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COVERING RANDOM POINTS IN A UNIT DISK

JENNIE C. HANSEN,* *Herriot-Watt University*
ERIC SCHMUTZ ** *** AND
LI SHENG,** *Drexel University*

Abstract

Let D be the punctured unit disk. It is easy to see that no pair x, y in D can cover D in the sense that D cannot be contained in the union of the unit disks centred at x and y. With this fact in mind, let $V_n = \{X_1, X_2, \dots, X_n\}$, where $X_1, X_2, \dots$ are random points sampled independently from a uniform distribution on D. We prove that, with asymptotic probability 1, there exist two points in V_n that cover all of V_n.

Keywords: Dominating set; random geometric graph; unit ball graph

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1. Introduction

For any $r > 0$ and any $p \in \mathbb{R}^2$, let $D_r(p)$ be the open (Euclidean) disk in $\mathbb{R}^2$ that is centered at p and has radius r. Let D be the punctured unit disk that is centered at the origin o, i.e. $D = D_1(o) - \{o\}$. If S and P are subsets of $\mathbb{R}^2$, we say that 'P covers S' if $S \subseteq \bigcup_{p \in P} D_1(p)$. (This use of the word 'cover' comes from combinatorics and is obviously related but not identical to the usual topological meaning.)

Now let $X_1, X_2, \dots$ be random points, chosen independently from a uniform distribution on a punctured unit disk D, and let $V_n = \{X_1, X_2, \dots, X_n\}$. We prove that, with asymptotic probability 1, V_n is covered by one of its two-member subsets. This result is surprising in light of the following three simple geometric observations. In short, the observations below state that *three* points of D are needed to cover D.

Observation 1. For $x \in D$, we have $D \nsubseteq D_1(x)$.

The second observation appears in [8].

Observation 2. If $x, y \in D$ then $D \nsubseteq D_1(x) \cup D_1(y)$.

For $k = 0, 1, 2$, let $p_k = \frac{1}{2}(\cos(2\pi k/3), \sin(2\pi k/3))$. Choose a positive number $\varepsilon < 1 - \sqrt{3}/2$, and let $\rho = 1 - \varepsilon$. Then we have the following observation.

Observation 3. We have $D \subseteq D_\rho(p_1) \cup D_\rho(p_2) \cup D_\rho(p_3)$.

Note that there is a bit of 'slack' in Observation 3: we have used disks of radius strictly less than 1.

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* Postal address: Actuarial Mathematics and Statistics Department and the Maxwell Institute for Mathematical Sciences, Herriot-Watt University, Riccarton, Edinburgh EH14 4AS, UK.
** Postal address: Mathematics Department, Drexel University, Philadelphia, PA 19104, USA.
*** Email address: schmutze@drexel.edu

The points $X_1, X_2, \dots$ 'fill out' all of D, i.e. with probability 1. The infinite set $\{X_1, X_2, \dots\}$ is dense in D. So, with probability 1, o, the center of the disk, is a limit point of the set $\{X_1, X_2, \dots\}$. Since $D \subseteq D_1(o)$, it is reasonable to ask whether, for large n, $V_n \subseteq D_1(X_i)$ for some $1 \le i \le n$. In Section 2 we prove that, with high probability, the answer is no: one point does not suffice. On the other hand, it follows easily from Observation 3 that, with asymptotic probability 1, three points from V_n will suffice to cover V_n. Briefly, with asymptotic probability 1, the small disks $D_\varepsilon(p_k)$ each contain at least one random point X_{i_k}. But then $D_1(X_{i_k})$ contains the entire sector $(2k-1)\pi/3 \le \theta \le (2k+1)\pi/3$, and $D \subseteq D_1(X_{i_0}) \cup D_1(X_{i_1}) \cup D_1(X_{i_2})$. (See [5] for more results like this.)

