RAMANUJAN CONGRUENCES FOR $p_{-k}(n)$

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1. Introduction. Let

$$f(x) = \prod_{r=1}^{\infty} (1 - x^r), \tag{1}$$

$$f^k(x) = \sum_{n=0}^{\infty} p_k(n)x^n \qquad (|x| < 1). \tag{2}$$

Thus $p_{-1}(n) = p(n)$ is just the partition function, for which Ramanujan **(4)** found congruence properties modulo powers of 5, 7, and 11. Ramanathan **(3)** considers the generalization of these congruences modulo powers of 5 and 7 for all k; unfortunately his results are incorrect, because of an error in his Lemma 4 on which his main theorems depend. This error is essentially a misquotation of the results of Watson **(5)**, which one may readily understand in view of Watson's formidable notation. Professor Ramanathan tells me he has been aware of the error for some time.

In this paper we shall prove the relevant results modulo powers of 2, 3, 5, 7, and 13; similar results certainly exist modulo powers of 11, but the technique of Atkin **(1)**, while sufficiently powerful to deal with a fixed small k, does not seem to extend to general k. We have in fact

THEOREM 1. *Let $k > 0$ and q be one of the primes* 2, 3, 5, 7, *or* 13. *Then if* $24m \equiv k \pmod{q^n}$ *we have* $p_{-k}(m) \equiv 0 \pmod{q^{\alpha n/2+\epsilon}}$ *where* $\epsilon = \epsilon(k) = O(\log k)$ *and* $\alpha = \alpha(k, q)$ *depending on q and the residue of k modulo* 24 *according to the following table:*

$k =$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
$q = 2$	–	–	–	–	–	–	–	3	–	–	–	–	–	–	–	3	–	–	–	–	–	–	–	3
$q = 3$	–	–	3	–	–	2	–	–	3	–	–	2	–	–	1	–	–	2	–	–	1	–	–	0
$q = 5$	2	1	1	1	2	2	1	1	1	1	1	0	0	0	1	1	0	0	0	1	1	0	0	0
$q = 7$	1	1	1	2	1	1	1	0	0	0	1	0	0	1	0	0	0	1	0	0	1	0	0	0
$q = 13$	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

The appearance of a blank in the table when $q = 2$ or 3 means that nothing is asserted, and in fact the structure of the modular relations is such that any results are unlikely; the problem is analogous to that of the parity of $p(n)$. We now consider how far Theorem 1 is "best possible."

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For a given q, k and n, let $d = d(q, k, n)$ be the largest power of q which divides $p_{-k}(m)$ for *all* m with $24m \equiv 1 \pmod{q^n}$. Theorem 1 states that $d \geqslant \alpha n/2 + O(\log k)$, for all k and n. For $q = 2$, 3, or 13, we have in fact $d = \alpha n/2 + O(\log k)$, and for any given k an algorithm can be given to determine d precisely in $O(\log k)$ steps. For $q = 5$ or 7 we can show that there *exist* k in each residue class modulo 24 such that $d = \alpha n/2 + O(\log k)$, and again for these k determine d precisely. In particular when $\alpha = 0$ there exist k with $d = 0$ for all n, with two exceptions.

However, it appears to the author that for $q = 5$ and 7 more powerful results must obtain, namely that given k and q there exists a $\beta = \beta(k, q)$ such that $d = \beta n/2 + O(\log k)$ for all n. A reasonable conjecture is that such a β is bounded by a constant independent of k, and that β depends only on q and the residue class of k modulo $24q^\delta$ for some fixed $\delta(q)$. However, the path to such a result is blocked by numerical accidents which seem difficult to overcome theoretically. The best result the author can obtain in this direction is

THEOREM 2. *Let* $k \equiv 9985 \pmod{29400}$ *and* $24m \equiv k \pmod{5^l 7^n}$. *Then*

$$p_{-k}(m) \equiv 0 \pmod{5^{2l+\epsilon} 7^{3n/2+\epsilon}}$$

where

$$|\epsilon| \leqslant 2[\log k/\log 5] + 5.$$

We do not give the proof of Theorem 2 in this paper, since it is analogous to that of Theorem 1 but involves some heavy details.

