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EXPLICIT FORMS OF LOCAL LIFTING FOR GL_2

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ABSTRACT. Let F be a local non-Archimedean field and let $\mathcal{S}(GL_2(F))$ be the set of equivalence classes of irreducible admissible representations of $GL_2(F)$. When K/F be a Galois field extension, there is a map, called *lifting*, from $\mathcal{S}(GL_2(F))$ to $\mathcal{S}(GL_2(K))$. We give an explicit form of lifting when K/F is a quadratic wildly ramified extension and the given representations are Weil supercuspidal. We also provide a comparison between Weil representations and induced representations of $GL_2(F)$.

0. **Introduction.** Let F be a local non-Archimedean field with residual characteristic p. We denote by W_F the absolute Weil group of F. Let K/F be a Galois field extension. Langlands conjectured a correspondence between the set of equivalence classes of irreducible admissible representations of $GL_N(F)$, say $\mathcal{S}\big(GL_N(F)\big)$, and the set of equivalence classes of N-dimensional semisimple Deligne representations of W_F, say $\mathcal{S}_N(W_F)$, which among other things, preserves invariants called *local constants*. From the conjectural correspondence, there is a map, called *base change lifting* or *lifting*, from $\mathcal{S}\big(GL_N(F)\big)$ to $\mathcal{S}\big(GL_N(K)\big)$ so that the following diagram commutes:

$$\begin{array}{ccc}
\mathcal{S}\big(GL_N(K)\big) & \xrightarrow{\text{Langlands correspondence}} & \mathcal{S}_N(W_K) \\
\text{lifting}\uparrow & & \uparrow\text{restriction} \\
\mathcal{S}\big(GL_N(F)\big) & \xrightarrow{\text{Langlands correspondence}} & \mathcal{S}_N(W_F)
\end{array}$$

When $N = 2$, the correspondence was known partially—the so called Weil (or oscillator) representation case—by Jacquet and Langlands in [JL], and the lifting was known in that case by Langlands [L]. Kutzko classified all of the supercuspidal representations of $GL_2(F)$ as induced representations which are represented by generic elements in $GL_2(F)$ and quasicharacters associated with the generic elements, and then constructed an explicit form of lifting in term of inducing data in case that K/F is tamely ramified. From this classification and tame lifting, he was able to complete the Langlands correspondence in the case $N = 2$, see [K1], [K2]. When N is equal to the residual characteristic, the tame lifting was studied by Henniart [H] and Kutzko and Moy [KM], and the lifting over a wild field extension with some restriction was studied by Moeglin [M]. To study the Langlands correspondence for general N and the Langlands correspondence over a global field, firstly we need an explicit lifting over a wild extension K/F.

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In this paper we characterize the generic elements of the lifting of Weil representations for GL_2 using purely local methods in case that K/F is a quadratic wildly ramified extension (hence the residual characteristic is 2), see Section 4. To prove the main Theorem 4.5, it is necessary to write the Weil representations in terms of the induced representation form, and a criterion is needed to tell if a given induced representation is a Weil representation. These are done in Section 2. In Section 1, we explain the Weil representations and the induced representations and in Section 3, we compute the difference of various field extensions and the dual blobs of characters.

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We use the following notations throughout:

F: a local non-Archimedean field;
O_F: the discrete valuation ring in F;
U_F: the multiplicative subgroup of O_F;
P_F: the maximal ideal of O_F;
π_F: a prime element of P_F;
$k_F = O_F/P_F$: the residual class field of F;
q_F: the cardinality of k_F;
ν_F: the valuation of F;
$\|\cdot\|_F$: the absolute value on F such that $\|x\|_F = q_F^{-\nu_F(x)}, x \in F^\times$.

1. **Constructions of supercuspidal representations of** GL_2. We introduce how to construct irreducible supercuspidal representations of $\mathrm{GL}_2(F)$. There are two methods. One is the Weil representation method and the other is the induced representation method. The Weil representation has been studied in [W], [JL] and [N]. We adapt here Kutzko's form of the Weil representation in [K3].

Let E/F be a field extension with the Galois group $\Gamma_{E/F}$. We denote by $d_{E/F}$ the exponent of the different of E/F. Given a quasicharacter θ of the multiplicative group $E^\times$ of E, the conductor $\mathfrak{f}(\theta)$ of θ is the smallest nonnegative integer m such that θ is trivial on U_E^m. Similarly, given an additive character ψ_E of E, the conductor $\mathfrak{f}(\psi_E)$ of ψ_E is the smallest integer m such that ψ_E is trivial on P_E^m. Let $\eta_{E/F}$ be a norm character of $F^\times$ with respect to E/F, that is, a nontrivial character of $F^\times$ which is trivial on $\mathrm{N}_{E/F}E^\times$ where $\mathrm{N}_{E/F}$ is the norm map of E/F. Let dx be the self-dual Haar measure on E for ψ_E. If G is a locally profinite set, we write $C_c^\infty(G)$ for the Schwartz space of G, *i.e.*, the space of locally constant compactly supported complex valued functions of G.

We fix, once for all, an additive character $\psi = \psi_F$ of F with $\mathfrak{f}(\psi) = 1$, and when the residual characteristic p of F is 2, ψ has the additional property that $\psi(x + x^2) = 1$ for x in O_F.

We define a Weil representation $\mathbf{W} = \mathbf{W}(E,\theta)$ of $\mathrm{GL}_2(F)$ on $C_c^\infty(F^\times)$ for a given quadratic separable extension E/F and a quasicharacter θ of $E^\times$ as follows:

$$\mathbf{W}\left(\begin{bmatrix} a & \\ & 1\end{bmatrix}\right)f(x) = f(ax), \tag{1.1}$$

$$\mathbf{W}\left(\begin{bmatrix} 1 & b \\ & 1\end{bmatrix}\right)f(x) = \psi(bx)f(x),$$

$$\mathbf{W}\left(\begin{bmatrix} d & \\ & d\end{bmatrix}\right)f(x) = \eta_{E/F}(d)\theta(d)f(x),$$

$$\mathbf{W}\left(\begin{bmatrix} & 1 \\ -1 & \end{bmatrix}\right)f(x) = \gamma_{E/F}\eta_{E/F}(x)\|x\|_E^{1/2} \cdot \int_E \theta^{-1}(\delta)\|\delta\|_E^{-1/2}\psi\,\mathrm{Tr}_{E/F}(x\delta)f\big(x\mathrm{N}_{E/F}(\delta)\big)\,d\delta,$$

where $f \in C_c^\infty(F^\times)$, $\gamma_{E/F} = \eta_{E/F}(a)\int_{U_F}\eta_{E/F}(\alpha)\psi(a\alpha)d\alpha\big/\big(\big|\int_{U_F}\eta_{E/F}(\alpha)\psi(a\alpha)d\alpha\big|\big)$, where a is a generator of $P_F^{\mathfrak{f}(\psi)-\mathfrak{f}(\eta_{E/F})}$ and $|\cdot|$ is the usual absolute value of a complex number.

NOTE. Let $C_\theta^\infty(F^\times \times E)$ be the space of functions $\bar{f}$ in $C_c^\infty(F^\times \times E)$ such that $\bar{f}(x\mathrm{N}_{E/F}\alpha, \beta\alpha^{-1}) = \|\alpha\|_E^{1/2}\theta(\alpha)\bar{f}(x,\beta)$ for $x \in F^\times$, $\alpha \in E^\times$ and $\beta \in E$. Then the map $C_\theta^\infty(F^\times \times E) \longrightarrow C_c^\infty(F^\times)$ defined by $\bar{f} \longmapsto f$ where $f(x) = \bar{f}(x,1)$ is an isomorphism. Therefore $\mathbf{W}(E,\theta)$ is equivalent to the usual Weil representation, see [K3].

The Weil representation $\mathbf{W}(E,\theta)$ has the properties, [JL].

PROPOSITION 1.2. *(1) If $\theta \neq \theta^\tau$ where $\Gamma_{E/F} = \langle\tau\rangle$, then $\mathbf{W}(E,\theta)$ is a supercuspidal representation.*

(2) If $\theta = \theta^\tau$, then $\mathbf{W}(E,\theta)$ is not supercuspidal.

(3) $\mathbf{W}(E,\theta_1) = \mathbf{W}(E,\theta_2)$ if and only if $\theta_2 = \theta_1$ or $\theta_2 = \theta_1^\tau$.

(4) When $E_1 \neq E_2$, $\mathbf{W}(E_1,\theta_1) = \mathbf{W}(E_2,\theta_2)$ if and only if $\theta_1^{\tau_1-1}$ and $\theta_2^{\tau_2-1}$ are of order 2 and $\theta_1 \circ N_{E_1E_2/E_1} = \theta_2 \circ N_{E_1E_2/E_2}$.

We will need the following lemma later.

LEMMA 1.3. *Let E be a quadratic separable extension field of F, and let $\Gamma_{E/F} = \langle\tau\rangle$. Let θ be a quasicharacter of $E^\times$. Then $\theta = \theta^\tau$ if and only if $\theta = \chi \circ N_{E/F}$ for some quasicharacter χ of $F^\times$.*

PROOF. Suppose that $\theta = \theta^\tau$. Since E/F is cyclic, the kernel of the norm map $N_{E/F} : E^\times \longrightarrow F^\times$ is the set $\{x^{\tau-1} \mid x \in E^\times\}$ so $\ker N_{E/F}$ is contained in $\ker\theta$. Define a quasicharacter χ on $N_{E/F}(E^\times)$ by $\chi\big(N_{E/F}(x)\big) = \theta(x)$ for $x \in E^\times$. It is well defined because $\ker N_{E/F} \subset \ker\theta$. Now we extend χ to $F^\times$. Then χ is a quasicharacter of $F^\times$ and $\theta = \chi \circ N_{E/F}$ on $E^\times$. The other direction is trivial. ∎

DEFINITION 1.4. If a Weil representation Π of $\mathrm{GL}_2(F)$ is of the form $\mathbf{W}(E,\theta)$ with E an unramified quadratic extension of F, then we say that Π is an *unramified* Weil representation. Otherwise we call it *ramified*.

PROPOSITION 1.5. *(1) Let* Π *be a ramified Weil representation of* $\mathrm{GL}_2(F)$*. Then there exists a quadratic ramified extension* E/F *and a quasicharacter* θ *of* $E^\times$ *such that* $\mathfrak{f}(\theta) \geq 2d_{E/F} - 1$ *and* $\mathfrak{f}(\theta) + d_{E/F}$ *is odd, and a character* χ *of* $F^\times$ *so that* Π *is equivalent to the representation* $\mathbf{W}(E, \theta) \otimes \chi \circ \det$*.*

(2) If there exist E', θ', χ' *with the above properties and if* $E' \neq E$ *then* $p = 2, \mathfrak{f}(\theta) = 2d_{E/F} - 1 = 2d_{E'/F} - 1 = \mathfrak{f}(\theta')$*, and* $\mathfrak{f}(\eta_{E/F} \cdot \eta_{E'/F}^{-1}) = d_{E/F}$*.*

PROOF. Lemma 2.2, in [K3]. ∎

It is known that every irreducible supercuspidal representation of $\mathrm{GL}_2(F)$ is induced from a finite dimensional representation of a compact open subgroup, see [K1]. We will describe this construction. Let V be a 2-dimensional vector space over F and $A = \mathrm{End}_F(V)$. An O_F*-lattice* in V is a free O_F-submodule of rank 2, and an O_F*-order* in A is an O_F-lattice in A as a 4-dimensional vector space that is also a subring of A. An O_F-order is called (left-) *principal* if its Jacobson radical is a principal (left-) ideal.

An O_F*-lattice chain* in V is a set $\mathcal{L} = \{L_i : i \in \mathbb{Z}\}$ of O_F-lattices L_i in V such that

(i) $L_i \supset L_{i+1}, L_i \neq L_{i+1}$ for every $i \in \mathbb{Z}$,

(ii) there exists an integer e such that $L_{i+e} = P_F L_i$ for every $i \in \mathbb{Z}$.

