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UNIQUENESS OF INVARIANT DENSITIES FOR CERTAIN RANDOM MAPS OF THE INTERVAL

BY

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ABSTRACT. A random map is a discrete time process in which one of a number of maps, $\mathcal{M}$, is chosen at random at each stage and applied. In this note we study a random map, where $\mathcal{M}$ is a set of piecewise linear Markov maps on [0, 1]. Sufficient conditions are presented which allow the determination of the unique absolutely continuous invariant measure of the process.

1. **Introduction**. In [6], a random map was defined to be "a discrete time process in which one of a number of functions is selected at random and applied." Let $T_1(x), \ldots, T_n(x)$ be maps of the unit interval and let $0 < p_i < 1$ be the probability of map T_i being applied at any given iteration. Then $\{T_i\}_{i=1}^n$ and $\{p_i\}_{i=1}^n$, $\Sigma_{i=1}^n p_i = 1$ constitute a random map T. If we select an $x_0 \in [0, 1]$, define $x_1 = T_i(x_0)$, where T_i has probability p_i, and we continue in this way, to obtain the orbit $\{x_{n+1} = T_i(x_n)\}$. A measure μ is called T-invariant if $\mu(A) = \Sigma_{i=1}^n p_i \mu(T_i^{-1}A)$ for each measurable set A.

In [6] sufficient conditions for a random map to have an absolutely continuous invariant measure are given. In the specific example $T_1 = x/2$ and $T_2(x) = 2x(\text{mod } 1)$ it is shown in [6] that the absolutely continuous invariant measure is unique. In this note we shall extend the uniqueness property to the class of piecewise linear Markov maps, and show how the unique absolutely continuous invariant measure can be determined.

2. **Markov Matrices**. A piecewise continuous map τ of an interval $I = [a_0, a_n]$ into itself is called Markov if there exist points $a_0 < a_1 < \ldots . < a_{n-1} < a_n$ such that for $i = 0, 1, \ldots, n-1$, $\tau_i = \tau|_{(a_i, a_{i+1})}$ is a homeomorphism onto some interval $(a_{j(i)}, a_{k(i)})$. The partition is referred to as a Markov partition. Let $\mathcal{C}$ denote the class of piecewise linear (not necessarily continuous) Markov maps on I. Let $I_i = (a_{i-1}, a_i)$ and suppose $\tau \in \mathcal{C}$ maps I_i onto $I_j \cup I_{j+1} \cup \ldots . \cup I_{j+k}$. Then the slope of the map τ on I_i is

$$m_{i,j} = \frac{a_{j+k} - a_{j-1}}{a_i - a_{i-1}}$$

We define the matrix $M = M_\tau$ by defining the ith row of M to have entries $1/m_{i,j}$ in

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columns $j, j+1, \ldots, j+k$, and 0 in all the remaining columns. For example, if τ is as shown in Figure 1,

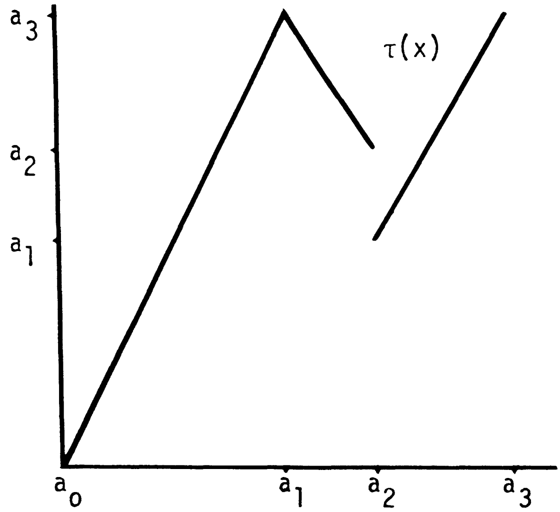

FIGURE 1.