Our main weapon in the proof of Theorem 1 is the irrational modular equation, which we use as in Watson **(5)**. However, we greatly simplify Watson's actual technique, and (for instance) our methods imply a three-page proof of Ramanujan's $p(n)$ congruence modulo a general power of 5. We give in § 2 a fairly detailed account for the case $q = 5$, and sketch in § 3 the basic formulae required for the other values of q.

The informed reader will of course recognize the general form of our results in terms of modular functions on $\Gamma_0(q)$ and $\Gamma_0(q^2)$, and that the unifying factor in the primes we consider is that for all of them the genus of $\Gamma_0(q)$ is zero. However, we prefer not to introduce this aspect since (it appears) one cannot obtain the full number-theoretic details without the use of modular equations in an essentially elementary way; a "more powerful" method such as that of Atkin **(1)** for the case $q = 11$ is in fact less effective. Of course the modular equations themselves are needed, and the natural proof of these is by equating coefficients on both sides of an identity whose form is known by modular theory. However for $q = 2$, 3, and 5, these equations can be found using no more than Euler's and Jacobi's series without much work.

2.1. We now let $q = 5$ and define a linear operator $U = U_5$ acting on any power series $F(x) = \sum_{n \geqslant N} a(n) x^n$ by

$$UF(x) = \sum_{5n \geqslant N} a(5n) x^n. \tag{3}$$

Clearly

$$U(\alpha_1 F_1(x) + \alpha_2 F_2(x)) = \alpha_1 UF_1(x) + \alpha_2 UF_2(x), \tag{4}$$

$$U(F_1(x)\cdot F_2(x^5)) = F_2(x)\cdot UF_1(x). \tag{5}$$

If $\omega^5 = 1$, $\omega \neq 1$, it is easily seen that

$$5UF(x) = \sum_{r=0}^{4} F(\omega^r x^{1/5}). \tag{6}$$

We also define a valuation $\pi = \pi_5$ by

$$5^{\pi(b)} \mid b, \qquad 5^{\pi(b)+1} \nmid b, \tag{7}$$

for any integer b. If $b = 0$, we write conventionally $\pi(0) = \infty$ and regard any inequality $\pi(0) > a$ as valid. Clearly

$$\pi(bc) = \pi(b) + \pi(c), \tag{8}$$

$$\pi(b + c) \geqslant \min(\pi(b), \pi(c)) \text{ with equality unless } \pi(b) = \pi(c). \tag{9}$$

Finally we extend π to any power series $F(x) = \sum_{n \geqslant N} a(n)x^n$ by writing

$$\pi(F(x)) = \min_{n\geqslant N} \pi(a(n)).$$

If $F_1(x)$ has leading coefficient unity, then

$$\pi(F_1(x)\cdot F_2(x)) = \pi(F_2(x)). \tag{10}$$

2.2. We now write

$$g(x) = xf^6(x^5)/f^6(x), \qquad \phi(x) = xf(x^{25})/f(x). \tag{11}$$

Then $t = \phi(x^{1/5})$ and $g(x)$ are connected by the equation*

$$t^5 - g(x)(25t^4 + 25t^3 + 15t^2 + 5t + 1) = 0. \tag{12}$$

It is clear that all the roots of (12), regarded as an equation in t, are given by

$$t = \phi(\omega^r x^{1/5}), \qquad r = 0 \text{ to } 4, \text{ where } \omega^5 = 1,\ \omega \neq 1.$$

Thus if S_r denotes the sum of the rth powers of the roots of (12), we have, by (6),

$$5U\phi^r(x) = S_r. \tag{13}$$

By Newton's formula we find, writing $g = g(x)$, that

$$\begin{aligned}
S_1 &= 5^2 g,\\
S_2 &= 5^4g^2 + 2.5^2g,\\
S_3 &= 5^6g^3 + 3.5^4g^2 + 9.5g,\\
S_4 &= 5^8g^4 + 4.5^6g^3 + 22.5^3g^2 + 4.5g,\\
S_5 &= 5^{10}g^5 + \underline{5}.5^8g^4 + \underline{40}.5^5g^3 + \underline{20}.5^3g^2 + \underline{5}g.
\end{aligned} \tag{14}$$

*According to Watson, (11) appears unproved in Ramanujan's note books, and Watson **(5)** gives an elementary proof. The earliest reference I can find is Weber **(6, p. 256)**.