The integer $e = e(\mathcal{L})$ is called the *period* of $\mathcal{L}$. Hence e is 1 or 2. If $\mathcal{L} = \{L_i\}$ and $\mathcal{L}' = \{L_i'\}$ are O_F-lattice chains in V, we say that $\mathcal{L}$ and $\mathcal{L}'$ are called to be *equivalent* if there is an integer k such that $L_i' = L_{i+k}$ for all integers i and write $\mathcal{L} \sim \mathcal{L}'$. Let $\mathcal{A}_\mathcal{L}$ be the set of g in A satisfying $gL_i \subset L_i$ for all integers i, then $\mathcal{A} = \mathcal{A}_\mathcal{L}$ is an O_F-order in A. We denote by $\mathcal{P}_\mathcal{A} = \mathcal{P}_{\mathcal{A}_\mathcal{L}}$ the Jacobson radical of $\mathcal{A}$. Write $e(\mathcal{A}_\mathcal{L}) = e(\mathcal{L})$. Then we have the propositions, see for example [BF].

PROPOSITION 1.6. *(1)* $\mathcal{A}_\mathcal{L}$ *is a principal order.*

(2) Every principal O_F*-order in* A *is of the form* $\mathcal{A}_\mathcal{L}$ *for some lattice chain* $\mathcal{L}$*. We may recover the lattice chain* $\mathcal{L}$ *from the order* $\mathcal{A}$*, up to shift in the index, and* $\mathcal{L}$ *is precisely the set of all* $\mathcal{A}$*-lattices in* V*.*

(3) As a fractional ideal of A*, the radical* $\mathcal{P}_\mathcal{A}$ *is invertible and we have*

$$\mathcal{P}_\mathcal{A}^n = \mathrm{End}_{O_F}^n(\mathcal{L}) \quad \textit{for every } n \in \mathbb{Z},$$

where $\mathrm{End}_{O_F}^n(\mathcal{L}) = \{g \in A : gL_i \subset L_{i+n} \textit{ for every } i \in \mathbb{Z}\}$*.*

We define a sequence of compact open subgroups of GL_2 by

$$\mathcal{U}_\mathcal{A} = \mathcal{U}_\mathcal{A}^0 = \mathcal{A}^\times, \text{ and } \mathcal{U}_\mathcal{A}^n = 1 + \mathcal{P}_\mathcal{A}^n \quad \text{for integers } n \geq 1.$$

We set

$$\mathcal{K}_\mathcal{A} = \{x \in G : x^{-1}\mathcal{A}x = \mathcal{A}\}.$$

This is a maximal open compact-mod-center subgroup of GL_2, and every maximal open compact-mod-center subgroup of GL_2 is of the form, for some principal order. The $\mathcal{U}_\mathcal{A}^n$ for $n \geq 0$, are normal subgroups of $\mathcal{K}_\mathcal{A}$ and in particular $\mathcal{U}_\mathcal{A}$ is the unique maximal compact subgroup of $\mathcal{K}_\mathcal{A}$.

PROPOSITION 1.7. *Let $\mathcal{L}$ and $\mathcal{L}'$ be O_F-lattices in V. Then*

(1)

$$\mathcal{L}' \sim \mathcal{L} \iff \mathcal{A}_{\mathcal{L}'} = \mathcal{A}_{\mathcal{L}}.$$

(2)

$$\begin{aligned} e(\mathcal{L}') = e(\mathcal{L}) &\iff \mathcal{L}' = g\mathcal{L} \quad \textit{for some } g \in A \\ &\iff \mathcal{A}_{\mathcal{L}'} = g\mathcal{A}_{\mathcal{L}} g^{-1} \quad \textit{for some } g \in A \\ &\iff \mathcal{K}_{\mathcal{A}_{\mathcal{L}'}} = g\mathcal{K}_{\mathcal{A}_{\mathcal{L}}} g^{-1} \quad \textit{for some } g \in A. \end{aligned}$$

In this paper we are interested in ramified supercuspidal representations of GL_2. We describe how to construct all of the ramified supercuspidal representations. For the unramified supercuspidal representation case, see [GK], [K1]. Let us fix a principal O_F-order $\mathcal{A}$ with $e(\mathcal{A}) = 2$. It is easy to check that all principal O_F-orders with period 2 are conjugate each other by $GL_2(F)$.

DEFINITION 1.8. An element $\bar{b}$ in A is said to be *$\mathcal{A}$-generic* of level $1 - 2n$ for some integer n if

(i) $\bar{E} = F[\bar{b}]$ is a quadratic ramified field extension of F in A,
(ii) $\bar{E}^\times \subset \mathcal{K}_{\mathcal{A}}$, and
(iii) $\nu_{\bar{E}}(\bar{b}) = 1 - 2n$.

Let r and n be integers satisfying $n > r \geq [(n+1)/2] > 0$, where $[x]$ denotes the greatest integer $\leq x$ for $x \in \mathbb{R}$. We then have a canonical isomorphism

$$\mathcal{P}^r_{\mathcal{A}} / \mathcal{P}^n_{\mathcal{A}} \xrightarrow{\approx} \mathcal{U}^r_{\mathcal{A}} / \mathcal{U}^n_{\mathcal{A}}$$

given by $k \longmapsto 1 + k$. This leads to an isomorphism

$$\mathcal{P}^{\mathfrak{f}(\psi)-n}_{\mathcal{A}} / \mathcal{P}^{\mathfrak{f}(\psi)-r}_{\mathcal{A}} \xrightarrow{\approx} (\mathcal{U}^r_{\mathcal{A}} / \mathcal{U}^n_{\mathcal{A}})^\wedge,$$

where "hat" denotes the topological dual which is the set of characters of the finite abelian group. Explicitly this is given by

$$\bar{b} + \mathcal{P}^{\mathfrak{f}(\psi)-r}_{\mathcal{A}} \longmapsto \psi_{A,\bar{b}} = \psi_{\bar{b}} \quad \text{for } \bar{b} \in \mathcal{P}^{\mathfrak{f}(\psi)-n}_{\mathcal{A}},$$

where $\psi_{\bar{b}}(k) = \psi \circ \operatorname{tr}_{A/F}\big(\bar{b}(k-1)\big)$ for $k \in \mathcal{U}^r_{\mathcal{A}}$.

We have similar properties for a field extension E instead of $\mathcal{A}$. Let ψ_E be an additive character of E with conductor $\mathfrak{f}(\psi_E)$. For the same r and n, we have an isomorphism

$$P^{\mathfrak{f}(\psi_E)-n}_E / P^{\mathfrak{f}(\psi_E)-r}_E \xrightarrow{\approx} (U^r_E / U^n_E)^\wedge \quad \text{given by } b + P^{\mathfrak{f}(\psi_E)-r}_E \longmapsto \psi_{E,b},$$

for $b \in P^{\mathfrak{f}(\psi)-n}_E$ where $\psi_{E,b}(k) = \psi_E b(k-1)$ for $k \in U^r_E$.

In particular, for a given quasicharacter θ of $E^\times$ with conductor $\mathfrak{f}(\theta)$, we may view θ as a character of $U^{[(\mathfrak{f}(\theta)+1)/2]}_E / U^{\mathfrak{f}(\theta)}_E$. Hence there is a coset $b + P^{\mathfrak{f}(\psi_E)-[(\mathfrak{f}(\theta)+1)/2]}_E$ such that $\theta(k) = \psi_{E,b}(k)$, for $k \in U^{[(\mathfrak{f}(\theta)+1)/2]}_E$. We call the coset $b + P^{\mathfrak{f}(\psi_E)-[(\mathfrak{f}(\theta)+1)/2]}_E$ by a *dual blob* of θ with respect to ψ_E, and when confusion is unlikely, for convenience we say the element b a dual blob of θ with respect to ψ_E. The terminology "dual blob" comes from [Ho].

PROPOSITION 1.9. *(1) With notations as above, let n be a positive integer and* $\bar{b}$ *a* $\mathcal{A}$*-generic element of level* $1-2n$. *Let* $\bar{E} = F[\bar{b}]$. *Let* $\bar{\theta}$ *be a quasicharacter of the subgroup* $\bar{E}^\times$ *of* $\mathrm{GL}_2(F)$ *such that* $\bar{\theta}(k) = \psi\,\mathrm{Tr}_{\bar{E}/F}\,\bar{b}(k-1)$ *for* $k \in U^n_{\bar{E}}$. *Then the complex valued function* $\bar{\theta}\psi_{\bar{b}}$ *of* $\bar{E}^\times \mathcal{U}^n_{\mathcal{A}}$ *defined by* $\bar{\theta}\psi_{\bar{b}}(ku) = \bar{\theta}(k)\psi_{\bar{b}}(u)$ *for* $k \in \bar{E}^\times$, $u \in \mathcal{U}^n_{\mathcal{A}}$ *is a well-defined quasicharacter of* $\bar{E}^\times\mathcal{U}^n_{\mathcal{A}}$ *which induces an irreducible supercuspidal representation of* $\mathrm{GL}_2(F)$, *denoted* $\Pi(\mathcal{A},\psi,\bar{b},\bar{\theta})$.

(2) Given an irreducible ramified supercuspidal representation Π *of* $\mathrm{GL}_2(F)$ *there exist* $\mathcal{A},\psi,\bar{b},\bar{\theta}$ *as above and a character* χ *of* $F^\times$ *such that* $\Pi \cong \Pi(\mathcal{A},\psi,\bar{b},\bar{\theta}) \otimes \chi \circ \det$.

(3) $\Pi(\mathcal{A},\psi,\bar{b},\bar{\theta}_1) \cong \Pi(\mathcal{A},\psi,\bar{b},\bar{\theta}_2)$ *if and only if* $\bar{\theta}_2 = \bar{\theta}_1$.

(4) $\Pi(\mathcal{A},\psi,\bar{b}_1,\bar{\theta}_1) \cong \Pi(\mathcal{A},\psi,\bar{b}_2,\bar{\theta}_2)$ *if and only if there exists an element g in* $\mathcal{K}_{\mathcal{A}}$ *such that*

(i) $\bar{b}_2 \equiv g\bar{b}_1g^{-1} \pmod{\mathcal{P}^{1-n}_{\mathcal{A}}}$,

(ii) $\bar{\theta}_2\psi_{\bar{b}_2} = (\bar{\theta}_1\psi_{\bar{b}_1})^g$.

PROOF. See Proposition 1.3.1 in [KM]. ∎

2. **Correspondence between two constructions.** In this section we give a connection between the constructions of the Weil representation method and the induced representation method for $\mathrm{GL}_2(F)$. A given Weil ramified representation can be described as an induced representation.

We fix a ramified quadratic extension E/F and a quasicharacter θ of $E^\times$ for which $\mathfrak{f}(\theta) \geq 2d_{E/F} - 1$ and $\mathfrak{f}(\theta) + d_{E/F}$ is an odd number, see Proposition 1.5. We set $n = n(E,\theta) = \big(\mathfrak{f}(\theta) + d_{E/F} - 1\big)/2$. We denote by $b = b(\psi\,\mathrm{Tr}_{E/F},\theta)$ the dual blob in E for θ with respect to $\psi\,\mathrm{Tr}_{E/F}$, that is $\theta(k) = \psi\,\mathrm{Tr}_{E/F}\big(b(k-1)\big)$ for $k \in U_E^{[(\mathfrak{f}(\theta)+1)/2]}$, and by $c_\psi = c_{\psi,E/F}$ the dual blob in F for the norm character $\eta_{E/F}$ of $F^\times$ with respect to ψ, that is $\eta_{E/F}(k) = \psi\big(c_\psi(k-1)\big)$ for $k \in U_F^{[(\mathfrak{f}(\eta_{E/F})+1)/2]}$.