then

$$M = \begin{pmatrix} \dfrac{a_1 - a_0}{a_3 - a_0} & \dfrac{a_1 - a_0}{a_3 - a_0} & \dfrac{a_1 - a_0}{a_3 - a_1} \\ 0 & 0 & \dfrac{a_2 - a_1}{a_3 - a_2} \\ 0 & \dfrac{a_3 - a_2}{a_3 - a_1} & \dfrac{a_3 - a_2}{a_3 - a_1} \end{pmatrix}$$

Notice that the non-negative entries in each row are equal and contiguous. If $|d\tau_i/dx| > 1$, then the non-negative entries in the ith row will be less than 1. Let $\mathcal{M}$ denote the class of matrices derived from Markov maps $\tau \in \mathcal{C}$. We shall refer to $M \in \mathcal{M}$ as a Markov matrix. Markov matrices are studied in [2, 3, 4]. In [3] it is shown that if $M \in \mathcal{M}$ then M is similar to a stochastic matrix via a simple diagonal matrix. Hence M has 1 as an eigenvalue of maximum modulus. Furthermore, if M is also irreducible, then the algebraic and geometric multiplicities of the eigenvalue 1 are also 1. This proves the existence of a large class of non-negative, not necessarily stochastic, matrices whose eigenvalue of maximum modulus is 1. We also note that for each $\tau \in \mathcal{C}$, $M_\tau \in \mathcal{M}$ is uniquely defined, but for each $M \in \mathcal{M}$ there are 2^n different Markov maps which induce the same M, since the slope on each I_i can be either positive or negative without altering M. Finally, we remark that if the partition is equal, M_τ is stochastic.

3. **Frobenius–Perron operator**. Let τ be a non-singular piecewise continuous, piecewise C^2 map on a partition $a_0 < a_1 < \ldots . < a_{n-1} < a_n$ of I. The

Frobenius–Perron operator $P_\tau : \mathcal{L}_1 \to \mathcal{L}_1$ is defined by

$$P_\tau f(x) = \frac{\mathrm{d}}{\mathrm{d}x} \int_{\tau^{-1}[0,x]} f(s)\,\mathrm{d}s,$$

where $\mathcal{L}_1$ is the space of Lebesque integrable functions on I. It is well known [1] that the function $f^* \in \mathcal{L}_1$ is invariant under τ, i.e.,

$$\int_{\tau^{-1}(A)} f(s)\mathrm{d}s = \int_A f(s)\mathrm{d}s, \tag{1}$$

where A is any Lebesque measurable set of I, if and only if f^* is a fixed point of P_τ. By $\tau^{-1}(A)$ we mean the set $\{x \in I : \tau(x) \in A\}$.

For $\tau \subset \mathcal{C}$ it is shown in [2] that

$$M_\tau = P_\tau|_{\mathcal{L}},$$

where $\mathcal{L} \subset \mathcal{L}_1$ is the space of step functions on the partititon defined by $a_0 < a_1 < \ldots . < a_{n-1} < a_n$. If the matrix M_τ has 1 an eigenvalue, then the associated non-negative eigenvector $\pi(x)$, viewed as a step function on $[a_0, a_n]$, is an invariant density under τ, i.e., (1) is satisfied for $\pi(x)$, and

$$\pi M_\tau = P_\tau \pi(x)$$

where $\pi = (\pi_1, \pi_2, \ldots, \pi_n)$, $\pi_i = \pi(x)|_{[a_{i-1}, a_i]}$.

4. **Existence of absolutely continuous invariant measure for general random maps**. Let τ: $I \to I$ be a non-singular transformation, and let X be a random variable on I with probability density function $f \in \mathcal{L}_1(I)$. Then the random variable $\tau(X)$ has a probability density function (by virtue of the Radon–Nikodyn Theorem) and it is given by $P_\tau f$, where P_τ is the Frobenius–Perron operator defined in section 2.