Since for $R > 5$ we have

(15) $$S_R = 5^2 g S_{R-1} + 5^2 g S_{R-2} + 3.5 g S_{R-3} + 5 g S_{R-4} + g S_{R-5},$$

it is clear that S_r is a polynomial in g all of whose coefficients are divisible by 5. However, we shall find it more convenient to write

(16) $$S_r = \sum_{\rho=1}^{\infty} a_{r\rho} g^{\rho},$$

where in fact*

(17) $$a_{r\rho} \neq 0 \quad \text{if and only if } [(r+4)/5] \leqslant \rho \leqslant r.$$

We now prove

LEMMA 1.

$$\pi(a_{r\rho}) \geqslant \left[\frac{5\rho - r + 1}{2}\right].$$

Proof. We proceed by strong induction on r. By inspection, Lemma 1 holds for $1 \leqslant r \leqslant 5$, and for $\rho = 1$ with $r > 5$ (since then $a_{r1} = 0$). Assuming Lemma 1 for all ρ and all $r < R$, for some $R > 5$, we obtain from (15)

$$a_{R\rho} = 5^2 a_{R-1,\rho-1} + 5^2 a_{R-2,\rho-1} + 3.5 a_{R-3,\rho-1} + 5 a_{R-4,\rho-1} + a_{R-5,\rho-1}$$

for $\rho > 1$, and so by (8) and (9) we have

$$\pi(a_{R\rho}) \geqslant \min_{1 \leqslant \sigma \leqslant 5} \left\{\left[\frac{5(\rho-1) - (R-\sigma) + 1}{2}\right] + \left[\frac{6-\sigma}{2}\right]\right\} \geqslant \left[\frac{5\rho - R + 1}{2}\right]$$

since

$$[b/2] + [c/2] \geqslant [(b + c - 1)/2],$$

and this gives Lemma 1 with $r = R$, completing the proof.

Lemma 1 does not take advantage of the underlined powers of 5 in (14), but it usually suffices for our inductions. We also need

(18) $$\pi(a_{r\rho}) \geqslant 1 \text{ always},$$

which was stated after (15), and

(19) $$\pi(a_{r\rho}) = 1 \qquad \text{for } \rho = [(r + 4/5] \text{ and } r \equiv 3, 4, \text{ or } 5 \pmod 5),$$

which can be seen easily by the method used to prove Lemma 1.

We now show that

(20) $$5U\{\phi^k(x)\cdot g^r(x)\} = g^{-r}(x)\cdot S_{k+6r} \qquad (k \geqslant 0, r \geqslant 0, k + r \geqslant 1).$$

For

$$\begin{aligned} U\{\phi^k(x)\cdot g^r(x)\} &= U\{\phi^{k+6r}(x)\cdot g^{-r}(x^5)\} \\ &= g^{-r}(x)\cdot U\phi^{k+6r}(x) \qquad \text{by (5)} \\ &= 5^{-1} g^{-r}(x)\cdot S_{k+6r} \qquad \text{by (13)}. \end{aligned}$$

*Throughout this paper we use square brackets $[x]$ to denote the integral part of x.