We fix a principal order $\mathcal{A}_n$ by setting that for every integer k

$$\mathcal{P}^k_n = \begin{bmatrix} P_F^{[(k+1)/2]} & P_F^{1-n+[k/2]} \\ P_F^{n+[k/2]} & P_F^{[(k+1)/2]} \end{bmatrix}.$$

Let $\bar{b} = \begin{bmatrix} & -N_{E/F}b \\ 1 & \mathrm{Tr}_{E/F}\,b + c_\psi \end{bmatrix}$ which is in A. Then $\bar{b}$ is an $\mathcal{A}_n$-generic element of level $1-2n$ and $\bar{E} = F[\bar{b}]$ is a quadratic field extension of F, see (1.8).

A connection between two exponents $d_{E/F}$ and $d_{\bar{E}/F}$ of the difference of E and $\bar{E}$ over F is given by:

LEMMA 2.1. $d_{E/F} = \min\big(d_{\bar{E}/F}, 2(n+1)/3\big)$.

PROOF. Since $\nu_E(b\pi_F^n) = \nu_{\bar{E}}(\bar{b}\pi_F^n) = 1$, we have that

$$d_{E/F} = \min\Big(2\big(\nu_F(\mathrm{Tr}_{E/F}\,b) + n\big), 2\nu_F(2) + 1\Big), \tag{2.2}$$

$$d_{\bar{E}/F} = \min\Big(2\big(\nu_F(\mathrm{Tr}_{E/F}\,b + c_\psi) + n\big), 2\nu_F(2) + 1\Big).$$

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First we claim that $d_{E/F} \leq d_{\bar{E}/F}$. If $\nu_F(\mathrm{Tr}_{E/F} b) \leq \nu_F(c_\psi)$, then we compare equations of (2.2) and have that $d_{E/F} \leq d_{\bar{E}/F}$. Suppose $\nu_F(\mathrm{Tr}_{E/F} b) > \nu_F(c_\psi)$. Then noting that $\nu_F(c_\psi) = 1 - d_{E/F}$, we have $2\big(\nu_F(\mathrm{Tr}_{E/F} b) + n\big) > 2\big(\nu_F(c_\psi) + n\big) = 2(n+1) - 2d_{E/F} \geq d_{E/F}$. Hence $d_{E/F} = 2\nu_F(2) + 1$. Now $2\big(\nu_F(\mathrm{Tr}_{E/F} b + c_\psi) + n\big) = 2\big(\nu_F(c_\psi) + n\big) \geq d_{E/F} = 2\nu_F(2) + 1$, hence $d_{\bar{E}/F} = d_{E/F}$. Therefore we have that $d_{\bar{E}/F} \geq d_{E/F}$.

Secondly we claim that $d_{\bar{E}/F} > d_{E/F}$ if and only if $3d_{\bar{E}/F} > 2(n+1)$. If $d_{\bar{E}/F} > d_{E/F}$, then by (2.2) $d_{E/F}$ is even and $\nu_F(\mathrm{Tr}_{E/F} b) = \nu_F(c_\psi)$, hence $d_{E/F} = 2(n+1)/3$, and $d_{\bar{E}/F} > 2(n+1)/3$. If $d_{\bar{E}/F} > 2(n+1)/3$, then $d_{\bar{E}/F} > d_{E/F}$ because $d_{E/F} \leq 2(n+1)/3$. So that the second claim holds. This claim says that $d_{\bar{E}/F} \leq d_{E/F}$ if and only if $3d_{\bar{E}/F} \leq 2(n+1)$. With the first claim the lemma follows. ∎

We will prove a key proposition for the connection between two constructions of supercuspidal representations which is an improved version of Proposition 2.3 in [K3]. We keep the same notation as above.

PROPOSITION 2.3. *$\mathbf{W}(E,\theta)(k)f_o = \psi\,\mathrm{Tr}_{A/F}\,\bar{b}(k-1)f_o$ for $k \in U_{\bar{E}}^{1+n-[(d_{E/F}+1)/2]}\mathcal{U}_{\mathfrak{A}_n}^n$, where f_o is the characteristic function of $U_F^{[(n+1)/2]}$ in $C_c^\infty(F^\times)$.*

PROOF. It is suffice to prove the equation for the elements in $U_{\bar{E}}^{1+n-[(d_{E/F}+1)/2]}$ and in $\mathcal{U}_{\mathfrak{A}_n}^n$ separately, because of the multiplicative property of $\mathbf{W}(E,\theta)$ and $\psi\,\mathrm{Tr}_{A/F}$. When k is an element of $\mathcal{U}_{\mathfrak{A}_n}^n$, it is true by Proposition 2.3 in [K3]. Here we prove the statement for k in $U_{\bar{E}}^{1+n-[(d_{E/F}+1)/2]}$. We set $\mathbf{W} = \mathbf{W}(E,\theta)$ and $d = d_{E/F}$. Suppose first that k is an element in $U_{\bar{E}}^{1+n-[(d+1)/2]} \cap F^\times$. Then k is of the form $1+w$ where $w \in P_F^{[(2+n-[(d+1)/2])/2]}$. By (1.1) $\mathbf{W}(1+w)f_o(x) = \eta_{E/F}(1+w)\theta(1+w)f_o(x)$. Since $1+n-[(d+1)/2] = \big[\big(\mathfrak{f}(\theta)+1\big)/2\big]$, $\nu_E(w) \geq \big[\big(\mathfrak{f}(\theta)+1\big)/2\big]$, and since $1+n \geq 3d/2$, $\big[\big(2+n-[(d+1)/2]\big)/2\big] \geq [(d+1)/2]$, hence $\nu_F(w) \geq \big[\big(\mathfrak{f}(\eta_{E/F})+1\big)/2\big]$. Let $s = \mathrm{Tr}_{E/F}\, b$, $\Delta = \mathrm{N}_{E/F} b$, and $c = c_\psi$. Therefore the equation becomes

$$\begin{aligned}
\mathbf{W}(1+w)f_o(x) &= (\psi c w)(\psi\,\mathrm{Tr}_{E/F}\, b w)f_o(x) \\
&= \psi\big((s+c)w\big)f_o(x) \\
&= \psi\,\mathrm{Tr}_{\bar{E}/F}(\bar{b}w)f_o(x).
\end{aligned}$$

Now let k be an element in $U_{\bar{E}}^{1+n-[(d+1)/2]}$ and not in $F^\times$. Then k is of the form $1+\bar{b}v+w$ where v and w are elements in F. From the first case we may assume k is of the form $1+\bar{b}v$ which is in $U_{\bar{E}}^{1+n-[(d+1)/2]}$. We write $1+\bar{b}v$ as a product of convenient matrices

$$\begin{aligned}
1+\bar{b}v &= \begin{bmatrix} 1 & -\Delta v \\ v & 1+(s+c)v \end{bmatrix} \\
&= \begin{bmatrix} & 1 \\ -1 & \end{bmatrix}^{-1} \begin{bmatrix} D & \\ & 1 \end{bmatrix} \begin{bmatrix} 1 & -v/D \\ & 1 \end{bmatrix} \begin{bmatrix} & 1 \\ -1 & \end{bmatrix} \begin{bmatrix} 1 & -\Delta v \\ & 1 \end{bmatrix},
\end{aligned}$$

where $D = \det(1+\bar{b}v) = 1+(s+c)v+\Delta v^2$.

To complete the proof, it is enough to show that

$$\mathbf{W}\left(\begin{bmatrix} D & \\ & 1 \end{bmatrix}\begin{bmatrix} 1 & -v/D \\ & 1 \end{bmatrix}\begin{bmatrix} & 1 \\ -1 & \end{bmatrix}\begin{bmatrix} 1 & -\Delta v \\ & 1 \end{bmatrix}\right)f_o(x) \\ = \psi\operatorname{Tr}_{\bar{E}/F}\big(\bar{b}(\bar{b}v)\big)\mathbf{W}\left(\begin{bmatrix} & 1 \\ -1 & \end{bmatrix}\right)f_o(x). \tag{2.4}$$

It is a straightforward computation, using (1.1), that the left side of the equation is

$$\mathbf{W}\left(\begin{bmatrix} D & \\ & 1 \end{bmatrix}\begin{bmatrix} 1 & -v/D \\ & 1 \end{bmatrix}\begin{bmatrix} & 1 \\ -1 & \end{bmatrix}\begin{bmatrix} 1 & -\Delta v \\ & 1 \end{bmatrix}\right)f_o(x) \\ = \psi(-vx)\gamma_{E/F}\eta_{E/F}(Dx)\|Dx\|_E^{1/2} \\ \cdot \int_E \theta^{-1}(\delta)\|\delta\|_E^{-1/2}\psi(-\Delta v D x \mathrm{N}_{E/F}\delta)\psi\operatorname{Tr}_{E/F}(Dx\delta)f_o(Dx\mathrm{N}_{E/F}\delta)\,d\delta.$$

We may check that $1+sv+\Delta v^2 = N_{E/F}(1+bv)$, $\nu_F(cv)\geq d$, and so $\eta_{E/F}(D) = \eta_{E/F}(1+sv+\Delta v^2)(1+cv) = 1$, and since $D\in U_F^{[(1+n+[d/2])/2]}$ we have that $D\cdot x\mathrm{N}_{E/F}\delta \in U_F^{[(n+1)/2]}$ if and only if $x\mathrm{N}_{E/F}\delta \in U_F^{[(n+1)/2]}$. Also $\nu_F\big(-\Delta v(D-1)(x\mathrm{N}_{E/F}\delta)\big) \geq \big(1-n/2-[(d-1)/2]/2\big)+\big[(1+n-[d/2])/2\big]+0 > 0$. So $\psi\big(-\Delta v(D-1)(x\mathrm{N}_{E/F}\delta)\big) = 1$. Note that f_o is the characteristic function of $U_F^{[(n+1)/2]}$. Putting these all together we have that

$$\mathbf{W}\left(\begin{bmatrix} D & \\ & 1 \end{bmatrix}\begin{bmatrix} 1 & -v/D \\ & 1 \end{bmatrix}\begin{bmatrix} & 1 \\ -1 & \end{bmatrix}\begin{bmatrix} 1 & -\Delta v \\ & 1 \end{bmatrix}\right)f_o(x) \\ = \psi(-vx)\gamma_{E/F}\eta_{E/F}(x)\|x\|_E^{1/2} \\ \cdot \int_{x\mathrm{N}_{E/F}\delta\in U_F^{[(n+1)/2]}} \theta^{-1}(\delta)\|\delta\|_E^{-1/2}\psi(-\Delta v x\mathrm{N}_{E/F}\delta)\psi\operatorname{Tr}_{E/F}(Dx\delta)\,d\delta. \tag{2.5}$$

On the other hand, the right side of the equation (2.4) becomes

$$\psi\operatorname{Tr}_{\bar{E}/F}\big(\bar{b}(\bar{b}v)\big)\mathbf{W}\left(\begin{bmatrix} & 1 \\ -1 & \end{bmatrix}\right)f_o(x) \\ = \psi\operatorname{Tr}_{\bar{E}/F}\big(\bar{b}(\bar{b}v)\big)\gamma_{E/F}\eta_{E/F}(x)\|x\|_E^{1/2} \\ \cdot \int_{x\mathrm{N}_{E/F}\delta\in U_F^{[(n+1)/2]}} \theta^{-1}(\delta)\|\delta\|_E^{-1/2}\psi\operatorname{Tr}_{E/F}(x\delta)\,d\delta. \tag{2.6}$$

For convenience we denote by $I_L(x)$ the integral part of (2.5), *i.e.*,

$$I_L(x) = \int_{x\mathrm{N}_{E/F}\delta\in U_F^{[(n+1)/2]}} \theta^{-1}(\delta)\|\delta\|_E^{-1/2}\psi(-\Delta v x\mathrm{N}_{E/F}\delta)\psi\operatorname{Tr}_{E/F}(Dx\delta)\,d\delta$$

and by $I_R(x)$ the integral part of (2.6), *i.e.*,

$$I_R(x) = \int_{x\mathrm{N}_{E/F}\delta\in U_F^{[(n+1)/2]}} \theta^{-1}(\delta)\|\delta\|_E^{-1/2}\psi\operatorname{Tr}_{E/F}(x\delta)\,d\delta.$$

We will need a lemma:

