^{ax}X^{bx}\Psi_\gamma(A). \tag{7.67}$$

Again, this expression needs to be regularized. At a formal level we see that Wilson loops are eigenstates of the metric operator if the loops considered are smooth. Notice that the metric only has support distributionally along the direction of the tangent to the loop. Moreover, the metric has only one non-vanishing component, the one along the loop. Therefore it is a degenerate metric. Now, this statement is still meaningless in a diffeomorphism invariant context, but it actually can be given a rigorous meaning with a little elaboration. Consider the Hamiltonian constraint for general relativity with a cosmological constant, given by expression (7.54). The only difference with the vacuum constraint is the term involving the determinant of the spatial metric. This term can be promoted to the connection representation with similar regularization difficulties as the rest of the constraint. It is easy to see that the additional term formally annihilates a Wilson loop based on a smooth loop. Therefore the determinant of the three-metric vanishes for these states, as is expected for a degenerate metric. Since these states are annihilated by the vacuum Hamiltonian constraint *and* the determinant of the three-metric, this means they are states for an arbitrary value of the cosmological constant! That spells serious trouble. General relativity with and general relativity without a cosmological constant are *very* different theories, and one does not expect them to share a common set of states, except for special situations, such as for degenerate metrics.

It turns out, one can improve the situation a little using intersections. One can find some solutions to the Hamiltonian constraint for the intersecting case by considering linear combinations of holonomies in such a way that the contributions at the intersections cancel [134, 136, 26]. However, unexpectedly, this is not enough to construct non-degenerate solutions. All the solutions constructed in this fashion, if they satisfy the Hamiltonian constraint, are also annihilated by the determinant of the metric [26]. This, plus the fact that they do not satisfy the diffeomorphism constraint, shows that these solutions are of little physical use in this context. They were, however, very important historically as motivational objects for the study of loops. We will show later how, when one works in the loop representation, it is possible to generate solutions to all the constraints that, although still based on loops, do not have this degeneracy problem.

7.4.3 Triads to the left and the Chern–Simons form

If one orders the constraints with the triads to the left, there is potential for a problem: as we have said, apparently in this factor ordering the diffeomorphism constraint fails to generate diffeomorphisms on the wavefunctions. This would be a reason to abandon this ordering altogether. However, by considering a very generic regularized calculation one can prove that the diffeomorphism constraint actually generates diffeomorphisms, so this is not a problem [137]. Besides, there is the advantage that when one considers the constraint algebra, one obtains (these are only formal unregulated results) the correct closure [51].

Let us see how the regularized version of the constraint in this factor ordering generates diffeomorphisms. We consider a point-split version of the diffeomorphism constraint,

$$\hat{C}(\vec{N}) = \lim_{\epsilon\to 0} \int d^3x d^3y N^a(x) f_\epsilon(x-y) \frac{\delta}{\delta A_b^i(x)} F_{ab}^i(y), \tag{7.68}$$

where $\lim_{\epsilon\to 0} f_\epsilon(x-y) = \delta(x-y)$. This expression differs from that in the factor ordering with triads to the right by the term in which the functional derivative acts on F_{ab}^i, $\int d^3y f_\epsilon(x-y)\delta F_{ab}^i(y)/\delta A_a^i(x)$. When the functional derivative acts on the portion of F_{ab}^i linear in A_a^i one gets a contribution of the form $\int d^3x N^a(x) \int d^3y \partial_b \delta(x-y) f_\epsilon(x-y)$. If one considers a regulator that is symmetric in x, y, $f_\epsilon(x-y) = f_\epsilon(y-x)$, this contribution vanishes. The action of the functional derivative on the term quadratic in the connections vanishes due to the antisymmetry of the structure constants ϵ^{ijk} of $SU(2)$. We have therefore proved that the expression for the constraint with the triads to the left coincides, if one considers symmetric regulators, with the expression with the triads to the left. Since the former generates diffeomorphisms on the wavefunctions the latter does so as well. Therefore the diffeomorphism constraint regains its natural geometric interpretation and can be solved by considering wavefunctionals of the connection $\Psi[A]$ that are invariant under diffeomorphisms.