Finally, it follows from Observation 2 that, for all i and j, $D \nsubseteq D_1(X_i) \cup D_1(X_j)$. Nevertheless, we prove that only two points of V_n are needed to cover V_n; with asymptotic probability 1, there are two points X_i and X_j in V_n such that $V_n \subseteq D_1(X_i) \cup D_1(X_j)$.

2. Coverage by one point

In this section we prove a *general* coverage result which holds for any dimension $m \ge 2$. Let $d_m(\cdot,\cdot)$ denote the Euclidean distance in $\mathbb{R}^m$. Suppose that $X_1, X_2, X_3, \dots$ is an infinite sequence of random points chosen independently from a uniform distribution in a unit ball in $\mathbb{R}^m$. We say that $x \in \mathbb{R}^m$ *covers* $V_n = \{X_1, X_2, \dots, X_n\}$ if $d_m(x, X_i) < 1$ for each $1 \le i \le n$. Call X_n a *dominator* if and only if X_n covers V_n.

Theorem 1. *With asymptotic probability 1, no point in V_n will cover all of V_n.*

Proof. For positive real numbers r and positive integers $m \ge 2$, let $\mu_m(r)$ denote the volume of a ball of radius r in $\mathbb{R}^m$, i.e.

$$\mu_m(r) = r^m \mu_m(1) = \frac{\pi^{m/2} r^m}{\Gamma(m/2+1)}.$$

Let $L(r)$ denote the volume of the intersection of two unit balls in $\mathbb{R}^m$ whose centers are a distance r apart. If the distance from the point X_n to the origin is r then the conditional probability that the ith point X_i is within distance 1 of X_n is $L(r)/\mu_m(1)$. The distance between the origin and the random point X_n is a random variable with density $f(r) = \mu'_m(r)/\mu_m(1) = mr^{m-1}$. Hence,

$$\Pr(X_n \text{ is a dominator}) = \int_0^1 f(r)\left(\frac{L(r)}{\mu_m(1)}\right)^{n-1} \mathrm{d}r.$$

We split the integral into two. Let

$$\xi = \frac{4(\log n)\mu_m(1)}{(n-1)\mu_{m-1}(1)}.$$

Then

$$\Pr(X_n \text{ is a dominator}) = I_1 + I_2,$$

where

$$I_1 = m\int_0^\xi r^{m-1}\left(\frac{L(r)}{\mu_m(1)}\right)^{n-1} \mathrm{d}r$$

and

$$I_2 = m\int_\xi^1 r^{m-1}\left(\frac{L(r)}{\mu_m(1)}\right)^{n-1} \mathrm{d}r.$$

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For the first piece, we use the trivial estimate $L(r)/\mu_m(1) \le 1$: for $m \ge 2$,

$$I_1 \le m \int_0^{\xi} r^{m-1}\, \mathrm{d}r = \xi^m = O\biggl(\frac{\log^2 n}{n^2}\biggr). \tag{1}$$

To estimate I_2, we use the following well-known formula for $L(r)$:

$$L(r) = 2\int_{r/2}^{1} \mu_{m-1}(\sqrt{1-x^2})\, \mathrm{d}x = 2\mu_{m-1}(1)\int_{r/2}^{1} (1-x^2)^{(m-1)/2}\, \mathrm{d}x. \tag{2}$$

It is intuitively obvious that $L(r)$ is decreasing, and this is easily confirmed by differentiating the right-hand side of (2) to obtain

$$L'(r) = -\mu_{m-1}(1)\biggl(1 - \frac{r^2}{4}\biggr)^{(m-1)/2} \le 0 \quad \text{for } 0 \le r \le 1. \tag{3}$$

Since $L(r) \le L(\xi)$ for all $r \ge \xi$, and since f is a density function, we have

$$I_2 \le \biggl(\frac{L(\xi)}{\mu_m(1)}\biggr)^{n-1} \int_{\xi}^{1} f(r)\, \mathrm{d}r \le \biggl(\frac{L(\xi)}{\mu_m(1)}\biggr)^{n-1}. \tag{4}$$