LEMMA 2.7. *If* $x\delta b^{-1} \notin U_E^{n-[d/2]}$, *then both of* $I_L(x)$ *and* $I_R(x)$ *are zero.*

PROOF OF LEMMA. We have that $\mathrm{N}_{E/F}(U_E^{1+n-[(d+1)/2]}) \subset U_F^{[(n+1)/2]}$, we can change the variable δ to $\delta(1+u)$ where $u \in P_E^{1+n-[(d+1)/2]}$ and do the double integral. Then

$$I_R(x) = (\text{nonzero constant}) \cdot I'_R(x),$$

where

$$I'_R(x) = \int_{u \in P_E^{1+n-[(d+1)/2]}} \int_{x\mathrm{N}_{E/F}\delta \in U_F^{[(n+1)/2]}} \theta^{-1}\big(\delta(1+u)\big) \|\delta(1+u)\|_E^{-1/2} \cdot \psi \operatorname{Tr}_{E/F}\big(x\delta(1+u)\big)\, d\delta\, du.$$

We have that

$$I'_R(x) = \int_{x\mathrm{N}_{E/F}\delta \in U_F^{[(n+1)/2]}} \theta^{-1}(\delta)\|\delta\|_E^{-1/2} \psi \operatorname{Tr}_{E/F}(x\delta) \cdot \int_{u \in P_E^{1+n-[(d+1)/2]}} \theta^{-1}(1+u)\|1+u\|_E^{-1/2} \psi \operatorname{Tr}_{E/F}(x\delta u)\, du\, d\delta.$$

Notice that $\theta^{-1}(1+u) = \theta(1-u) = \psi \operatorname{Tr}_{E/F}(-bu)$ for $u \in P_E^{1+n-[(d+1)/2]}$, we simplify further that

$$I'_R(x) = \int_{x\mathrm{N}_{E/F}\delta \in U_F^{[(n+1)/2]}} \theta^{-1}(\delta)\|\delta\|_E^{-1/2} \psi \operatorname{Tr}_{E/F}(x\delta) \cdot \int_{u \in P_E^{1+n-[(d+1)/2]}} \psi \operatorname{Tr}_{E/F}\big((x\delta - b)u\big)\, du\, d\delta.$$

Since the conductor of $\psi \operatorname{Tr}_{E/F}$ equals to $2-d$, we have $\int_{u \in P_E^{1+n-[(d+1)/2]}} \psi \operatorname{Tr}_{E/F}\big((x\delta - b)u\big)\, du = 0$ unless $(x\delta - b)u \in P_E^{2-d}$ equivalently unless $(x\delta - b) \in u^{-1}P_E^{2-d} = P_E^{1-n-[d/2]}$. Therefore $x\delta b^{-1} \notin U_E^{n-[d/2]}$ implies $I_R(x)$ is zero.

Now we check the $I_L(x)$. Using the same change of variable, we get

$$I_L(x) = (\text{nonzero constant}) \cdot I'_L(x),$$

where

$$I'_L(x) = \int_{u \in P_E^{1+n-[(d+1)/2]}} \int_{x\mathrm{N}_{E/F}\delta \in U_F^{[(n+1)/2]}} \theta^{-1}\big(\delta(1+u)\big) \|\delta(1+u)\|_E^{-1/2} \cdot \psi\Big(-\Delta v x \mathrm{N}_{E/F}\big(\delta(1+u)\big)\Big) \psi \operatorname{Tr}_{E/F}\big(Dx\delta(1+u)\big)\, d\delta\, du.$$

We get

$$\begin{aligned} I'_L(x) = &\int_{x\mathrm{N}_{E/F}\delta \in U_F^{[(n+1)/2]}} \theta^{-1}(\delta)\|\delta\|_E^{-1/2} \psi\big(-\Delta v x \mathrm{N}_{E/F}(\delta)\big) \psi \operatorname{Tr}_{E/F}(Dx\delta) \\ &\cdot \int_{u \in P_E^{1+n-[(d+1)/2]}} \theta^{-1}(1+u)\|1+u\|_E^{-1/2} \\ &\cdot \psi\Big(-\Delta v x \mathrm{N}_{E/F}(\delta)\big(\mathrm{N}_{E/F}(1+u) - 1\big)\Big) \\ &\cdot \psi \operatorname{Tr}_{E/F}(Dx\delta u)\, du\, d\delta \\ = &\int_{x\mathrm{N}_{E/F}\delta \in U_F^{[(n+1)/2]}} \theta^{-1}(\delta)\|\delta\|_E^{-1/2} \psi\big(-\Delta v x \mathrm{N}_{E/F}(\delta)\big) \psi \operatorname{Tr}_{E/F}(Dx\delta) \\ &\cdot \int_{u \in P_E^{1+n-[(d+1)/2]}} \psi \operatorname{Tr}_{E/F}\big((Dx\delta - b)u\big)\, du\, d\delta, \end{aligned}$$

because $\nu_F\big(-\Delta vx\mathrm{N}_{E/F}(\delta)(\mathrm{N}_{E/F}(1+u)-1)\big) = \big(1-n/2-[(d+1)/2]/2\big) + [(1+n+[d/2])/2] > 0$. It is easy to see that $x\delta - b \notin P_E^{1-n-[d/2]}$ implies $Dx\delta - b \notin P_E^{1-n-[d/2]}$, because suppose on the contrary that $Dx\delta - b \in P_E^{1-n-[d/2]}$. Since $(D-1)b \in P_E^{2[(1-n-[d/2])/2]+1-2n} \subset P_E^{1-n-[d/2]}$, we have $D(x\delta - b) = Dx\delta - b - (D-1)b \in P_E^{1-n-[d/2]}$, that is $x\delta - b \in P_E^{1-n-[d/2]}$.

Therefore $\int_{u\in P_E^{1+n-[(d+1)/2]}} \psi\,\mathrm{Tr}_{E/F}\big((Dx\delta - b)u\big)\,du = 0$ unless $x\delta - b \in P_E^{1-n-[d/2]}$. Hence $x\delta b^{-1} \notin U_E^{n-[d/2]}$ implies $I_L(x)$ is zero. ∎

Now we consider the equations (2.5) and (2.6) under the restriction to $x\delta b^{-1} \in U_E^{n-[d/2]}$. Then the equation (2.5) is

$$\begin{aligned}
&\mathbf{W}\left(\begin{bmatrix} D & \\ & 1\end{bmatrix}\begin{bmatrix} 1 & -v/D \\ & 1\end{bmatrix}\begin{bmatrix} & 1 \\ -1 & \end{bmatrix}\begin{bmatrix} 1 & -\Delta v \\ & 1\end{bmatrix}\right) f_o(x)\\
&\quad = \psi(-vx)\gamma_{E/F}\eta_{E/F}(x)\|x\|_E^{1/2}\\
&\qquad \cdot \int_{\substack{x\mathrm{N}_{E/F}\delta \in U_F^{[(n+1)/2]}\\ x\delta b^{-1}\in U_E^{n-[d/2]}}} \theta^{-1}(\delta)\|\delta\|_E^{-1/2}\psi(-\Delta vx\mathrm{N}_{E/F}\delta)\psi\,\mathrm{Tr}_{E/F}(Dx\delta)\,d\delta.
\end{aligned}$$

We may write $Dx\delta = x\delta + (D-1)b - (D-1)(x\delta - b)$ and $\nu_E\big((D-1)(x\delta - b)\big) \geq 2-d$, hence $\psi\,\mathrm{Tr}_{E/F}(Dx\delta) = \big(\psi\,\mathrm{Tr}_{E/F}(x\delta)\big)\big(\psi\,\mathrm{Tr}_{E/F}(D-1)b\big)$. Therefore

$$\begin{aligned}
&\mathbf{W}\left(\begin{bmatrix} D & \\ & 1\end{bmatrix}\begin{bmatrix} 1 & -v/D \\ & 1\end{bmatrix}\begin{bmatrix} & 1 \\ -1 & \end{bmatrix}\begin{bmatrix} 1 & -\Delta v \\ & 1\end{bmatrix}\right) f_o(x)\\
&\quad = \psi(-vx)\psi\,\mathrm{Tr}_{E/F}\big((D-1)b\big)\gamma_{E/F}\eta_{E/F}(x)\|x\|_E^{1/2}\\
&\qquad \cdot \int_{\substack{x\mathrm{N}_{E/F}\delta \in U_F^{[(n+1)/2]}\\ x\delta b^{-1}\in U_E^{n-[d/2]}}} \theta^{-1}(\delta)\|\delta\|_E^{-1/2}\psi(-\Delta vx\mathrm{N}_{E/F}\delta)\psi\,\mathrm{Tr}_{E/F}(x\delta)\,d\delta.
\end{aligned}$$

We make a change of variable δ to bux^{-1} with $u \in U_E^{n-[d/2]}$,

$$\begin{aligned}
&\mathbf{W}\left(\begin{bmatrix} D & \\ & 1\end{bmatrix}\begin{bmatrix} 1 & -v/D \\ & 1\end{bmatrix}\begin{bmatrix} & 1 \\ -1 & \end{bmatrix}\begin{bmatrix} 1 & -\Delta v \\ & 1\end{bmatrix}\right) f_o(x)\\
&\quad = \psi(-vx)\psi\,\mathrm{Tr}_{E/F}\big((D-1)b\big)\gamma_{E/F}\eta_{E/F}(x)\|x\|_E^{1/2}\\
&\qquad \cdot \int_{\substack{\Delta x^{-1}\mathrm{N}_{E/F}u \in U_F^{[(n+1)/2]}\\ u\in U_E^{n-[d/2]}}} \theta^{-1}(bux^{-1})\|x^{-1}bu\|_E^{-1/2}\\
&\qquad \cdot \psi\big(-\Delta vx\mathrm{N}_{E/F}\big(bux^{-1}\big)\big)\psi\,\mathrm{Tr}_{E/F}\big(x(bux^{-1})\big)\,du.
\end{aligned}$$

Here we note $\nu_F\big(\Delta x^{-1}\big) = 0$ and so $\nu_F\big(-\Delta v\Delta x^{-1}(\mathrm{N}_{E/F}u - 1)\big) \geq \big(1 - n/2 - [(d+1)/2]/2\big) + \big[(1+n+[d/2])/2\big] > 0$. Hence $\psi\big(-\Delta vx\mathrm{N}_{E/F}(bux^{-1})\big) = \psi(-\Delta v\Delta x^{-1})$

$\psi(-\Delta v\Delta x^{-1}(\mathrm{N}_{E/F}u-1)) = \psi(-\Delta^2 vx^{-1})$. Therefore we have

$$\begin{aligned}&\mathbf{W}\left(\begin{bmatrix} D & \\ & 1\end{bmatrix}\begin{bmatrix} 1 & -v/D \\ & 1\end{bmatrix}\begin{bmatrix} & 1 \\ -1 & \end{bmatrix}\begin{bmatrix} 1 & -\Delta v \\ & 1\end{bmatrix}\right)f_o(x)\\ &= \psi(-vx)\psi(-\Delta^2 vx^{-1})\psi\operatorname{Tr}_{E/F}\big((D-1)b\big)\gamma_{E/F}\eta_{E/F}(xb^{-1})\|xb^{-1}\|_E^{1/2}\\ &\quad\cdot\int_{\substack{\Delta x^{-1}\mathrm{N}_{E/F}u\in U_F^{[(n+1)/2]}\\ u\in U_E^{n-[d/2]}}}\theta^{-1}(u)\|u\|_E^{-1/2}\psi\operatorname{Tr}_{E/F}(bu)\,du.\end{aligned} \tag{2.8}$$

On the other hand, the equation (2.6) becomes

$$\begin{aligned}&\psi\operatorname{Tr}_{\bar E/F}\big(\bar b(\bar b v)\big)\mathbf{W}\left(\begin{bmatrix} & 1 \\ -1 & \end{bmatrix}\right)f_o(x)\\ &= \psi\operatorname{Tr}_{\bar E/F}\big(\bar b(\bar b v)\big)\gamma_{E/F}\eta_{E/F}(x)\|x\|_E^{1/2}\\ &\quad\cdot\int_{\substack{\Delta x^{-1}\mathrm{N}_{E/F}u\in U_F^{[(n+1)/2]}\\ u\in U_E^{n-[d/2]}}}\theta^{-1}(\delta)\|\delta\|_E^{-1/2}\psi\operatorname{Tr}_{E/F}(x\delta)\,d\delta.\end{aligned}$$