2.3. We now fix some $k > 0$ and define

$$L_1(x) = U\phi^k(x) = \sum_{s=1}^{\infty} b_{1s}g^{s+k_1}(x), \qquad k_1 = [(k-1)/5],$$

$$(21) \qquad L_{2n}(x) = UL_{2n-1}(x) = \sum_{s=1}^{\infty} c_{ns}g^{s+l_n}(x), \quad l_n = [k_n/5],$$

$$L_{2n+1}(x) = U\{\phi^k(x)\cdot L_{2n}(x)\} = \sum_{s=1}^{\infty} b_{n+1,s}g^{s+k_{n+1}}(x), \qquad k_{n+1} = [(k+l_n)/5],$$

The first two terms of the equations define $L_n(x)$ inductively, and the third term is justified by (17). The infinite series in fact terminate and b_{n1} and c_{n1} are non-zero; examination of the leading power of x in $L_n(x)$ justifies the values of k_n and l_n.

Reverting to our original problem, we now write $p_{-k}(m) = 0$ if m is negative. Let

$$(22) \qquad \lambda_{2n-1} = \lambda_{2n} = -k(5^{2n}-1)/24 \qquad (n \geqslant 1),$$

so that

$$24\lambda_n \equiv 1 \pmod{5^n}.$$

Then we have

LEMMA 2. *For* $n \geqslant 1$

$$(23) \qquad f^{-k}(x^5)\cdot L_{2n-1}(x) = \sum_{m=1+k_n}^{\infty} p_{-k}(5^{2n-1}m + \lambda_{2n-1})\cdot x^m,$$

$$(24) \qquad f^{-k}(x)\cdot L_{2n}(x) = \sum_{m=1+l_n}^{\infty} p_{-k}(5^{2n}m + \lambda_{2n})\cdot x^m.$$

Proof. We have

$$f^{-k}(x^5)\cdot L_1(x) = f^{-k}(x^5)U\left\{f^k(x^{25})\cdot \sum_{m=0}^{\infty} p_{-k}(m)\cdot x^{m+k}\right\}$$
$$= \sum_{m=1+k_1}^{\infty} p_{-k}(5m-k)x^m \qquad \text{by (5)},$$

which is (23) for $n = 1$. Next, assuming (23) for n, (24) for n is an immediate deduction. Finally, assuming (24) for n we have

$$f^{-k}(x^5)L_{2n+1}(x) = f^{-k}(x^5)\cdot U\left\{f^k(x^{25})\cdot \sum_{m=1+l_n}^{\infty} p_{-k}(5^{2n}m + \lambda_{2n})x^{m+k}\right\}$$
$$= \sum_{m=1+k_{n+1}}^{\infty} p_{-k}(5^{2n}(5m-k) + \lambda_{2n})\cdot x^m,$$

which is (23) for $n+1$ since

$$\lambda_{2n+1} = -5^{2n}k + \lambda_{2n}.$$

This completes the proof of Lemma 2 by induction.

2.4. We now define

$$\theta(b) = 1 \qquad \text{if } b \equiv 1 \text{ or } 2 \pmod 5,$$
$$\theta(b) = 0 \qquad \text{if } b \equiv 3, 4, \text{ or } 5 \pmod 5.$$

Then by (18) and Lemma 1 we have

(25) $$\pi(a_{b\rho}) \geqslant \theta(b) + 1 \qquad \text{for } \rho = [(b+4)/5],$$

and a detailed calculation shows that we have *equality* in (25) unless $b \equiv 11$ or $17 \pmod{25}$.

We also define, for $n \geqslant 1$,

(26) $$A_{2n-1} = \theta(k) + \sum_{r=1}^{n-1} \{\theta(6k_r + 6) + \theta(6l_r + 6 + k)\},$$
$$A_{2n} = A_{2n-1} + \theta(6k_n + 6).$$