To estimate the right-hand side of (4), note that from (3) it follows that there exists some $0 < c_\xi < \xi$ such that

$$L(\xi) = L(0) + L'(c_\xi)\xi = \mu_m(1) - \mu_{m-1}(1)\biggl(1 - \frac{c_\xi^2}{4}\biggr)^{(m-1)/2}\xi. \tag{5}$$

Since $0 < c_\xi < \xi = o(1)$, we have $(1 - c_\xi^2/4)^{(m-1)/2} > \frac{1}{2}$ for all sufficiently large n. So it follows from (5) that

$$L(\xi) \le \mu_m(1) - \frac{\mu_{m-1}(1)\xi}{2} \tag{6}$$

for all sufficiently large n. Substituting (6) into the right-hand side of (4), we obtain

$$I_2 \le \biggl(1 - \frac{\xi\mu_{m-1}(1)}{2\mu_m(1)}\biggr)^{n-1} = O\biggl(\frac{1}{n^2}\biggr). \tag{7}$$

Combining our estimates (1) and (7) for I_1 and I_2, respectively, we conclude that, for some positive constant c and all sufficiently large n,

$$\Pr(X_n \text{ is a dominator}) < \frac{c\log^2 n}{n^2}.$$

Finally, the X_is are identically distributed, so, for $1 \le i < n$, the probability that X_i covers V_n is equal to the probability that X_n is a dominator. Therefore, by Boole's inequality, the probability that one of the X_is in V_n covers all of V_n is at most $c\log^2 n/n$.

Remark 1. A stronger statement than Theorem 1 is

$$\Pr(\{\text{for infinitely many } n,\ V_n \text{ is covered by one of it members}\}) = 0.$$

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We thank a thorough referee for the following argument. Define the events

$$E_n = \{X_n \text{ is a dominator}\} \quad \text{and} \quad F_n = \{d_m(X_n, X_k) > 1 \text{ for some } k > n\}.$$

Also define $E = \{E_n \text{ occurs for infinitely many } n\}$ and $G = E^c \cap F_1 \cap F_2 \cap F_3 \cdots$ (where the superscript 'c' denotes complementation.). On G, V_n is covered by one of its members for at most finitely many n. In the proof of Theorem 1 we showed that $\Pr(E_n) \le c \log^2 / n^2$. Therefore $\Pr(E^c) = 1$. Since no $x \ne o$ can cover the sample space, we have $\Pr(F_n) = 1$ for all n. Hence, $\Pr(G) = 1$.

3. A geometric lemma

The remaining results in this paper are proved under the assumption that the dimension $m = 2$. Recall that Observation 2 states that a unit disk centered at a point o cannot be completely covered with two unit disks having centers at points other than o: $D_1(o) \not\subseteq D_1(q) \cup D_1(u)$ for $q \ne o \ne u$. The purpose of this section is to prove Lemma 1, below, which provides an upper bound for the area of the uncovered region $D_1(o) \cap (D_1(q) \cup D_1(u))^c$. A heuristic indication of this lemma's significance is the following: the smaller the uncovered region, the more likely it is that none of the random points will fall in that uncovered region. If no random points fall in the uncovered region then q and u cover V_n.

Some notation is needed to state Lemma 1. For any $r > 0$ and any $v \in \mathbb{R}^2$, let $\partial D_r(v)$ be the circle of radius r that bounds the open disk $D_r(v)$. Fix $b \ge 3$, and define

$$L_b = \lfloor b^{1/3} (\log b)^2 \rfloor, \qquad \delta_b = \frac{1}{b^{1/3} \log b}, \quad \text{and} \quad \theta_b = \frac{\pi}{L_b},$$