The change of variable δ to bux^{-1} gives

$$\begin{aligned}&\psi\operatorname{Tr}_{\bar E/F}\big(\bar b(\bar b v)\big)\mathbf{W}\left(\begin{bmatrix} & 1 \\ -1 & \end{bmatrix}\right)f_o(x)\\ &= \psi\operatorname{Tr}_{\bar E/F}\big(\bar b(\bar b v)\big)\gamma_{E/F}\eta_{E/F}(x)\|x\|_E^{1/2}\\ &\quad\cdot\int_{\substack{\Delta x^{-1}\mathrm{N}_{E/F}u\in U_F^{[(n+1)/2]}\\ u\in U_E^{n-[d/2]}}}\theta^{-1}(bux^{-1})\|bux^{-1}\|_E^{-1/2}\psi\operatorname{Tr}_{E/F}(xbux^{-1})\,du\\ &= \psi\operatorname{Tr}_{\bar E/F}\big(\bar b(\bar b v)\big)\gamma_{E/F}\eta_{E/F}(x)\|x\|_E^{1/2}\theta^{-1}(xb^{-1})\|xb^{-1}\|_E^{1/2}\\ &\quad\cdot\int_{\substack{\Delta x^{-1}\mathrm{N}_{E/F}u\in U_F^{[(n+1)/2]}\\ u\in U_E^{n-[d/2]}}}\theta^{-1}(u)\|u\|_E^{-1/2}\psi\operatorname{Tr}_{E/F}(bu)\,du.\end{aligned} \tag{2.9}$$

Now the proof is completed from (2.4) (2.8) and (2.9) if we show that

$$\psi(-vx)\psi\big(-\Delta^2 vx^{-1}\big)\psi\operatorname{Tr}_{E/F}\big((D-1)b\big) = \psi\operatorname{Tr}_{\bar E/F}\big(\bar b(\bar b v)\big), \tag{2.10}$$

when $\Delta x^{-1}\mathrm{N}_{E/F}u\in U_F^{[(n+1)/2]}$ and $u\in U_E^{n-[d/2]}$.

From $\mathrm{N}_{E/F}u \in U_F^{[(n+[d/2])/2]}$, we have $\Delta x^{-1}\in U_F^{[(n+1)/2]}$. Since $\nu_F(\Delta) = 1-2n$, $\nu_F(x) = 1-2n$ and $x-\Delta\in U_F^{1-2n+[(n+1)/2]}$. We have then $\nu_F\big(-vx^{-1}(x-\Delta)^2\big)\geq 3n/2-[d/2]/2+2n-1+2\big(1-2n+[(n+1)/2]\big) > 0$, $\nu_F(scv)\geq[(d+1)/2]-n+1-d+3n/2-[d/2]/2>0$ and $\nu_F(s\Delta v^2)\geq[(d+1)/2]-n+1-2n+3n-[d/2]>0$, hence the left side of the equation is

$$\begin{aligned}&\psi(-vx)\psi(-\Delta^2 vx^{-1})\psi\operatorname{Tr}_{E/F}\big((D-1)b\big)\\ &= \psi\big(-vx^{-1}(x-\Delta)^2\big)\psi(vx^{-1}2x\Delta)\psi\Big(\big((s+c)v+\Delta v^2\big)s\Big)\\ &= \psi(-2\Delta v+s^2v+scv+s\Delta v^2)\\ &= \psi(-2\Delta v+s^2 v).\end{aligned}$$

We note that $\bar{b}^2 = \begin{bmatrix} -\Delta & -\Delta(s+c) \\ s+c & -\Delta+(s+c)^2 \end{bmatrix}$, $\nu_F(2scv) > 0$ and $\nu_F(c^2v) = 2-2d+3n/2-[d/2]/2 > 0$, therefore the right side of the equation (2.10) is

$$\begin{aligned}\psi\,\mathrm{Tr}_{\bar{E}/F}(\bar{b}^2 v) &= \psi(-2\Delta v + s^2 v + 2scv + c^2 v)\\ &= \psi(-2\Delta v + s^2 v).\end{aligned}$$

This completes the proof. ∎

COROLLARY 2.11. *With notation as above, there exists a quasicharacter $\bar{\theta}_e$ of $\bar{E}^\times$ with the property $\bar{\theta}_e(k) = \psi\,\mathrm{Tr}_{\bar{E}/F}\,\bar{b}(k-1)$ for $k \in U_{\bar{E}}^{1+n-[(d_{E/F}+1)/2]}$ so that $\mathbf{W}(E,\theta)$ is equivalent with $\Pi(\mathcal{A}_n,\psi,\bar{b},\bar{\theta}_e)$.*

PROOF. The map $k \mapsto \psi\,\mathrm{Tr}_{A/F}\,\bar{b}(k-1)$ is a character of $U_{\bar{E}}^{1+n-[(d_{E/F}+1)/2]}\mathcal{U}^n_{\mathcal{A}_n}$ by Proposition 1.9. Using Proposition 1.5, we apply the proof of Corollary 2.4 in [K3] here. ∎

We denote by $\Pi(\mathcal{A}_n,\psi,\bar{b},\bar{\theta}_e)$ a supercuspidal representation as in (1.9) with additional property that $\bar{\theta}_e(k) = \psi\,\mathrm{Tr}_{\bar{E}/F}\,\bar{b}(k-1)$ for $k \in U_{\bar{E}}^{1+n-[(d+1)/2]}$ where $d = \min(d_{\bar{E}/F}, 2(n+1)/3)$. Since $d \geq (\mathfrak{f}(\bar{\theta}_e)+1)/2$, every irreducible ramified supercuspidal representation is of the form $\Pi(\mathcal{A}_n,\psi,\bar{b},\bar{\theta}_e)$.

The generic element $\bar{b}$ is constructed from the element b which is a dual blob of θ with respect to $\psi\,\mathrm{Tr}_{E/F}$. Since the conductor of θ is $1+2n-[d_{E/F}/2]$, b is unique modulo $P_E^{1-n-[d_{E/F}/2]}$, or $\mathrm{Tr}_{E/F}\,b$ is unique modulo $P_F^{[(1-n+[(d_{E/F}+1)/2])/2]}$ and $\mathrm{N}_{E/F}b$ is unique modulo $P_F^{1-n-[d_{E/F}/2]}$. We define the following, which extends the definition in Section 3, [K3].

DEFINITION 2.12. An $\mathcal{A}_n$-generic element $\bar{b}$ of level $1-2n$ is called *Weil generic* if there exists a quadratic ramified extension E/F with $2(n+1) \geq 3d_{E/F}$ and an element b in E with $\nu_E(b) = 1-2n$ such that

$$\text{(2.13)}\qquad \begin{aligned}&\text{(i) } \mathrm{tr}\,\bar{b} \equiv \mathrm{Tr}_{E/F}\,b + c_{\psi,E/F} \pmod{P_F^{[(1-n+[(d_{E/F}+1)/2])/2]}},\\ &\text{(ii) } \det\bar{b} \equiv N_{E/F}b \pmod{P_F^{1-n-[d_{E/F}/2]}}.\end{aligned}$$

PROPOSITION 2.14. *Let $\Pi = \Pi(\mathcal{A}_n,\psi,\bar{b},\bar{\theta}_e)$ be a supercuspidal representation. Then Π is a Weil representation if and only if $\bar{b}$ is Weil-generic.*

We need several lemmas to prove the proposition.

LEMMA 2.15. *Suppose that the pair (E,b) satisfies the condition (2.13). Suppose that E_1/F is a ramified quadratic extension and for some element b_1 in E_1 $\mathrm{Tr}_{E_1/F}\,b_1 \equiv \mathrm{Tr}_{E/F}\,b \pmod{P_F^{[(1-n+[(d_{E/F}+1)/2])/2]}}$ and $N_{E_1/F}b_1 \equiv N_{E/F}b \pmod{P_F^{1-n-[d_{E/F}/2]}}$. Then the pair (E_1,b_1) also satisfies the conditions (2.13).*

PROOF. It is enough to show that $c_{E_1/F} \equiv c_{E/F} \pmod{P_F^{[(1-n+[(d_{E/F}+1)/2])/2]}}$. The exponents of two differences are determined as $d_{E_1/F} =$

$\min\Big(2\big(\nu(\mathrm{Tr}_{E_1/F}\, b_1)+n\big), 2\nu_F(2)+1\Big)$ and $d_{E/F} = \min\Big(2\big(\nu(\mathrm{Tr}_{E/F}\, b)+n\big), 2\nu_F(2)+1\Big)$. From the congruent relation $\mathrm{Tr}_{E_1/F}\, b_1 \equiv \mathrm{Tr}_{E/F}\, b \pmod{P_F^{[(1-n+[(d_{E/F}+1)/2])/2]}}$, if $\nu_F(\mathrm{Tr}_{E/F}\, b) \geq \big[\big(1-n+[(d_{E/F}+1)/2]\big)/2\big]$, then from $2(n+1) \geq 3d_{E/F}$ we have $2\big(\nu(\mathrm{Tr}_{E/F}\, b_1)+n\big) > d_{E/F}$, hence $d_{E/F} = 2\nu_F(2)+1$ and also $d_{E_1/F} = 2\nu_F(2)+1 = d_{E/F}$. If $\nu_F(\mathrm{Tr}_{E/F}\, b) < \big[\big(1-n+[(d_{E/F}+1)/2]\big)/2\big]$, then $\nu_F(\mathrm{Tr}_{E_1/F}\, b_1) = \nu_F(\mathrm{Tr}_{E/F}\, b)$, hence $d_{E_1/F} = d_{E/F}$ always.

Again from $2(n+1) \geq 3d_{E/F}$ a simple computation show that $\big[\big(1-n+[(d_{E/F}+1)/2]\big)/2\big] \leq 1-[(d_{E/F}+1)/2]$. Therefore from the duality relation, in other to show that $c_{E_1/F} \equiv c_{E/F} \pmod{P_F^{[(1-n+[(d_{E/F}+1)/2])/2]}}$, we only need to check that $\eta_{E_1/F} \equiv \eta_{E/F}$ on $U_F^{1-[(1-n+[(d_{E/F}+1)/2])/2]}$. The character $\eta_{E/F}$ under the restriction to $U_F^{1-[(1-n+[(d_{E/F}+1)/2])/2]}$ is completely determined by the data $\mathfrak{f}(\eta_{E/F}) = d_{E/F}$, $\eta_{E/F}^2 = 1$ and $\eta_{E/F}(1 + x\,\mathrm{Tr}_{E/F}\, b + x^2\mathrm{N}_{E/F} b) = 1$ for x in F with $2\nu_F(x) \geq 2n - \big[\big(1-n+[(d_{E/F}+1)/2]\big)/2\big]$. Since $\nu_F(x\,\mathrm{Tr}_{E_1/F}\, b_1 - x\,\mathrm{Tr}_{E/F}\, b) \geq \Big(2n - \big[\big(1-n+[(d_{E/F}+1)/2]\big)/2\big]\Big)/2 + \big[\big(1-n+[(d_{E/F}+1)/2]\big)/2\big] > d_{E/F}$ and $\nu_F(x^2\mathrm{N}_{E_1/F} b_1 - x^2\mathrm{N}_{E/F} b) \geq 2n - \big[\big(1-n+[(d_{E/F}+1)/2]\big)/2\big] + 1 - n - [d_{E/F}/2] > d_{E/F}$, $\eta_{E_1/F}$ satisfies the same data hence we are done. ∎

Let E/F be quadratic ramified with $3d_{E/F} \leq 2(n+1)$ and b an element of E with $\nu_E(b) = 1-2n$. We denote by $\mathbf{W}(E,b)$ the set of equivalent classes of representations $\mathbf{W}(E,\theta)$ where θ is a character of $E^\times$ such that θ has a dual blob b and $\eta_{E/F}(\pi_F)\theta(\pi_F) = 1$ for some fixed prime element π_F in F, so that we fix the central character of $\mathbf{W}(E,\theta)$ which is trivial at π_F, see (1.1).