Finally let

(27) $$5^{A_{2n-1}} b'_{ns} = b_{ns}, \qquad 5^{A_{2n}} c'_{ns} = c_{ns}.$$

We show that all the b'_{ns} and c'_{ns} are integers, which follows from

LEMMA 3. *We have*

(28) $$\pi(b'_{ns}) \geqslant \max(0, [(5s-6)/2]),$$

(29) $$\pi(c'_{ns}) \geqslant \max(0, [(5s-6)/2]).$$

Proof. First, we have

$$5^{\theta(k)+1} b'_{1s} = a_{k, s+k_1}$$

so that

$$\pi(b'_{1s}) = \pi(a_{k,s+k_1}) - 1 - \kappa(k)$$
$$\geqslant [(5s + 5k_1 - k - 1 - 2\theta(k))/2] \qquad \text{by Lemma 1}$$
$$\geqslant [(5s-6)/2]$$

since

$$5k_1 - k - 1 - 2\theta(k) \geqslant -6$$

(by examination of cases modulo 5).

Further

$$\pi(b'_{11}) \geqslant 0 \qquad \text{by (25)}.$$

We now assume (28) for some n and all s. Since

$$5c_{ns} = \sum_{\sigma=1}^{\infty} b_{n\sigma} a_{6(\sigma+k_n),\ s+l_n+\sigma+k_n} \qquad \text{by (20) and (21)}$$

we obtain

$$\pi(c'_{ns}) \geqslant -1 - \theta(6k_n + 6) + \min_{\sigma\geqslant 1}\{\max(0, [(5\sigma-6)/2])$$
$$+ [(5s - \sigma + 5l_n - k_n + 1)/2]\}$$

and the $\min_{\sigma \geqslant 1}$ is attained at $\sigma = 1$ (since increasing σ by 1 increases the first term in curly brackets by at least 2, and decreases the second term by at most 1) so that

$$\begin{aligned}\pi(c'_{ns}) &\geqslant [(5s + 5l_n - k_n - 2 - 2\theta(6k_n + 6))/2] \\ &\geqslant [(5s - 6)/2],\end{aligned}$$

since

$$5l_n - k_n - 2\theta(6k_n + 6) \geqslant -4$$

(by examination of cases modulo 5). Further

$$\begin{aligned}\pi(c'_{n1}) &\geqslant \min\{0, \min_{\sigma \geqslant 2}\{[(5\sigma - 6)/2] + [(4 - \sigma + 5l_n - k_n - 2\theta(6k_n + 6))/2]\} \\ &\geqslant \min\{0, 2 - 1\} = 0.\end{aligned}$$

Thus we have (29) for n and all s. A similar argument shows that (29) for all s and some n implies (28) for all s and $n + 1$, which proves Lemma 3 by induction.

2.5. We now define B_n $(= B_n(k))$ as the largest integer such that $p_{-k}(m) \equiv 0 \pmod{5^{B_n}}$ for *all* m with $24m \equiv k \pmod{5^n}$. By (10), (22), and Lemma 2 we have

$$B_n = \pi(L_n(x)).$$

It is clear from the definitions of b_{ns} and c_{ns} that in fact

$$B_{2n-1} = \min_{s \geqslant 1} b_{ns},$$

$$B_{2n} = \min_{s \geqslant 1} c_{ns},$$

and so by Lemma 3 we have

$$B_n \geqslant A_n.$$

Before proceeding to compute A_n in a suitable form, we make some remarks on the "best possible" aspect of Theorem 1. If, for all n, we have

$$\pi(b'_{n1}) = \pi(c'_{n1}) = 0,$$

then clearly $B_n = A_n$. Now the induction of Lemma 3 shows that if we have *equality* in our use of (25) at all stages of the argument this will certainly be valid. Thus we have

LEMMA 4. *Suppose that none of* k, $6k_r + 6$ $(r \geqslant 1)$, $6l_r + 6 + k$ $(r \geqslant 1)$, *are congruent to* 11 *or* 17 (mod 25). *Then for all* n *we have* $B_n = A_n$.