where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x. We are essentially going to partition $D_{\delta_b}(o)$ into $2L_b$ sectors as follows. (It is not strictly correct to call this a partition of $D_{\delta_b}(o)$ since the origin was omitted, the bounding circle was included, and some pairs of sectors have a nonempty intersection (with zero area).) For integers i such that $0 \le i < L_b$, let Q_i be the sector consisting of those points $(x, y) = (r \cos\theta, r \sin\theta)$ whose polar coordinates satisfy $0 < r \le \delta_b$ and $(i - \frac{1}{2})\theta_b \le \theta \le (i + \frac{1}{2})\theta_b$. Similarly, let U_i consist of the points with $0 < r \le \delta_b$ and $(i - \frac{1}{2})\theta_b \le \theta - \pi \le (i + \frac{1}{2})\theta_b$. Note that the sectors Q_i and U_i are located symmetrically with respect to o. Let $\tilde{q}_i \in Q_i$ and $\tilde{u}_i \in U_i$ be the extreme points whose polar coordinates are respectively $(\delta_b, (i - \frac{1}{2})\theta_b)$ and $(\delta_b, (i + \frac{1}{2})\theta_b + \pi)$. Finally, for any points $u, w \in D_1(o)$, let $A(u, w)$ denote the area of $(D_1(u) \cup D_1(w))^c \cap D_1(o)$, i.e. the area of the region in $D_1(o)$ that is not covered by $D_1(u) \cup D_1(w)$. The main result in this section is stated as follows.

Lemma 1. *There exists a uniform constant $C > 0$ (independent of the parameter b) such that, for $0 \le i < L_b$, and for all $q_i \in Q_i$ and $u_i \in U_i$, we have*

$$A(q_i, u_i) \le A(\tilde{q}_i, \tilde{u}_i) \le \frac{C}{b \log^3 b}.$$

We state four facts below which together imply Lemma 1. For the first three facts, proofs have been omitted because they are obvious geometrically once they are understood. For the first fact, we observe that, for any $q, u \in D_1(o)$, the omitted area $A(q, u)$ increases if we move one (or both) of the two points q and u away from the origin along a radial line.

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Fact 1. Let q, q' and u, u' be four points in $D_1(o)$ such that q lies on the line segment $\overline{o, q'}$ and u lies on the line segment $\overline{o, u'}$. Then $A(q', u') \geq A(q, u)$.

Fact 2. Suppose that $p, q \in \mathbb{R}^2$ are such that $d_2(p, q) < 2$. Let a and b be the two points where the circles $\partial D_1(p)$ and $\partial D_1(q)$ intersect. Then, $\overline{a, b} \perp \overline{p, q}$, and the two line segments $\overline{a, b}$ and $\overline{p, q}$ intersect at their midpoints.

Fact 3. Let o_1 and o_2 be two points on the circle $x^2 + y^2 = \delta_b^2$. Then, $A(o_1, o_2)$ is a decreasing function of $\angle o_1 o o_2$.

Fact 4. Uniformly for $0 \leq i < L_b$, we have $A(\tilde{q}_i, \tilde{u}_i) = O(1/b \log^3 b)$.

Proof. Without loss of generality, let $i = 0$. To simplify the notation, define $x_b = \delta_b \cos(-\frac{1}{2}\theta_b)$ and $y_b = \delta_b \sin(-\frac{1}{2}\theta_b)$. Let (ξ, η) be the point in the first quadrant where the circles $x^2 + y^2 = 1$ and $(x - x_b)^2 + (y - y_b)^2 = 1$ intersect. Then

$$
\begin{aligned}
A(\tilde{q}_0, \tilde{u}_0) &\leq 4 \int_0^{\xi} \sqrt{1 - x^2} - \left(y_b + \sqrt{1 - (x - x_b)^2}\right) \mathrm{d}x \\
&= -4 y_b \xi + 4 \int_0^{\xi} \frac{-2 x x_b + x_b^2}{\sqrt{1 - x^2} + \sqrt{1 - (x - x_b)^2}} \,\mathrm{d}x.
\end{aligned}
$$

Hence, we have

$$
A(\tilde{q}_0, \tilde{u}_0) = O(\xi y_b) + O(x_b \xi^2) + O(x_b^2 \xi). \tag{8}
$$