LEMMA 2.16. *The set $\mathbf{W}(E,b)$ consists of $(q-1)q^{n-[(d_{E/F}+1)/2]}$ distinct representations if $2(n+1) > 3d_{E/F}$ and $(q-1)q^{n-[(d_{E/F}+1)/2]}/2$ distinct representations if $2(n+1) = 3d_{E/F}$.*

PROOF. The number of choice for θ is $[U_E : U_E^{1+n-[(d_{E/F}+1)/2]}] = (q-1)q^{n-[(d_{E/F}+1)/2]}$. Let τ be the nontrivial element of the Galois group of E/F. Then $\theta = \theta^\tau$ if and only if $b \equiv b^\tau \pmod{P_E^{1-n-[d_{E/F}/2]}}$ if and only if $\nu_E(b - b^\tau) \geq 1-n-[d_{E/F}/2]$ if and only if $-2n + d_{E/F} \geq 1-n-[d_{E/F}/2]$ if and only if $3d_{E/F} \geq 2(n+1)$. When $3d_{E/F} = 2(n+1)$, the number of distinct representations in $\mathbf{W}(E,b)$ is $(q-1)q^{n-[(d_{E/F}+1)/2]}/2$ by Proposition (1.2) (3). ∎

LEMMA 2.17. *Suppose that there are two pairs (E_1,b_1) and (E_2,b_2) which satisfy the condition (2.13). If either $\mathrm{Tr}_{E_1/F}\, b_1 \not\equiv \mathrm{Tr}_{E_2/F}\, b_2 \pmod{\mathrm{Tr}_{E/F} P_E^{1-n-[d_{E/F}/2]}}$ or $N_{E_1/F} b_1 \not\equiv N_{E_2/F} b_2 \pmod{N_{E/F} P_E^{1-n-[d_{E/F}/2]}}$, then the two sets $\mathbf{W}(E_1,b_1)$ and $\mathbf{W}(E_2,b_2)$ are disjoint.*

PROOF. See Lemma 3.4 in [K3]. ∎

PROOF OF PROPOSITION 2.14. Suppose $\bar{b}$ is Weil-generic. Denote by $\mathbf{W}(\bar{b})$ the union of $\mathbf{W}(E,b)$'s such that b satisfies the condition (2.13). It is easy to check that

$$[P_F^{[(1-n+[(d_{E/F}+1)/2])/2]} \times P_F^{1-n-[d_{E/F}/2]} : \mathrm{Tr}_{E/F} P_E^{1-n-[d_{E/F}/2]} \times N_{E/F} P_E^{1-n-[d_{E/F}/2]}]$$

is equal to 1 if $2(n+1) > 3d_{E/F}$ and is equal to 2 if $2(n+1) = 3d_{E/F}$. From Lemma 2.16 and Lemma 2.17, the cardinality of $\mathbf{W}(\bar{b})$ is $(q-1)q^{n-[(d_{E/F}+1)/2]}$. Let $\Pi(\mathcal{A}_n, \psi, \bar{b})$ be the set of equivalent classes of representations of the form $\Pi(\mathcal{A}_n, \psi, \bar{b}, \bar{\theta}_e)$ such that $\Pi(\mathcal{A}_n, \psi, \bar{b}, \bar{\theta}_e)$ has a central character which is trivial on π_F. From Corollary 2.11, the set $\mathbf{W}(\bar{b})$ is contained in $\Pi(\mathcal{A}_n, \psi, \bar{b})$. But from Proposition 1.9, (2), the cardinality of $\Pi(\mathcal{A}_n, \psi, \bar{b})$ is $[U_{\bar{E}} : U_{\bar{E}}^{1+n-[(d_{E/F}+1)/2]}] = (q-1)q^{n-[(d_{E/F}+1)/2]}$, that is, two sets are equal. Hence we are done. ∎

We summarize the conclusion of this section.

THEOREM 2.18. *Suppose that $\Pi(\mathcal{A}_n, \psi, \bar{b}_w, \bar{\theta}_e)$ is an irreducible Weil ramified supercuspidal representation with Weil-generic $\bar{b}_w$. Let E be a ramified quadratic extension of F and b an element of E with relation (2.13). Then there exists a quasicharacter θ of $E^\times$ such that the dual blob of θ is b and the representation $\mathbf{W}(E,\theta)$ is equivalent to $\Pi(\mathcal{A}_n, \psi, \bar{b}_w, \bar{\theta}_e)$.*

3. **Preliminary results.** Let F be a 2-adic field and L a Galois totally ramified field extension of F with (2,2)-Galois type, *i.e.*, $\Gamma_{L/F} \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. Let $d_{L/F}$ be the exponent of the different of L/F. Then from Galois theory there are three intermediate fields of L/F, say E_1, E_2 and E_3. Each one is a quadratic extension of F.

PROPOSITION 3.1. *We fix E_3 such that $d_{L/E_3} \geq d_{L/E_1}$ and $d_{L/E_3} \geq d_{L/E_2}$. Then we have the following:*

(1) $d_{L/E_1} = d_{L/E_2} = d_{E_3/F}$.
(2) $d_{E_1/F} = d_{E_2/F}$.
(3) $d_{L/E_3} = 2d_{E_1/F} - d_{L/E_1}$.
(4) $d_{L/E_1}, d_{L/E_2}, d_{L/E_3}, d_{E_3/F}$ and $d_{L/F}$ are all even numbers.
(5) $d_{E_1/F} = 2, 4, \ldots, 2\nu_F(2)$ or $2\nu_F(2) + 1$.

PROOF. Since $\Gamma_{L/F}$ is finite abelian, by the Hasse-Arf theorem, every jump occurs at an integer in the upper filtration of the Galois group, see [S, IV, Section 3], so say $\Gamma_{L/F}^{s-1} \supsetneq \Gamma_{L/F}^{s-1+}$ and $\Gamma_{L/F}^{s-1+t} \supsetneq \Gamma_{L/F}^{s-1+t+}$, where s and t are integers. The integer s is greater than or equal to 2 because the extension L/F is wildly ramified and the integer t is greater than or equal to 0. Let K_3 be the intermediate field that corresponds to $\Gamma_{L/F}^{s-1+t}$, *i.e.*, $\Gamma_{L/F}^{s-1+t} = \Gamma_{L/K_3}$. Since $\Gamma_{L/F}^{s-1+t} = \Gamma_{L/F,s-1+2t} = \Gamma_{L/K_3,s-1+2t}$, and $\Gamma_{L/K_3,s+2t} = 1$, we have $d_{L/K_3} = s + 2t$, see [S, IV,Section 3]. Suppose that an element σ of $\Gamma_{L/F}$ is not in $\Gamma_{L/F}^{s-1+t}$. Then $\{1, \sigma\}$ is a subgroup of $\Gamma_{L/F}$, with the corresponding subfield, say K_1. Since $\sigma \notin \Gamma_{L/F}^{s-1+}$, $\Gamma_{L/K_1}^{s-1} \supsetneq \Gamma_{L/K_1}^{s} = 1$, therefore $d_{L/K_1} = s$. The other subgroup, say Γ_{L/K_2}, of $\Gamma_{L/F}$ has the same property as Γ_{L/K_1}, so $d_{L/K_2} = s$. Here d_{L/K_3} is the largest. Hence in terms of E_i we have that $d_{L/E_1} = s$, $d_{L/E_2} = s$, $d_{L/E_3} = s + 2t$.

From properties of subgroups of a Galois group, $\Gamma_{E_3/F} \cong \Gamma_{L/F}/\Gamma_{L/E_3} \cong \{1,\sigma\}$, and $\Gamma^{s-1}_{E_3/F} \supsetneq \Gamma^{s}_{E_3/F} = 1$, we have that $d_{E_3/F} = s$. It is known that $d_{L/F} = d_{L/E_i} + 2d_{E_i/F}$ for each i, because the ramification index of E_i/F is 2, so that $d_{L/F} = 3s + 2t$, and $d_{E_1/F} = d_{E_2/F} = s+t$. This proves (1), (2), (3) and (4). The property (5) is well known. ∎

PROPOSITION 3.2. *Let L be a quadratic ramified extension of E. Let ψ_E be an additive character of E with conductor $\mathfrak{f}(\psi_E)$, and let θ be a multiplicative quasicharacter of $E^\times$ with conductor $\mathfrak{f}(\theta)$. Let b be a dual blob of θ with respect to ψ_E. Then*

(1) $\psi_E \operatorname{Tr}_{L/E}$ is an additive character of E with conductor $2\mathfrak{f}(\psi_E)-d_{L/E}$, and $\theta\circ \mathrm{N}_{L/E}$ is a multiplicative quasicharacter of $L^\times$. The conductor $\mathfrak{f}(\theta\circ\mathrm{N}_{L/E})$ of $\theta\circ\mathrm{N}_{L/E}$ is $2\mathfrak{f}(\theta)-d_{L/E}$ if $\mathfrak{f}(\theta) > d_{L/E}$, $\mathfrak{f}(\theta)$ if $\mathfrak{f}(\theta) < d_{L/E}$ and less than or equal to $\mathfrak{f}(\theta)$ if $\mathfrak{f}(\theta) = d_{L/E}$.

(2) Assume that $\mathfrak{f}(\theta)$ is not equal to $d_{L/E}$. There is an element β in L such that $b + \beta$ is a dual blob of $\theta \circ \mathrm{N}_{L/E}$ with respect to $\psi_E \operatorname{Tr}_{L/E}$ and $\nu_L(\beta) = 2\mathfrak{f}(\psi_E) - d_{L/E} - \mathfrak{f}(\theta)$.

PROOF. (1) It is known that $\operatorname{Tr}_{L/E}(P_L^t) = P_E^{[(t+d_{L/E})/2]}$ for every integer t, see [S]. So $\psi_E \operatorname{Tr}_{L/E}(P_L^{2\mathfrak{f}(\psi_E)-d_{L/E}}) = \psi_E(P_E^{\mathfrak{f}(\psi_E)}) = 1$ and $\psi_E \operatorname{Tr}_{L/E}(P_L^{2\mathfrak{f}(\psi_E)-d_{L/E}-1}) = \psi_E(P_E^{\mathfrak{f}(\psi_E)-1}) \neq 1$. Hence $\mathfrak{f}(\psi_E \operatorname{Tr}_{L/E}) = 2\mathfrak{f}(\psi_E) - d_{L/E}$. From [S, V, Section 3], $\mathrm{N}_{L/E}(U_L^t) = U_E^{[(t+d_{L/E})/2]}$ if $t \geq d_{L/E}$ and $\mathrm{N}_{L/E}(U_L^t) = U_E^t$ if $t < d_{L/E} - 1$ and $\mathrm{N}_{L/E}(U_L^{d_{L/E}-1}) \subset U_E^{d_{L/E}-1}$. With a similar argument we have the result of the second part.