It follows from Lemma 2 (or can be seen directly from (21)) that k_n is the least non-negative integer such that

$$5^{2n-1}(1 + k_n) > k(5^{2n} - 1)/24$$

or

$$1 + k_n > 5k/24 - k/5^{2n-1}.$$

Hence

$$k_n = \left[\frac{5k}{24} - \frac{k}{5^{2n-1}}\right]$$

$$= \left[\frac{5k-1}{24}\right] = K, \quad \text{say},$$

provided $k/5^{2n-1} \leqslant 1/24$,
or

$$2n - 1 \geqslant \log(24k)/\log 5.$$

Similarly,

$$l_n = \left[\frac{k}{24} - \frac{k}{5^{2n}}\right]$$

$$= \left[\frac{k-1}{24}\right] = \left[\frac{K}{5}\right] = L, \quad \text{say},$$

provided that

$$2n \geqslant \log(24k)/\log 5.$$

We now define

$$\alpha = \alpha(k) = \theta(6K + 6) + \theta(6L + 6 + k).$$

Since increasing k by 24 increases $6K + 6$ and $6L + 6 + k$ each by 30, the values of $\theta(K + 6)$ and $\theta(6L + 6 + k)$ remain the same, so that α depends only on the residue of k modulo 24; examination of cases gives the list for $q = 5$ in Theorem 1.

Finally (26) and the above results on k_n and l_n show that

$$A_{2n-1} \geqslant \alpha(n - n_0),$$

$$A_{2n} \geqslant \alpha(n - n_0),$$

where

$$n_0 = [\log(24k)/\log 5],$$

which implies that

$$A_n \geqslant \alpha n/2 + O(\log k),$$

and so the results for $q = 5$ in Theorem 1.

For individual values of k we can of course get more precise results; two interesting cases are

(30) if $24m \equiv 5 \pmod{5^n}$, then $p_{-5}(m) \equiv 0 \pmod{5^{[(3n-3)/2]}}$,

(31) if $24m \equiv 25 \pmod{5^n}$, then $p_{-25}(m) \equiv 0 \pmod{5^{[(3n-6)/2]}}$.

To complete our results as to the best possible nature of Theorem 1 (as explained in the Introduction) we require to show that for each residue class modulo 24 there exists a k satisfying the hypothesis of Lemma 4. We can find no elegant method for this; calculation gives k = 1, 2, 3, 4, 53, 6, 7, 8, 9, 10, 59, 12, 13*, 14*, 15, 16, 65*, 18*, 19*, 20, 21, 70*, 23*, 24*. For $k \equiv 12 \pmod{24}$, one can prove $A_n \geqslant 1$ always although $\alpha = 0$; $k = 12$ has $B_n = A_n = 1$

for all n. The values with an asterisk, for which $\alpha = 0$, have in fact $B_n = A_n = 0$ for all n. A crude probability argument suggests that there should be an infinity of k in each residue class modulo 24 satisfying the hypothesis of Lemma 4, but the author cannot prove this.

2.6. The properties of $p_{-k}(m)$ which we have established are, despite the complexity of the proof, only the first step towards a series of deeper and more interesting properties. Thus, as in Atkin and O'Brien **(2)**, we can prove for $p(m)$ the following result:

(32) *for all* $n \geqslant 1$ *there exists a constant* k_n *not divisible by* 5 *such that for all* $m \geqslant 1$ *we have*

$$\frac{p(5^{n+2}m + \lambda \quad)}{5^{n+2}} \equiv k_n \cdot \frac{p(5^n m + \lambda_n)}{5^n} \qquad (\text{mod } 5^{[5n/2]+1})$$

where

$$\begin{aligned} \lambda_n &= -(5^n - 1)/24 && (n \text{ even}), \\ &= -(5^{n+1} - 1)/24 && (n \text{ odd}). \end{aligned}$$

Results analogous to (32) exist for all $p_{-k}(m)$; in a sense the results of this paper conceal these recurrence properties.

