## LETTER TO THE EDITOR

Dear Editor,

*Correction to the appendix of*
*'Three characterizations of population strategy stability'*

Hines (1980) in a discussion of three characterizations of biological population strategy stability states that for $\boldsymbol{W}$ an $m \times m$ matrix the two statements

(1) $\boldsymbol{W} + \boldsymbol{W}^{\mathrm{T}}$ is a positive definite matrix, and

(2) for all positive definite symmetric matrices $\boldsymbol{Q}$, the eigenvalues of $\boldsymbol{QW}$ have positive real parts

are equivalent. In the demonstration that statement 2 implies statement 1, he claims that 'If $\boldsymbol{x}^{\mathrm{T}}\boldsymbol{W}\boldsymbol{x} = 0$ for $\boldsymbol{x} \neq 0$, 0 is an eigenvalue of $\boldsymbol{W}, \ldots$', ignoring the possibility that $\boldsymbol{W}\boldsymbol{x}$ might be orthogonal to $\boldsymbol{x}$. Because of this possibility statement (2) implies only

(1′) $\boldsymbol{W} + \boldsymbol{W}^{\mathrm{T}}$ is a positive semi-definite matrix.

The corresponding second statement, equivalent to (1′), is

(2′) for all positive definite symmetric matrices $\boldsymbol{Q}$, the eigenvalues of $\boldsymbol{QW}$ have non-negative real parts.

As before, statement (1) does imply statement (2) (which implies statements (1′) and (2′), which imply each other).

How are the conclusions in Hines (1980) affected by this change? One conclusion, that the second ESS condition (Maynard Smith (1974), Hines (1980)) implies a certain stability, depends on the fact that (1) implies (2), and so is unaltered. Another conclusion, that the requirement that stability occur for arbitrary covariance matrices implies this second ESS condition, depended on (2) implying (1). Hence that conclusion is unjustified. In the notation of Hines (1980), page 336, which we use without further comment, $\boldsymbol{A}$ is forced by the requirement just indicated to be negative semi-definite (i.e. $\boldsymbol{x}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{x} \leqq 0$) rather than negative definite, the second ESS condition.

For the purpose of Hines (1980), at least, the practical consequence of this semi-definiteness is slight. If for some payoff matrix $\boldsymbol{A}$, $\boldsymbol{x}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{x}$ is a negative semi-definite quadratic form, but not a negative definite one, the negative semi-definiteness is not stable under perturbations of $\boldsymbol{A}$. Instead, some perturbation of $\boldsymbol{A}$, say to $\boldsymbol{A}^{(p)}$ can render $\boldsymbol{x}^{\mathrm{T}}\boldsymbol{A}^{(p)}\boldsymbol{x} > 0$ for some (suitably restricted) $\boldsymbol{x}$, thereby implying that the corresponding equilibrium strategy will be unstable. In such cases, therefore, an additional condition, one of stability under perturbations of $\boldsymbol{A}$, implies that $\boldsymbol{x}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{x}$ is strictly negative definite, the conclusion at issue.

We are grateful to S. Karlin who suggested to one of us that the validity of the claim that statements (1) and (2) were equivalent be checked.

Yours sincerely,

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## References

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MAYNARD SMITH, J. (1974) The theory of games and the evolution of animal conflicts. *J. Theoret. Biol.* **47**, 209–221.