Consider now the stationary stochastic process defined by

$$X_{n+1} = \alpha\tau(X_n) + (1-\alpha)\,\gamma(X_n) \tag{2}$$

where τ and γ are non-singular transformations from I into I and α is a random variable which assumes the value 1 with probability $0 \le \lambda \le 1$ and the value 0 with probability $1-\lambda$. Then

$$\begin{aligned}\operatorname{Prob}\{X_{n+1} \le x\} &= \operatorname{Prob}\{\alpha\tau(X_n) + (1-\alpha)\,\gamma(X_n) \le x\} \\ &= \operatorname{Prob}\{\alpha\tau(X_n) + (1-\alpha)\,\gamma(X_n) \le x \,|\, \alpha = 1\}\operatorname{Prob}\{\alpha = 1\} \\ &\quad + \operatorname{Prob}\{\alpha\tau(X_n) + (1-\alpha)\,\gamma(X_n) \le x \,|\, \alpha = 0\}\operatorname{Prob}\{\alpha = 0\}\end{aligned}$$

i.e.,

$$\operatorname{Prob}\{X_{n+1} \le x\} = \lambda \operatorname{Prob}\{\tau(X_n) \le x\} + (1-\lambda)\operatorname{Prob}\{\gamma(X_n) \le x\} \tag{3}$$

Let X_n have probability density function f. Differentiating both sides of (3), we get:

$$\mathcal{P}_\lambda f(x) = \lambda P_\tau f(x) + (1-\lambda) P_\gamma f(x),$$

where $\mathcal{P}_\lambda f$ denotes the probability density function of X_{n+1}. A fixed point of $\mathcal{P}_\lambda$ can therefore be interpreted as the density of the stationary measure for the stochastic process (3).

DEFINITION: A transformation $\tau: I \to I$ is piecewise C^2 if there exists a partition $0 = a_0 < a_1 < \ldots . < a_p = 1$ of I such that for each integer $i = 1, \ldots, p$, the restriction τ_i of τ to (a_{i-1}, a_i) is a C^2 function which can be extended to the closed interval $[a_{i-1}, a_i]$ as a C^2 function. τ need not be continuous at the points $\{a_i\}$. The following result is a special case of Theorem 1 of [6].

LEMMA 1. *Let τ and γ be nonsingular, piecewise C^2 transformations from I into I. If* $\inf|(\tau^k)'(x)| > 1$ *and* $\inf|(\gamma^l)'(x)| > 1$ *for some integers k and l, then $\mathcal{P}_\lambda = \lambda P_\tau + (1-\lambda)P_\gamma$ has a fixed point $f^* \in \mathcal{L}_1$ for each* $0 \leq \lambda \leq 1$.

5. **Special random maps**. For any piecewise C^2 map $\tau: I \to I$, it can be shown that P_τ has the following representation [1]:

$$P_\tau f(x) = \sum_{i=1}^{n} f(\psi_i(x))\sigma_i(x)\chi_i(x) \tag{4}$$

where $\psi_i = \tau_i^{-1}$, $\sigma_i(x) = |\psi_i'(x)|$ and χ_i is the characteristic function of the interval $J_i = \tau_i(I_i)$, $\{I_i\}_{i=1}^n$ being the subintervals of the partition for which τ is piecewise C^1.

LEMMA 2. *Let $\tau, \gamma \in \mathcal{C}$ have a common Markov partition $\mathcal{J} = \{I_i\}_{i=1}^n$ of I and let* $\alpha = \inf|\tau'(x)| > 1$, $\beta = \inf|\gamma'(x)| > 1$. *Then every fixed point $\pi(x) = \pi_\lambda(x)$ of $\mathcal{P}_\lambda$* $= \lambda P_\tau + (1-\lambda)P_\gamma$, $0 \leq \lambda \leq 1$, *is a step function on $\mathcal{J}$.*

PROOF: Fix $0 \leq \lambda \leq 1$, using (4), we obtain:

$$\lambda \sum_{i=1}^{n} \pi(\tau_i^{-1}(x)) \frac{1}{|\tau_i'(x)|} \chi_{\tau_i(I_i)}(x) + (1-\lambda) \sum_{i=1}^{n} \pi(\gamma_i^{-1}(x)) \frac{1}{|\gamma_i'(x)|} \chi_{\gamma_i(I_i)}(x) = \pi(x)$$