Finally (32) itself is merely the ramified case $\varphi = 5$ of a general "multiplicative" property that exists for all primes $\varphi > 3$. The details of this property are very complex for $p(m)$, and we therefore give an example in terms of $\gamma(m)$, where

$$g(x) = xf^6(x^5)/f^6(x) = \sum_{m=1}^{\infty} \gamma(m) x^m.$$

Thus $\gamma(m)$ is the coefficient of a function on $\Gamma_0(5)$, whereas $p(m)$ is the coefficient of a form of half-integral dimension. Then we have for $n \geqslant 1$

$$(33) \qquad \begin{aligned} &\gamma(5^n m) \equiv 0 \pmod{5^n} \qquad (m \geqslant 1), \\ &\gamma(5^n) \not\equiv 0 \pmod{5^{n+1}}, \end{aligned}$$

as can easily be seen by the methods of this paper. If now

$$\delta(m) = \gamma(5^n m)/\gamma(5^n),$$

the division being valid in any ring of integers modulo 5^α by (33), we have

$$(34) \qquad \delta(5m) \equiv \delta(5)\, \delta(m) \pmod{5^{3n}},$$

analogous to (32), and finally

$$(35) \qquad \delta(m\varphi) - \delta(m)\delta(\varphi) + \varphi^{-1}\, \delta(m/\varphi) \equiv 0 \pmod{5^{3n}}$$

for any prime $\varphi \neq 5$, where $\delta(\mu) = 0$ if μ is non-integral.

It is clear that (35) is analogous to the *exact* multiplicative relations arising in the theory of Hecke operators for modular forms of *negative* dimension; our forms have positive or zero dimension.

3. We now write down the formulae analogous to those of § 2 for the remaining values of q, indicating briefly the various minor changes that appear.

3.1. $q = 2$. Let

$$g(x) = xf^{24}(x^2)/f^{24}(x), \qquad \phi(x) = xf^8(x^4)/f^8(x).$$

Then

$$t^2 = g(x)(2^4 t + 1) \text{ where } t = \phi(x^{\frac{1}{2}}).$$

We have

$$2U_2\phi^r(x) = S_r = \sum_{\rho=1}^{\infty} a_{r\rho} g^\rho,$$

where

$$\pi(a_{r\rho}) \geqslant 8\rho - 4r, \qquad \pi(ar_\rho) \geqslant 1$$

and in fact $a_{r\rho} \neq 0$ if and only if $[(r+1)/2] \leqslant \rho \leqslant r$. If $\theta(b) = 0$ for b even and $\theta(b) = 3$ for b odd, we obtain, after defining $L_n(x)$ as in (21), the result

$$\pi(L_n(x)) = A_n,$$

where

$$A_{2n-1} = \theta(k) + \sum_{r=1}^{n-1} \{\theta(3k_r + 3) + \theta(3l_r + 3 + k)\},$$

$$A_{2n} = A_{2n-1} + \theta(3k_n + 3).$$

Further

$$K = [(2k-1)/3], \qquad L = [(k-1)/3],$$

and

$$\alpha = \theta(3K + 3) + \theta(3L + 3 + k) = 3 \text{ always},$$

by examination of cases modulo 3.

These results lead, of course, to congruence properties of $p_{-8k}(m)$ as in Theorem 1. We note that there are no complications with B_n as in § 2.5 and here Theorem 1 is best possible, since for $\rho = [(r+1)/2]$ we have equality in $\pi(a_{r\rho}) = \theta(r) + 1$.

3.2. $q = 3$. Let

$$g(x) = xf^{12}(x^3)/f^{12}(x), \qquad \phi(x) = xf^3(x^9)/f^3(x).$$

Then

$$t^3 = g(x)(3^3t^2 + 3^2t + 1) \quad \text{where} \quad t = \phi(x^{1/3}).$$

We have

$$3U_3\phi^r(x) = S_r = \sum_{\rho=1}^{\infty} a_{r\rho}\, g^\rho,$$

where

$$\pi(a_{r\rho} \geqslant [(9\rho - 3r + 1)/2], \qquad \pi(a_{r\rho}) \geqslant 1,$$

and in fact $a_{r\rho} \neq 0$ if and only if $[(r+2)/3] \leqslant \rho \leqslant r$. If $\theta(b) = 2, 1, 0$ according as $b \equiv 1, 2, 0 \pmod 3$, we obtain after defining $L_n(x)$ as in (21) the result