In this ordering, Wilson loops do not solve the Hamiltonian constraint. However, there is a very interesting and rich solution one can construct. Consider the following state, a function of the Chern–Simons form built with the Ashtekar connection,

$$\Psi_\Lambda[A] = \exp\left(-\frac{6}{\Lambda}\int \tilde{\epsilon}^{abc}\mathrm{Tr}[A_a\partial_b A_c + \tfrac{2}{3}A_aA_bA_c]\right). \tag{7.69}$$

This functional has the property that the triad equals the magnetic field constructed from the Ashtekar connection (in the language of Yang–Mills theory, the electric field equals the magnetic field),

$$\frac{\delta}{\delta A_a^i}\Psi_\Lambda[A] = \frac{3}{\Lambda}\tilde{\epsilon}^{abc}F_{bc}^i\Psi_\Lambda[A]. \tag{7.70}$$

Moreover, it is well known that this functional is invariant under (small) gauge transformations and diffeomorphisms. One can check that it is annihilated by the corresponding constraints (with the proviso of the symmetric regulator in the diffeomorphism constraint introduced above). What may come as a surprise is that it is actually annihilated by the Hamiltonian constraint with a cosmological constant. This is easy to see. Consider the constraint

$$\hat{\mathcal{H}} = \epsilon_{ijk}\frac{\delta}{\delta A_a^i}\frac{\delta}{\delta A_b^j}F_{ab}^k - \frac{\Lambda}{6}\epsilon_{ijk}\underset{\sim}{\epsilon}_{abc}\frac{\delta}{\delta A_a^i}\frac{\delta}{\delta A_b^j}\frac{\delta}{\delta A_c^k} \tag{7.71}$$

and notice that the rightmost derivative of the determinant of the metric reproduces the term on the left when acting on the wavefunction. Notice that the result holds without even considering the action of the other derivatives, and therefore is very robust *vis a vis* regularization. This result was noticed independently by Ashtekar [53] and Kodama [54]. A nice feature of this result is that the metric is non-degenerate in the sense that we discussed in the previous section. The metric is just given by the trace of the product of two magnetic fields built with the Ashtekar connection. Such a property holds classically for spaces of constant curvature. This has led some authors to suggest that this wavefunction is associated with the DeSitter geometry [55].

The reader may question the relevance of the Chern–Simons state. First of all, it is only one state. Moreover, a similar state is present in Yang–Mills theory (this is easy to see, since the Hamiltonian is $E^2 + B^2$ and adjusting constants one gets for the corresponding state $E = iB$) and is known to be non-physical since it is not normalizable. This is true, but it is also true that the nature of a theory defined on a fixed background as a Yang–Mills theory is expected to be radically different from that of a theory invariant under diffeomorphisms, such as general relativity. Therefore normalizability under the inner product of one theory does not necessarily imply or rule out normalizability under the inner product of the other. The non-normalizability in the Yang–Mills context is under the Fock inner product, and it is expected that inner products of that kind will not have any relevance in the context of general relativity. At the moment, however, the normalizability or not of any state in general relativity cannot be decided, since we lack an inner product for the theory.

It is remarkable that the Chern–Simons form, which is playing such a

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prominent role in particle physics nowadays, should have such a singular role in general relativity. It is the only non-trivial state in the connection representation that we know that may have something to do with a non-degenerate geometry. We will also see in chapters 10 and 11 that the state plays a prominent role in the progress made in finding states in the loop representation and has opened up new connections between general relativity, topological field theories and knot theory.

There are more things one could say about the connection representation. There is the compelling work of Ashtekar, Balachandran and Jo [61] concerning the CP violation problem and the partial success (in the linearized theory) of Ashtekar [56] in addressing the issue of time. We do not have space here to do justice to these pieces of work and we refer the reader to the relevant literature. In particular, a good summary of these topics appears in the book by Ashtekar [2].

7.5 Conclusions

We have formulated gravity canonically and discussed the general features of its canonical quantization. We have discussed the difficulties associated with the traditional metric variables and introduced a new set of variables that allows some progress in the definition of the quantum constraint equations and their solutions. We have discussed some of the factor ordering and regularization issues and set the stage for the introduction of a loop representation, which we will do in the following chapter.