Notice that $|\tau_i'(x)|$ and $|\beta_i'(x)|$ are both constant on I_i, since $\tau, \gamma \in \mathcal{C}$. Now let I_k be any interval of the partition $\mathcal{J}$ and let $x, y \in I_k$ be distinct and fixed. Then $\chi_{\tau_i(I_i)}(x) = \chi_{\tau_i(I_i)}(y)$ for all i and $\chi_{\gamma_i(I_i)}(x) = \chi_{\gamma_i(I_i)}(y)$ for all i. Thus,

$$\begin{aligned}\pi(x) - \pi(y) = &\sum_{i=1}^{n} \frac{1}{|\tau_i'|} [\pi(\tau_i^{-1}(x)) - \pi(\tau_i^{-1}(y))] \chi_{\tau_i(I_i)}(x) \\ &+ (1-\lambda) \sum_{i=1}^{n} \frac{1}{|\gamma_i'|} [\pi(\gamma_i^{-1}(x)) - \pi(\gamma_i^{-1}(y))] \chi_{\gamma_i(I_i)}(x)\end{aligned}$$

Let i_1 vary over all integers $i \in \{1, 2, \ldots, n\}$ such that $x \in \tau_i(I_i)$ and let j_1 vary over all integers $i \in \{1, 2, \ldots, n\}$ such that $x \in \gamma_i(I_i)$. Then,

$$\begin{aligned}\pi(x) - \pi(y) = \lambda &\sum_{i_1} \frac{1}{|\tau_{i_2}'|} [\pi(\tau_{i_1}^{-1}(x)) - \pi(\tau_{i_1}^{-1}(y))] \\ &+ (1-\lambda) \sum_{j_1} \frac{1}{|\gamma_{j_1}'|} [\pi(\gamma_{j_1}^{-1}(x)) - \pi(\gamma_{j_1}^{-1}(y))]\end{aligned}$$

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Similarly, for each i_1 and j_1,

$$\pi(\tau_{i_1}^{-1}(x)) - \pi(\tau_{i_1}^{-1}(y)) = \lambda \sum_{i_2} \frac{1}{|\tau'_{i_2}|} [\pi(\tau_{i_2}^{-1}\tau_{i_1}^{-1}(x)) - \pi(\tau_{i_2}^{-1}\tau_{i_1}^{-1}(y))]$$
$$+ (1-\lambda) \sum_{j_2} \frac{1}{|\gamma'_{j_1}|} [\pi(\gamma_{j_2}^{-1}\tau_{i_1}^{-1}(x)) - \pi(\gamma_{j_2}^{-1}\tau_{i_1}^{-1}(y))],$$

where i_2 varies over all integers $i \in \{1, 2, \ldots, n\}$ such that $\tau_{i_1}^{-1}(x) \in \tau_i(I_i)$ and j_2 varies over all integers $i \in \{1, 2, \ldots, n\}$ such that $\tau_{i_1}^{-1}(x) \in \gamma_i(I_i)$. In an identical manner, we get,

$$\pi(\gamma_{j_1}^{-1}(x)) - \pi(\gamma_{j_2}^{-1}(y)) = \lambda \sum_{l_2} \frac{1}{|\tau'_{l_2}|} [\pi(\tau_{l_2}^{-1}\gamma_{j_1}^{-1}(x)) - \pi(\tau_{l_2}^{-1}\gamma_{j_1}^{-1}(y))]$$
$$+ (1-\lambda) \sum_{k_2} \frac{1}{|\gamma'_{k_2}|} [\pi(\gamma_{k_2}^{-1}\gamma_{j_1}^{-1}(x)) - \pi(\gamma_{k_2}^{-1}\gamma_{j_1}^{-1}(y))].$$