$$\pi(L_n(x)) = A_n,$$

where

$$A_{2n-1} = \theta(k) + \sum_{r=1}^{n-1} \{\theta(4k_r + 4) + \theta(4l_r + 4 + k)\},$$

$$A_{2n} = A_{2n-1} + \theta(4k_n + 4).$$

Further

$$K = [(3k-1)/8], \qquad L = [(k-1)/8],$$

and

$$\alpha = \theta(4K + 4) + \theta(4L + 4 + k).$$

Examination of cases modulo 8 now gives the results for $p_{-3k}(m)$ in Theorem 1; as for $q = 2$ these are best possible.

3.3. $q = 7$. Let

$$g(x) = xf^4(x^7)/f^4(x), \qquad \phi(x) = x^2 f(x^{49})/f(x).$$

Then

$$t^7 = g(x)(7^2t^6 + 5.7t^5 + 7t^4) + g^2(x)(7^3t^6 + 7^3t^5 + 3.7^2t^4 + 7^2t^3 + 3.7t^2 + 7t + 1),$$

where $t = \phi(x^{1/7})$. We have

$$7U_7\phi^r(x) = S_r = \sum_{\rho=1}^{\infty} a_{r\rho}\, g^\rho,$$

where

$$\pi(a_{r\rho}) \geqslant [(7\rho - 2r + 3)/4], \qquad \pi(a_{r\rho}) \geqslant 1,$$

and in fact $a_{r\rho} \neq 0$ if and only if $[(2r+6)/7] \leqslant \rho \leqslant 2r$. In (21) we have the modification

$$k_1 = [(2k-1)/7], \qquad l_n = [k_n/7], \qquad k_{n+1} = [(2k + l_n)/7].$$

If $\theta(b) = 1$ if $b \equiv 1$ or $4 \pmod 7$ and $\theta(b) = 0$ otherwise, we obtain

$$\pi(L_n(x)) \geqslant A_n,$$

where

$$A_{2n-1} = \theta(k) + \sum_{r=1}^{n-1} \{\theta(4k_r + 4) + \theta(4l_r + 4 + k)\},$$

$$A_{2n} = A_{2n-1} + \theta(4k_n + 4).$$

Further

$$K = [(7k-1)/24], \qquad L = [(k-1)/24],$$

and

$$\alpha = \theta(4K + 4) + \theta(4L + 4 + k).$$

An examination of cases modulo 24 now gives the results in Theorem 1. As for $q = 5$, we have the possibility that $B_n > A_n$, but as in Lemma 4 we have that $B_n = A_n$ provided none of k, $4k_r + 4$, $4l_r + 4 + k$, are congruent to 39 or 43 modulo 49. As for $q = 5$ we find the following k with $B_n = A_n$, an asterisk again denoting that $A_n = 0$ for all n: $k = 1, 2, 3, 4, 5, 6, 31, 8, 9^*, 10^*, 35, 12^*, 37^*, 14, 63^*, 16^*, 17^*, 18, 19^*, 20^*, 21, 70^*, 23^*, 24^*$. For $k \equiv 8 \pmod{24}$ one can prove that $A_n \geqslant 1$ always. The value $k = 8$ gives $B_n = A_n = 1$ for all n.

3.4. $q = 13$. Here we give very few details, since a full account of the modular equation is given in (6). We have

$$K = [(13k - 1)/24], \qquad L = [(k - 1)/24],$$

and

$$\alpha = \theta(2K + 2) + \theta(2L + 2 + k), \tag{36}$$

where $\theta(b) = 1$ if $b \equiv 10 \pmod{13}$ and $\theta(b) = 0$ otherwise, with the restriction that if *both* θ in (36) are 1 we only have $\alpha = 1$. This leads to the single result for $k \equiv 8 \pmod{24}$. As a compensation we have $B_n = A_n$ always, and Theorem 1 is best possible.

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