Note that $x_b^2 + y_b^2 = \delta_b^2 = 1/b^{2/3} \log^2 b$, that $\xi^2 + \eta^2 = 1$, that $(\xi - x_b)^2 + (\eta - y_b)^2 = 1$, that $x_b = \delta_b(1 + O(\theta_b^2))$, and that $y_b = (-\delta_b \theta_b / 2)(1 + O(\theta_b^2))$. Combining these equations, we obtain $\xi = O(\delta_b)$. Substituting these estimates into (8), we obtain

$$
A(\tilde{q}_0, \tilde{u}_0) = O\left(\frac{1}{b \log^3 b}\right).
$$

4. Two point dominating sets

Recall that the dimension $m = 2$. In this section we consider the problem of covering the set $V_n = \{X_1, X_2, \ldots, X_n\}$, where the X_is are chosen independently and uniform randomly from the punctured disk $D = D_1(o) - \{o\}$, by two points $X_i, X_j \in V_n$. Assume that $n \geq 3$, and recall the definitions for L_n, U_i, and Q_i in the previous section (with $b = n$). For $0 \leq i < L_n$, let $N(Q_i)$ and $N(U_i)$ respectively denote the number points in V_n that lie in Q_i and U_i. Let

$$
\tau_n = \sum_{i=0}^{L_n - 1} I_i,
$$

where the indicator variable $I_i = 1$ if and only if $N(U_i) = N(Q_i) = 1$. (Remark: we consider the event $\{N(U_i) = N(Q_i) = 1\}$ instead of the event $\{N(U_i) \geq 1,\ N(Q_i) \geq 1\}$ because it simplifies a conditioning argument later.)

Lemma 2. *We have*

$$
\Pr\left(\tau_n < \frac{n^{1/3}}{16 \log^6 n}\right) = O\left(\frac{\log^6 n}{n^{1/3}}\right).
$$

Proof. For $0 \le i < L_n$, let

$$p = \frac{\mathrm{Area}(Q_i)}{\mathrm{Area}(D_1(o))} = \frac{\delta_n^2}{2L_n} = \frac{1}{2n\log^4 n}\biggl(1 + O\biggl(\frac{1}{n^{1/3}\log^2 n}\biggr)\biggr).$$

Then

$$\mathrm{E}(I_i) = n(n-1)p^2(1-2p)^{n-2}, \tag{9}$$

and

$$\mathrm{E}(\tau_n) = L_n n(n-1)p^2(1-2p)^{n-2} = \frac{n^{1/3}}{4(\log n)^6}\biggl(1 + O\biggl(\frac{1}{n^{1/3}(\log n)^2}\biggr)\biggr).$$

Similarly, for $0 \le i, j < L_n$ such that $i \ne j$,

$$\mathrm{E}(I_i I_j) = n(n-1)(n-2)(n-3)p^4(1-4p)^{n-4}. \tag{10}$$

Since $\tau_n = \sum_{i=0}^{L_n-1} I_i$, and the I_is are identically distributed, we have

$$\mathrm{var}(\tau_n) = L_n(L_n-1)\,\mathrm{E}(I_1 I_2) + L_n\,\mathrm{E}(I_1) - (\mathrm{E}(\tau))^2.$$

Combining this identity with the expression for $\mathrm{E}(I_i)$ in (9), the expression for $\mathrm{E}(I_i I_j)$ in (10), and the definitions for L_n, δ_n, and p, we obtain

$$\mathrm{var}(\tau_n) = \mathrm{E}(\tau_n)\biggl(1 + O\biggl(\frac{1}{(\log n)^8}\biggr)\biggr).$$

The lemma now follows by Chebyshev's inequality.