(2) For an element $x \in P_L^{[(\mathfrak{f}(\theta\circ\mathrm{N}_{L/E})+1)/2]}$, $\mathrm{N}_{L/E}(1 + x) = 1 + \operatorname{Tr}_{L/E}(x) + \mathrm{N}_{L/E}(x) \in U_E^{[(\mathfrak{f}(\theta)+1)/2]}$. Therefore

$$\theta \circ \mathrm{N}_{L/E}(1 + x) = \psi_E b\big(\mathrm{N}_{L/E}(1 + x) - 1\big) = \big(\psi_E b \operatorname{Tr}_{L/E}(x)\big)\big(\psi_E b \mathrm{N}_{L/E}(x)\big).$$

The map $x \longmapsto \psi_E b\mathrm{N}_{L/E}(x)$ is an additive character on $P_L^{[(\mathfrak{f}(\theta\circ\mathrm{N}_{L/E})+1)/2]}$, because

$$\begin{aligned}\psi_E b\mathrm{N}_{L/E}(x + y) &= \psi_E b\big(\mathrm{N}_{L/E}(x) + \mathrm{N}_{L/E}(y) + \operatorname{Tr}_{L/E}(xy^\tau)\big)\\ &= \big(\psi_E b\mathrm{N}_{L/E}(x)\big)\big(\psi_E b\mathrm{N}_{L/E}(y)\big),\end{aligned}$$

where $\Gamma_{L/E} = \{1,\tau\}$, noting that $\operatorname{Tr}_{L/E}(xy^\tau) \in P_E^{\mathfrak{f}(\theta)}$. It is known that $\mathrm{N}_{L/E}(P_L^t/P_L^{t+1}) = P_E^t/P_E^{t+1}$ as a set, and so the conductor of this additive character is $\mathfrak{f}(\theta)$. By topological duality, there is an element β in L with $\nu_L(\beta) = 2\mathfrak{f}(\psi_E) - d_{L/E} - \mathfrak{f}(\theta)$ such that $\psi_E \operatorname{Tr}_{L/E}(\beta x) = \psi_E b\mathrm{N}_{L/E}(x)$ for $x \in P_L^{[(\mathfrak{f}(\theta\circ\mathrm{N}_{L/E})+1)/2]}$. Therefore we have that

$$\theta \circ \mathrm{N}_{L/E}(1 + x) = \psi_E \operatorname{Tr}_{L/E}\big((b + \beta)x\big) \quad \text{for } x \in P_L^{[(\mathfrak{f}(\theta\circ\mathrm{N}_{L/E})+1)/2]}.$$ ∎

Let F be a 2-adic field and let E and K be two different quadratic ramified Galois extensions of F. Suppose that the Galois group $\Gamma_{K/F}$ of K over F is $\{1,\tau\}$, and the Galois group $\Gamma_{E/F}$ of E over F is $\{1,\sigma\}$. Then the field KE is a $(2,2)$-type Galois extension of F with the Galois group $\{1,\tau,\sigma,\tau\sigma\}$.

PROPOSITION 3.3. *Let b be an element of E with $\nu_E(b) = 1-2n$ and write b as a sum of $a\pi_{KE}$ and h where a and h are elements of K. Then $\nu_{KE}(a\pi_{KE}) = 1-4n-d_{KE/K}+2d_{E/F}$ and $\nu_{KE}(h) = 2-4n$.*

PROOF. Since $\nu_E(b) = 1-2n, \nu_{KE}(b) = 2-4n$ which is an even number so $\nu_{KE}(h) < \nu_{KE}(a\pi_{KE})$ and $\nu_{KE}(h) = 2-4n$. We have $b = a\pi_{KE} + h$, so $b^\tau = a\pi_{KE}^\tau + h$. Therefore $\nu_{KE}(b^\tau - b) = \nu_{KE}\big(a(\pi_{KE}^\tau - \pi_{KE})\big) = \nu_{KE}(a) + d_{KE/K}$. We may write $b = x\pi_E + y$, where x and y are in F. Then $b^\tau = x\pi_E^\tau + y$, hence $\nu_E(b^\tau - b) = \nu_E\big(x(\pi_E^\tau - \pi_E)\big) = \nu_E(x) + d_{E/F} = -2n + d_{E/F}$. Comparing two equations, we have $\nu_{KE}(a) = -4n - d_{KE/K} + 2d_{E/F}$. ∎

We will study the element h in K and a quasicharacter of $K^\times$ derived from h. We have $\nu_K(h) = 1-2n$ and will assume $1+2n \geq d_{KE/E} + d_{E/F}$, which is the only case we need later. There is a quasicharacter ϱ of $K^\times$ whose dual blob is h with respect to $\psi\,\mathrm{Tr}_{K/F}$, so that the conductor of ϱ is $\mathfrak{f}(\varrho) = 1+2n-d_{K/F}$ and $\varrho(k) = \psi\,\mathrm{Tr}_{K/F}\big(h(k-1)\big)$ for $k \in U_K^{1+n-[(d_{K/F}+1)/2]}$. Then we have the following:

PROPOSITION 3.4. *The quasicharacter $\varrho \circ N_{KE/K}$ of $KE^\times$ has the conductor $2+4n-d_{KE/F}$. There is an element δ in KE such that $h+\delta$ is a dual blob of $\varrho \circ N_{KE/K}$ with respect to $\psi\,\mathrm{Tr}_{KE/F}$ and $\nu_{KE}(\delta) = 3-2n-d_{KE/K}-d_{K/F}$.*

PROOF. First show that $\mathfrak{f}(\varrho) > d_{KE/K}$, *i.e.*, $1+2n-d_{K/F}-d_{KE/K} > 0$. There are three cases for the values $d_{K/F}$, $d_{E/F}$ by (3.1) as follows: i) $d_{K/F} = s+t$, $d_{E/F} = s$; ii) $d_{K/F} = s$, $d_{E/F} = s+t$; and iii) $d_{K/F} = s+t$, $d_{E/F} = s+t$, where s is a positive even integer and t is a nonnegative integer. Then we have respectively in case i) $d_{KE/K} = s$, $d_{KE/E} = s+2t$; in case ii) $d_{KE/K} = s+2t$, $d_{KE/E} = s$; and in case iii) $d_{KE/K} = s$, $d_{KE/E} = s$. Using the assumption $1+2n-d_{KE/E}-d_{E/F} \geq 0$, and comparing case by case, we have $\mathfrak{f}(\varrho) > d_{KE/K}$. Therefore, by the Proposition 3.3, the results follow immediately, *i.e.*, $\mathfrak{f}(\varrho \circ \mathrm{N}_{KE/K}) = 2\mathfrak{f}(\varrho) - d_{KE/K} = 2+4n-d_{KE/F}$, and $\nu_{KE}(\delta) = 2\mathfrak{f}(\psi\,\mathrm{Tr}_{K/F}) - d_{KE/K} - \mathfrak{f}(\varrho \circ \mathrm{N}_{KE/K}) = 3-2n-d_{KE/K}-d_{K/F}$. ∎

4. **Lifting of supercuspidal representations of** GL_2. It is a conjecture of Langlands that there should be a natural bijection between the set $\mathcal{S}\big(\mathrm{GL}_N(F)\big)$ of equivalent classes of irreducible admissible representations of $\mathrm{GL}_N(F)$ and the set $\mathcal{S}_N(W_F)$ of equivalent classes of N-dimensional semisimple Deligne representations of W_F. Bernstein and Zelevinsky [Z] had shown that we may restrict our attention to the set $\mathcal{S}^o\big(\mathrm{GL}_N(F)\big)$ of equivalent classes of irreducible supercuspidal representations of $\mathrm{GL}_N(F)$ and the set $\mathcal{S}_N^o(W_F)$ of equivalent classes of N-dimensional irreducible continuous representations of W_F. The bijection Φ should satisfy the following conditions:

(i) $\Phi(\Pi \otimes \chi \circ \det) = \Phi(\Pi) \otimes \chi$, for every quasicharacter χ of $F^\times$,

(ii) $\omega_\Pi = \det \Phi(\Pi)$, where ω_Π is the central quasicharacter of Π,

(iii) $L\big(\Phi(\Pi)\big) = L(\Pi)$, $\epsilon\big(\Phi(\Pi)\big) = \epsilon(\Pi)$.

For the definition of L-functions and ϵ-factors, see [D1], [GJ]. Suppose K is a field extension of F. From the conjectural correspondence, there is a map called a *lifting* $\mathfrak{L}_{K/F}$

from $\mathcal{S}\big(\mathrm{GL}_N(F)\big)$ to $\mathcal{S}\big(\mathrm{GL}_N(K)\big)$ so that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{S}\big(\mathrm{GL}_N(K)\big) & \xrightarrow{\text{Langlands correspondence}} & \mathcal{S}_N(W_K) \\ \text{lifting}\uparrow & & \uparrow\text{restriction} \\ \mathcal{S}\big(\mathrm{GL}_N(F)\big) & \xrightarrow{\text{Langlands correspondence}} & \mathcal{S}_N(W_F) \end{array}$$

We will collect properties on the correspondence, restriction and lifting, for reference see [JL], [L], or for a convenient summary see [GL].

PROPOSITION 4.1. *Given a Weil representation* $\mathbf{W}(E,\theta)$ *of* $\mathrm{GL}_2(F)$*, the Langlands correspondence gives that*

$$\Phi\big(\mathbf{W}(E,\theta)\big) = \mathrm{Ind}_{W_E}^{W_F}\theta.$$

For the one dimensional case we have that:

PROPOSITION 4.2. *If* χ *is a quasicharacter of* $\mathrm{GL}_1(F)$*, then* $\mathfrak{L}_{K/F}(\chi) = \chi\circ \mathrm{N}_{K/F}$.

The restriction map of $\mathcal{S}_2(W_F)$ to $\mathcal{S}_2(W_K)$ gives the following:

PROPOSITION 4.3. *Let* K/F *be a quadratic extension. If a representation* σ *of* $\mathcal{S}_2(W_F)$ *is imprimitive, i.e.,* $\sigma = \mathrm{Ind}_{W_E}^{W_F}\theta$*, for some quadratic extension* E/F *and a quasicharacter* θ *of* $E^\times$*, then*

(1) $\mathrm{Res}_{W_K}^{W_F}(\mathrm{Ind}_{W_E}^{W_F}\theta) = \mathrm{Ind}_{W_{KE}}^{W_K}\theta\circ \mathrm{N}_{KE/E}$ *if* $E\neq K$*,*
(2) $\mathrm{Res}_{W_K}^{W_F}(\mathrm{Ind}_{W_E}^{W_F}\theta) = \theta\oplus\theta^\tau$ *if* $E = K$*,* $\Gamma_{K/F} = \langle\tau\rangle$*.*

PROPOSITION 4.4. *(1) Given a Weil representation* $\mathbf{W}(E,\theta)$ *of* $\mathrm{GL}_2(F)$ *and a quadratic extension* K/F *which is different from* E*,*

$$\mathfrak{L}_{K/F}\big(\mathbf{W}(E,\theta)\big) = \mathbf{W}(KE,\theta\circ \mathrm{N}_{KE/E}).$$

(2) Given an admissible representation Π *of* $\mathrm{GL}_2(F)$ *and a quasicharacter* χ *of* $F^\times$*,*

$$\mathfrak{L}_{K/F}(\Pi\otimes\chi\circ\det) = \mathfrak{L}_{K/F}(\Pi)\otimes\chi\circ N_{K/F}\circ\det.$$

It is known that when $K = E$, the $\mathfrak{L}_{K/F}\big(\mathbf{W}(E,\theta)\big)$ is not a supercuspidal representation. When $N = 2$, for the induced representation form, the lifting is known from the inducing data , see [K2], [P], except the case that F is 2-adic and K/F is wildly ramified. We characterize the generic element of the lifted representation in this remaining case.