Therefore, letting $\rho = \max(\alpha, \beta)$

$$\begin{aligned} |\pi(x) - \pi(y)| &\leq \frac{\lambda}{\rho} \sum_{i_1} |\pi(\tau_{i_1}^{-1}(x)) - \pi(\tau_{i_1}^{-1}(y))| \\ &+ \frac{(1-\lambda)}{\rho} \sum_{j_1} |\pi(\gamma_{j_1}^{-1}(x)) - \pi(\gamma_{j_1}^{-1}(x))| \\ &\leq \frac{\lambda^2}{\rho^2} \sum_{i_1} \sum_{i_2} |\pi(\tau_{i_2}^{-1}\tau_{i_1}^{-1}(x)) - \pi(\tau_{i_2}^{-1}\tau_{i_1}^{-1}(y))| \\ &+ \frac{\lambda(1-\lambda)}{\rho^2} \sum_{i_1} \sum_{j_2} |\pi(\tau_{j_2}^{-1}\tau_{i_1}^{-1}(x)) - \pi(\tau_{j_2}^{-1}\tau_{i_1}^{-1}(y))| \\ &+ \frac{(1-\lambda)\lambda}{\rho^2} \sum_{j_1} \sum_{l_2} |\pi(\tau_{l_2}^{-1}\gamma_{j_1}^{-1}(x)) - \pi(\tau_{l_2}^{-1}\gamma_{j_1}^{-1}(y))| \\ &+ \frac{(1-\lambda)^2}{\rho^2} \sum_{j_1} \sum_{k_2} |\pi(\gamma_{k_2}^{-1}\gamma_{j_1}^{-1}(x)) - \pi(\gamma_{k_2}^{-1}\gamma_{j_1}^{-1}(y))|. \end{aligned}$$

Continuing this procedure M times we see that

$$|\pi(x) - \pi(y)| \leq \sum_{m=0}^{M} \binom{M}{m} \lambda^m \frac{1}{\rho^M} L_M \tag{5}$$

where each L_M has the form:

$$\sum_{s_1} \cdots\cdots \sum_{s_M} |\pi(\mu_{s_M}^{-1} \cdots\cdots \mu_{s_1}^{-1}(x)) - \pi(\mu_{s_M}^{-1} \cdots\cdots \mu_{s_1}^{-1}(y))|$$

with $\mu_{s_k} = \tau_{s_k}$ or γ_{s_k}. Now, for each $m = 0, 1, \ldots, M$,

$$\{(\mu_{s_M}^{-1} \cdots\cdots \mu_{s_1}^{-1}(x), \mu_{s_M}^{-1} \cdots\cdots \mu_{s_1}^{-1}(y)\}_{s_1, s_2, \ldots, s_M}$$

is a collection of at most n^M nonoverlapping intervals. Hence, for all m

$$L_M \leq \bigvee_0^1 \pi$$

Consequently, for any M,

$$|\pi(x) - \pi(y)| \leq \frac{1}{\rho^M} \bigvee_0^1 \pi \tag{6}$$

Since $\rho > 1$ and π is of bounded variation, (7) implies that $\pi(x) = \pi(y)$, i.e. π is constant on I_k. Q.E.D.

THEOREM 1. *Let $\tau, \gamma \in \mathscr{C}$ have a common Markov partition $\mathscr{J} = \{I_i\}_{i=1}^n$ with* $\inf|\tau'| > 1$ *and* $\inf|\gamma'| > 1$. *Let A and B be the Markov matrices induced by τ and γ, respectively. Then the density of every invariant measure of the random Markov map*

$$\alpha\tau(x) + (1 - \alpha)\, \gamma(x),$$

where α is a 0–1 random variable with $\text{Prob}\{\alpha = 1\} = \lambda$, *is a step function $\pi(x)$ on $\mathscr{J}$, and the vector $\pi = (\pi_1, \ldots, \pi_n\}$, $\pi_i = \pi(x)|_{I_i}$ is a left eigenvector of the matrix*

$$\mathscr{C}_\lambda = \lambda A + (1 - \lambda)B$$

PROOF: Let $S \subset \mathscr{L}_1$ denote the step functions on the partition $\mathscr{J}$ of I. Then

$$\begin{aligned} \mathscr{P}_{\lambda|_S} &= \lambda P_{\tau|_S} + (1 - \lambda) P_{\gamma|_S} \\ &= \lambda A + (1 - \lambda)B \end{aligned}$$

By virtue of Lemma 1, $\mathscr{P}_\lambda$ admits a fixed point $\pi(x) \geq 0$. In view of Lemma 2, we know that $\pi \in S$. Hence π, viewed as a vector of $\mathscr{J}$, must be a left-eigenvector of $\mathscr{C}_\pi$. Q.E.D.