Theorem 2. *There exists a constant $c > 0$ such that, with probability greater than $1 - c/(\log n)^3$, there exist two points of V_n that cover V_n.*

Proof. Let

$$\mathcal{T}_n = \{i : 0 \le i \le L \text{ and } N(Q_i) = N(U_i) = 1\}.$$

If $\mathcal{T}_n \ne \varnothing$, define $Y_n = \min \mathcal{T}_n$ to be the smallest of the indices in $\mathcal{T}_n$; otherwise, if $\mathcal{T}_n = \varnothing$, set $Y_n = -1$. Define the indicator random variable W_n as $W_n = 1$ if and only if both the following conditions are satisfied:

- $\tau_n \ne 0$, i.e. $\mathcal{T}_n = \{i_1, i_2, \dots, i_{\tau_n}\}$ for some $i_1 < i_2 < \cdots < i_{\tau_n}$;
- the two points in $Q_{i_1} \cup U_{i_1}$ cover V_n.

Define $\mathcal{Z}_n$ to be set of points in V_n that lie within distance $\delta_n = 1/n^{1/3}\log n$ of the origin, and let Z_n be the number of these points. Also, let $\beta_n = 2n^{1/3}/(\log n)^2$. Then

$$\Pr(W_n = 0) \le \Pr(W_n = 0,\ \tau_n \ne 0,\ Z_n \le \beta_n) + \Pr(\tau_n = 0) + \Pr(Z_n > \beta_n).$$

Note that Z_n has a binomial distribution $Z_n \overset{\mathrm{D}}{=} \mathrm{Bin}(n, \delta_n^2)$, where '$\overset{\mathrm{D}}{=}$' denotes equality in distribution. Therefore, by Chernoff's inequality, $\Pr(Z_n \ge \beta_n) \le \exp(-\beta_n/8)$. By Lemma 2, $\Pr(\tau_n = 0) = O(\log^6 n/n^{1/3})$. Therefore,

$$\Pr(W_n = 0) \le \Pr(W_n = 0,\ \tau_n \ne 0,\ Z_n \le \beta_n) + O\biggl(\frac{\log^6 n}{n^{1/3}}\biggr). \tag{11}$$

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Next we decompose the first term on the right-hand side of (11) according to the value of Y_n:

$$\begin{aligned}
&\Pr(W_n = 0,\ \tau_n \neq 0,\ Z_n \leq \beta_n) \\
&\quad = \sum_{k=0}^{L_n-1} \Pr(W_n = 0 \mid Y_n = k,\ \tau_n \neq 0,\ Z_n \leq \beta_n) \Pr(Y_n = k, Z_n \leq \beta) \\
&\quad = \sum_{k=0}^{L_n-1} \Pr(W_n = 0 \mid Y_n = k,\ Z_n \leq \beta_n) \Pr(Y_n = k,\ Z_n \leq \beta_n).
\end{aligned}$$

We have

$$\begin{aligned}
&\Pr(W_n = 0 \mid Y_n = k,\ Z_n \leq \beta_n) \\
&\quad = \sum_{S} \Pr(W_n = 0 \mid \mathcal{Z}_n = S,\ Y_n = k) \Pr(\mathcal{Z}_n = S \mid Y_n = k,\ Z_n \leq \beta_n),
\end{aligned}$$

where the sum is over subsets $S \subseteq \{1, 2, \dots, n\}$ such that $2 \leq |S| \leq \beta_n$. It is enough to find a lower bound for $\Pr(W_n = 1 \mid \mathcal{Z}_n = S,\ Y_n = k)$.

To simplify notation, let $\gamma = A(\tilde{q}_0, \tilde{r}_0)$, and recall that $\gamma = O(1/n \log^3 n)$. In addition, define $|D_{\delta_n}(o)| = \pi / n^{2/3} (\log n)^2$ to be the area of the disk $D_{\delta_n}(o)$. An important observation is that, once we have specified $n - |S| =$ the number of points that fall *outside* $D_{\delta_n}(o)$, the locations in $D_{\delta_n}(o)^{\mathrm{c}}$ of these $n - |S|$ points are independent of the locations of the $|S|$ points in $D_{\delta_n}(o)$. Hence,