Recall some notations before we prove the main theorem. If $\Pi(\mathcal{A}_n,\psi,\bar{b}_w,\bar{\theta}_e)$ is a ramified Weil supercuspidal representation, then it is equivalent to $\mathbf{W}(E,\theta)$ for some quadratic ramified extension E/F and a quasicharacter θ of $E^\times$ such that if b is a dual blob of θ in E then $\bar{b}_w$ and b satisfy the relation (2.13), by Theorem 2.18. Let K be a quadratic ramified extension of F which is different from E. Then there is an element β in KE so that the quasicharacter $\theta\circ \mathrm{N}_{KE/E}$ of $KE^\times$ has a dual blob $b+\beta$. We may write $b = a\pi_{KE}+h$ with

$a, h \in K$. Then there is a quasicharacter ϱ of $K^\times$ whose dual blob is h and there is an element δ in KE so that the quasicharacter $\varrho \circ N_{KE/K}$ of $KE^\times$ has a dual blob $h+\delta$. (See Section 3). We denote a principal order $\tilde{\mathfrak{A}}_n$ in $\mathrm{GL}_2(K)$ by setting that for every integer k

$$\tilde{\mathfrak{P}}_n^k = \begin{bmatrix} P_K^{[(k+1)/2]} & P_K^{1-n+[k/2]} \\ P_K^{n+[k/2]} & P_K^{[(k+1)/2]} \end{bmatrix}.$$

THEOREM 4.5. *Let F be a 2-adic field. Suppose that Π is an irreducible Weil supercuspidal ramified representation of $\mathrm{GL}_2(F)$ of the form $\Pi = \Pi(\mathfrak{A}_n, \psi, \bar{b}_w, \bar{\theta}_e) \otimes \chi \circ \det$ where $\bar{b}_w$ is a Weil generic element and $\bar{\theta}_e$ is a quasicharacter of $\bar{E}^\times = F[\bar{b}_w]^\times$ such that $\bar{\theta}_e = \psi_{\bar{b}_w}$ on $U_{\bar{E}}^{1+n-[(d_{E/F}+1)/2]}$. Let K be a quadratic ramified extension of F which is different from E. Assume $2(n+1) \neq 2d_{K/F} + d_{E/F}$.*

Then the lifting $\mathfrak{L}_{K/F}(\Pi)$ of Π is an irreducible Weil supercuspidal ramified representation of $\mathrm{GL}_2(K)$ of the form $\mathfrak{L}_{K/F}(\Pi) = \Pi(\tilde{\mathfrak{A}}_N, \tilde{\psi}, \bar{\tilde{b}}, \bar{\tilde{\theta}}) \otimes \tilde{\chi} \circ \det$ for some quasicharacter $\bar{\tilde{\theta}}$ of $K[\bar{\tilde{b}}]^\times$ such that:

(i) if $2(n+1) > 2d_{K/F} + d_{E/F}$, then

$$N = 1 + 2n - d_{KE/K}/2 - d_{K/F} - d_{E/F},$$
$$\tilde{\psi} = \psi \,\mathrm{Tr}_{K/F}\, \pi_K^{1-d_{K/F}},$$
$$\bar{\tilde{b}} = \begin{bmatrix} & -\mathrm{N}_{KE/K}\tilde{b} \\ 1 & \mathrm{Tr}_{KE/K}\tilde{b} + \tilde{c}_{\tilde{\psi}} \end{bmatrix}, \quad \text{where } \tilde{b} = \pi_K^{d_{K/F}-1}(a\pi_{KE} + \beta - \delta) \text{ and}$$
$$\tilde{\chi} = \varrho \cdot \chi \circ N_{K/F}.$$

(ii) if $2d_{K/F} + 1 < 2(n+1) < 2d_{K/F} + d_{E/F}$, then

$$N = n,$$
$$\tilde{\psi} = \psi \,\mathrm{Tr}_{K/F}\, \pi_K^{1-d_{K/F}},$$
$$\bar{\tilde{b}} = \begin{bmatrix} & -\mathrm{N}_{KE/K}\tilde{b} \\ 1 & \mathrm{Tr}_{KE/K}\tilde{b} + \tilde{c}_{\tilde{\psi}} \end{bmatrix}, \quad \text{where } \tilde{b} = \pi_K^{d_{K/F}-1}(a\pi_{KE} + \beta - \delta) \text{ and}$$
$$\tilde{\chi} = \varrho \cdot \chi \circ N_{K/F}.$$

(iii) if $2(n+1) \leq 2d_{K/F} + 1$, then

$$N = n, \tilde{\psi} = \psi \,\mathrm{Tr}_{K/F}\, \pi_K^{1-d_{K/F}},$$
$$\bar{\tilde{b}} = \begin{bmatrix} & -\mathrm{N}_{KE/K}\tilde{b} \\ 1 & \mathrm{Tr}_{KE/K}\tilde{b} + \tilde{c}_{\tilde{\psi}} \end{bmatrix}, \quad \text{where } \tilde{b} = \pi_K^{d_{K/F}-1}(b + \beta) \text{ and}$$
$$\tilde{\chi} = \chi \circ N_{K/F}.$$

PROOF. Let $\Pi = \Pi(\mathfrak{A}_n, \psi, \bar{b}_w, \bar{\theta}_e)$. From the assumption that $\bar{b}_w$ is Weil generic, there exists a ramified quadratic field E and an element b of E so that b and $\bar{b}_w$ satisfy the relation (2.13). Then Π is equivalent with $\mathbf{W}(E, \theta)$ for some quasicharacter θ of $E^\times$ whose dual blob is b, by Theorem 2.18. From the Proposition 4.4, the lifting of $\mathbf{W}(E, \theta)$ is that $\mathfrak{L}_{K/F}\big(\mathbf{W}(E, \theta)\big) = \mathbf{W}(KE, \theta \circ \mathrm{N}_{KE/E})$.

Consider the case (iii) first: Suppose $2(n+1) \leq 2d_{K/F}+1$. Since $2(n+1) \geq 3d_{E/F}$, $d_{E/F} < d_{KE/F}$, so $\mathfrak{f}(\theta) = 1+2n-d_{E/F} < 2d_{K/F}-d_{E/F} = d_{KE/E}$. By Proposition 3.2, $\mathfrak{f}(\theta \circ \mathrm{N}_{KE/E}) = \mathfrak{f}(\theta) = 1+2n-d_{KE/K}$. Therefore $\mathfrak{f}(\theta \circ \mathrm{N}_{KE/E}) \geq 2d_{KE/K}-1$ and $\mathfrak{f}(\theta \circ \mathrm{N}_{KE/E})+d_{E/F}$ is odd. These two conditions imply that $\theta \circ \mathrm{N}_{KE/E}$ is not of the form $\rho \circ N_{KE/K}$ for some quasicharacter ρ of $K^{\times}$. Because if $\theta \circ \mathrm{N}_{KE/E} = \rho \circ N_{KE/K}$, then $\mathfrak{f}(\theta \circ \mathrm{N}_{KE/E}) = \mathfrak{f}(\rho \circ N_{KE/K})$ which is bigger than $d_{KE/K}$. So $\mathfrak{f}(\rho \circ N_{KE/K}) = 2\mathfrak{f}(\rho)-d_{KE/K}$. We have $\mathfrak{f}(\theta \circ \mathrm{N}_{KE/E}) = 2\mathfrak{f}(\rho)-d_{KE/K}$ which contradicts the fact that $\mathfrak{f}(\theta \circ \mathrm{N}_{KE/E})+d_{KE/K}$ is odd. By Lemma 1.3 and Proposition 1.2, the representation $\mathbf{W}(KE, \theta \circ \mathrm{N}_{KE/E})$ is a supercuspidal Weil representation. By Proposition 3.2, the dual blob of $\theta \circ \mathrm{N}_{KE/E}$ is $b+\beta$ with respect to $\psi\,\mathrm{Tr}_{KE/F}$ or $\pi_K^{d_{K/F}-1}(b+\beta)$ with respect to $\psi\,\mathrm{Tr}_{K/F}\,\pi_K^{1-d_{K/F}}\,\mathrm{Tr}_{KE/K}$. Here the conductor $\mathfrak{f}(\tilde{\psi}) = 1$ where $\tilde{\psi} = \psi\,\mathrm{Tr}_{K/F}\,\pi_K^{1-d_{K/F}}$. Since $\nu_{KE}(\pi_K^{d_{K/F}-1}(b+\beta)) = \nu_{KE}(\pi_K^{d_{K/F}-1}\beta) = 1-2n$, we may apply the Proposition 2.1, and we are done.

In case (i), we have $2(n+1) > 2d_{K/F}+d_{E/F}$. Write $b = a\pi_{KE}+h$ where a and h are elements of K. There exists a quasicharacter ϱ of $K^{\times}$ whose dual blob is h. Then the quasicharacter $\varrho \circ \mathrm{N}_{KE/E}$ has a dual blob $h+\delta$ for some δ in KE. (See Proposition 3.4). We have $\mathfrak{f}(\theta \circ \mathrm{N}_{KE/E}) = 2\mathfrak{f}(\theta)-d_{KE/E}$, and write $\theta \circ \mathrm{N}_{KE/E} = \tilde{\theta}\cdot\varrho \circ \mathrm{N}_{KE/E}$ where $\tilde{\theta} = \theta \circ \mathrm{N}_{KE/E}\cdot(\varrho \circ \mathrm{N}_{KE/E})^{-1}$. Since $b+\beta-(h+\delta) = a\pi_{KE}+\beta-\delta$, we have that $\tilde{\theta}(k) = \psi\,\mathrm{Tr}_{KE/F}(a\pi_{KE}+\beta-\delta)(k-1) = \psi\,\mathrm{Tr}_{K/F}\,\pi_K^{1-d_{K/F}}\,\mathrm{Tr}_{KE/K}\,\pi_K^{d_{K/F}-1}(a\pi_{KE}+\beta-\delta)(k-1)$ for $k \in U_{KE}^{\mathfrak{f}(\theta)-d_{KE/E}/2}$. Let $\tilde{b} = \pi_K^{d_{K/F}-1}(a\pi_{KE}+\beta-\delta)$. We may check that in this case $1+2n \geq d_{KE/E}+d_{E/F}$. Hence $\nu_{KE}(\tilde{b}) = \nu_{KE}(\pi_K^{d_{K/F}-1}a\pi_{KE}) = -1-4n-d_{KE/K}+2d_{K/E}+2d_{E/F}$. Now $\mathfrak{f}(\tilde{\theta}) = 2-d_{KE/K}-\nu_{KE}(\tilde{b}) = 3+4n+2d_{K/F}+2d_{E/F}$. We have $\mathfrak{f}(\tilde{\theta}) \geq 2d_{KE/K}-1$ and $\mathfrak{f}(\tilde{\theta})+d_{KE/K}$ is odd. Therefore $\mathbf{W}(KE,\tilde{\theta})$ is a Weil supercuspidal representation, and the generic element $\tilde{\tilde{b}} = \begin{bmatrix} & -\mathrm{N}_{KE/K}\tilde{b} \\ 1 & \mathrm{Tr}_{KE/K}\,\tilde{b}+\tilde{c}_{\tilde{\psi}} \end{bmatrix}$ is of level $1-2N$ where $N = 1+2n-d_{KE/K}/2-d_{K/F}-d_{E/F}$. This completes case (i).

Now case (ii): Suppose $2d_{K/F}+1 < 2(n+1) < 2d_{K/F}+d_{E/F}$. From $2(n+1) \geq 3d_{E/F}$, we have $d_{E/F} < d_{K/F}$. Hence by Proposition 3.1 $2d_{K/F} = d_{KE/E}+d_{E/F}$, so $1+2n \geq d_{KE/E}+d_{E/F}$. Therefore we take the same $\tilde{b}$, so $\tilde{\theta}(k) = \psi\,\mathrm{Tr}_{K/F}\,\pi_K^{1-d_{K/F}}\,\mathrm{Tr}_{KE/K}\,\tilde{b}(k-1)$ for $k \in U_{KE}^{\mathfrak{f}(\theta)-d_{KE/E}/2}$, and we have $\nu_{KE}(\tilde{b}) = \nu_{KE}(\pi_K^{d_{K/F}-1}\beta) = 1-2n+2d_{K/F}-d_{KE/E}-d_{E/F} = 1-2n$. Therefore $\mathfrak{f}(\tilde{\theta}) = 1+2n-d_{KE/K}$. We may check that $\mathfrak{f}(\tilde{\theta}) \geq 2d_{KE/K}-1$ and $\mathfrak{f}(\tilde{\theta})+d_{KE/K}$ is odd using $d_{KE/K} = d_{E/F}$. Hence $\tilde{b}$ produces the generic element $\tilde{\tilde{b}}$ of level $1-2N$ where $N = n$ in this case. This completes the proof of the theorem. ∎

In the case $2(n+1) = 2d_{K/F}+d_{E/F}$, $\mathfrak{f}(\tilde{\theta})$ could vary from 0 to $1+2n-d_{KE/K}$. Hence the lifted representation may be not supercuspidal. We need further information to analyze the lifted representation, for example the relation between θ and $\tilde{\theta}$.

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