COROLLARY 1: *If $\mathscr{C}_\pi$ is irreducible, π is unique (up to constant multiples).*

PROOF: Since A and B are similar to stochastic matrices, $r(A)$, the spectral radius of A, is 1, and $r(B) = 1$. It is easy to show that $r(C_\lambda) = 1$. Thus π is an eigenvector associated with the maximal eigenvalue. It then follows from the Perron–Frobenius Theorem that π is unique (up to constant multiples).

REMARK: The foregoing analysis can be easily generalized to n maps on the interval.

EXAMPLE: Let τ, γ: $[0, 1] \to [0, 1]$, defined on the common Markov partition $\mathscr{D} = \{0, 1/3, 2/3, 1\}$, be as shown in Figure 2. Since there are more than two discontinuities in the derivatives for both maps, Cor. 7 of [6] does not yield uniqueness. On the given partition, τ and γ induce the Markov matrices

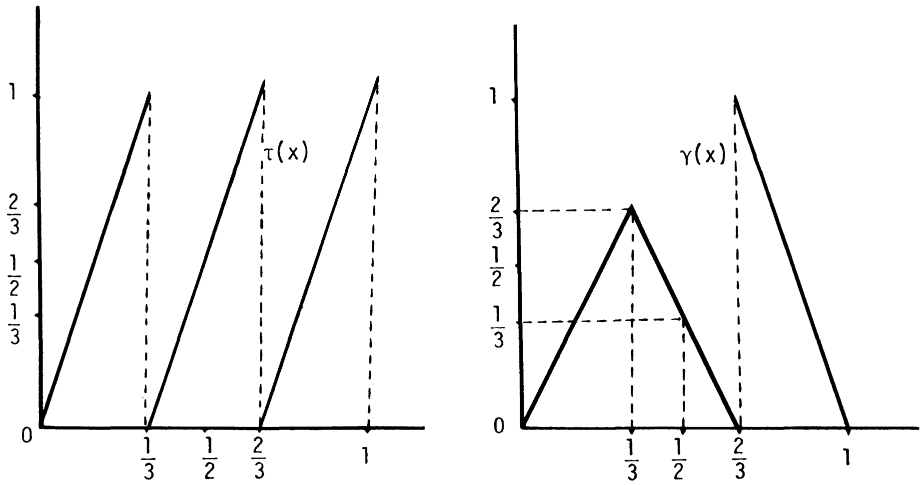

FIGURE 2.

$$A = \begin{pmatrix} 1/3 & 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 0 & 0 \\ 0 & 0 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 & 1/3 \end{pmatrix} \qquad B = \begin{pmatrix} 1/2 & 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 & 0 \\ 1/2 & 0 & 0 & 0 \\ 1/3 & 1/3 & 1/3 & 0 \end{pmatrix}$$

respectively. By Corollary 1, we know that for any $0 \le \lambda \le 1$, $C_\lambda = \lambda A + (1 - \lambda)B$ has a unique non-negative eigenvector π associated with the eigenvalue 1. For $\lambda = 1/2$,

$$C_\lambda = \begin{pmatrix} 1/6 & 5/6 & 5/6 & 5/6 \\ 1/6 & 5/12 & 1/4 & 0 \\ 1/4 & 0 & 1/6 & 1/6 \\ 1/3 & 1/3 & 1/3 & 1/3 \end{pmatrix}$$

The normalized left eigenvector for the eigenvalue 1 is $\pi_{1/2} = 3/29\ (12, 16, 16, 1)$.

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