$$\begin{aligned}
\Pr(W_n = 1 \mid \mathcal{Z}_n = S,\ Y_n = k) &\geq \frac{(1 - |D_{\delta_n}(o)|/\pi - \gamma/\pi)^{n-|S|}}{(1 - |D_{\delta_n}(o)|/\pi)^{n-|S|}} \\
&\geq \left(1 - \frac{C}{n(\log n)^3}\right)^{n-|S|} \\
&\geq 1 - \frac{C'}{(\log n)^3}
\end{aligned}$$

for some constants C and C' which are independent of $\mathcal{Z}_n$ and Y_n. Hence,

$$\Pr(W_n = 0) \leq \frac{c}{(\log n)^3}$$

for some positive constant c that does not depend on n.

We note that the result obtained in Theorem 2 depends on a delicate trade-off. We must choose small enough δ_n and large enough L_n to guarantee that, for any $q \in Q_i$ and any $u \in U_i$, where (Q_i, U_i) is a pair of opposite sectors of $D_{\delta_n}(o)$, there is a high probability that none of the points $X_1, X_2, \dots, X_n$ lie in the 'uncovered' region $(D_1(q) \cup D_1(r))^{\mathrm{c}} \cap D_1(o)$. On the other hand, δ_n must not be so small or L_n so large that we cannot find (with high probability) some pair of opposite sectors (Q_i, U_i) such that there exists some $X_j \in Q_i$ and $X_k \in U_i$.

We end this section with an observation that is not needed in this paper, but is worth mentioning because of its relevance in applications [7]. It is implicit in the proof of Theorem 2 that, with asymptotic probability 1, the two covering points can be chosen in such a way that the distance between them is less than 1. In the language of graph theory we say that the two points are a connected dominating set for the random unit disk graph with vertices $X_1, X_2, \dots, X_n$.

5. Other densities

It is not difficult to see that our conclusions do not hold for arbitrary densities. In particular, the following is an example of a density for which two points do *not* suffice. Choose a positive number r such that $1 + 2r < \sqrt{3}$. For $j = 0, 1, 2$, let $z_j = (\cos(2\pi j/3), \sin(2\pi j/3))$, and let $O_j = D_r(z_j) \cap D_1(o)$ be the set of points in the unit disk whose distance from z_j is less than r. Let $M = \text{Area}(O_j)$ be the common area of these three regions, and define $f(x, y) = 1/3M$ if $(x, y) \in O_j$ for some j (and $f(x, y) = 0$ otherwise). With asymptotic probability 1, each of the three regions contains at least one of the random points. A point in O_j cannot cover a point in O_i if $i \neq j$ because the distance between two such points is more than $\sqrt{3} - 2r > 1$. Therefore, with asymptotic probability 1, three points are required. We have not been able to characterize the densities f for which two points do in fact suffice. We conjecture that a sufficient condition is for f to be radially symmetric and weakly decreasing as a function of the distance to the origin. In other words, in polar coordinates $\partial f/\partial r \leq 0$ and $\partial f/\partial \theta = 0$.

6. Final comments

The problems in this paper originated in the context of mathematical models for wireless networks [2], [6], [9]. For that particular application, dimensions $m = 2$ (see [1]) and $m = 3$ (see [3] and [4]) are the only ones where the problems make sense. Nevertheless, we believe it is a very natural and interesting mathematical question to consider an arbitrary fixed dimension m: for a random set of points V_n in the unit ball in $\mathbb{R}^m$, how many points of V_n are needed to cover V_n? We proved in Section 2 of this paper that, in general, one point is not enough. Our main result answers the question only for dimension $m = 2$; when $m = 2$, two points suffice. We did prove in [6] that, when $m = 3$, the probability that there does *not* exist a four-point covering set is exponentially small. Therefore, for $m = 3$, the smallest covering set consists of either two, three or four points (with asymptotic probability 1 as n tends to ∞). Limited simulations by Patricia Stamets suggest that two or three points suffice when $m = 3$. However, attempts to prove this got bogged down in complicated calculations. We conjecture that, in general, m points suffice. But we have no idea how to handle this general case.

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