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SOME CRITERIA FOR THE REGULARITY OF A DIRECT PRODUCT OF REGULAR p-GROUPS

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Abstract

A sufficient condition, which may also be necessary, for the regularity of the direct product of two finite regular p-groups is obtained. In an attempt to decide the necessity of the condition, we look at two related questions. Firstly, does the regularity of the direct square of a regular p-group imply the regularity of all the finite direct powers; secondly, is the join of two regular varieties regular? Finally, the results obtained are used to give some precise answers in special cases.

0. Introduction

One of the aspects of finite regular p-groups which has not commanded a great deal of attention in the literature, but which appears to be of importance for a full understanding of the subject is the question of the regularity of the direct product of two regular groups. In 1939, just six years after the introduction of the concept of regularity by Hall (1934), Wielandt (see Huppert (1967; III, 10.3c)) gave an example of a regular 3-group with an irregular direct square. On the other hand, Grün (1954) showed that if a group is p-abelian (for definitions see section 1.1) then its direct product with any regular p-group is again regular; in Groves (1973) the converse of this is proved—that if some regular p-group has regular direct product with every other regular p-group, then that group is p-abelian. There are also a number of results concerning varieties and regular p-groups (see, for example, Groves (1973a) or Weischsel (1967)) which are relevant to the present topic insofar as a variety of p-groups in which every finite group is regular has the property that regularity is preserved by direct products within the variety.

The aim of this paper is to investigate various problems based on the question of deciding when the direct product of two regular p-groups is again regular. In Section 1, we introduce notation and give a complete statement and discussion of the main results; Sections 2–5 are devoted largely to proofs. Section 2 concerns the general question of direct products. In Sections 3 and 4, we look at the questions, arising from Section 2, of when the regularity of a direct square implies the regularity of all the finite direct powers (V-regularity) and when the V-regularity of two groups implies the regularity of their direct product. Finally, Section 5 is concerned with applying the results of the previous three sections to some specific cases of direct products of regular p-groups.

1. Preliminaries and statement of results

1.1. PRELIMINARIES. *Throughout this paper, p will denote a fixed odd prime number. All groups mentioned are finite p-groups and all varieties are locally finite varieties of p-groups*, except where it is clearly indicated to the contrary. A finite p-group G is said to be *regular* if, for all $g, h \in G$, there exist commutators $d_1, \cdots, d_n$ in g and h such that $(gh)^p = g^p h^p d_1^p \cdots d_n^p$. A p-group is *p-abelian* if it is regular and if commutators have order dividing p. A variety of p-groups is regular if each finite group in the variety is regular. A finite p-group is V-regular if it is regular and all its direct powers are regular or, alternatively, if the variety it generates is regular. For basic facts concerning regular groups we refer to Huppert (1967; III).

Where varieties of groups are discussed, we refer to Neumann (1967) for notation and basic results. We differ however, in using $\underline{\underline{V}}(G)$ to denote the verbal subgroup of the group G corresponding to the variety $\underline{\underline{V}}$ and in not reserving G, H for relatively free groups. For convenience, we mention the names of the three varieties we will have most occasion to use:

$\underline{\underline{A}}$ is the variety of abelian groups,

$\underline{\underline{N}}_c$ is the variety of nilpotent groups of class at most c,

$\underline{\underline{B}}_{p^n}$ is the variety of groups of exponent dividing n.

We shall denote by X a fixed free group of countably infinite rank with the free generating set $\{x_1, x_2, \cdots\}$; X_2 will denote the free subgroup freely generated by $\{x_1, x_2\}$. If

$$\{1\} \to R \to F \to G \to \{1\}$$

(where F is a free group) is an exact sequence defining a presentation of a group G, and if $\underline{\underline{V}}$ is any variety, then a *$\underline{\underline{V}}$-relator* of G (with respect to the given presentation) is any element of $\underline{\underline{V}}(F) \cap R$. If H is a subgroup of a group G and K is a normal subgroup of H, then H/K is a *section* of G; it is *proper* unless $H = G$ and $K = \{1\}$. Finally, $C_p \, wr \, C_p$ denotes the restricted wreath product of two cyclic groups of order p.

1.2. The first, and fundamental, problem is to decide what factors can cause the irregularity of a direct product of two regular groups. We begin by looking at two 'least criminals' and showing that the respective subgroups generated by the p'th powers of commutators have substantially the same structure.

THEOREM A1. *Let G_1 and G_2 be regular groups and suppose that $G_1 \times G_2$ is irregular. Then G_1 and G_2 have sections H_1 and H_2 with 2-generator presentations in which the $\underline{\underline{B}}_p\underline{\underline{A}}$-relators of H_1 are precisely those of H_2.*

Looking at the sufficiency of this condition for irregularity, we find that the condition is close to being sufficient: but new problems arise.

THEOREM A2. *Let H_1 and H_2 be regular groups with 2-generator presentations in which the $\underline{\underline{B}}_p\underline{\underline{A}}$-relators of H_1 are precisely those of H_2. If $H_1 \times H_2$ is regular, then $H_1 \times H_1$ and $H_2 \times H_2$ are both regular.*

1.3. From Theorem A2, the following question arises:

(1) *if $G \times G$ and $H \times H$ are both regular, is $G \times H$ regular?*

It would seem likely that, in general, the answer to this is negative even though positive answers may be obtainable by a restriction to, for example, metabelian groups. To enable some approach to the question, we split the question into two somewhat different questions, a positive answer to both of which would imply a positive answer to our original question:

(2)
i) *if $G \times G$ is regular, is every finite direct power of G regular (that is, is G V-regular)?*
ii) *if G, H are V-regular, is $G \times H$ regular?*

(We leave question ii) for the next part of this discussion.)

If G is regular, but not V-regular, then G contains a section minimal with respect to being V-irregular; we shall call such a group a RMVI group. Consequently, in looking for an answer to question 2 i) we focus our attention on RMVI groups. Such groups appear likely to be widespread and of relatively complicated structure; we can, however, relate some of them to simpler groups.

LEMMA B1. *If H is a* RMVI *group, then either*
i) $H \in \operatorname{var}(C_p \, wr \, C_p) \vee \underline{\underline{U}}$; *for some regular variety $\underline{\underline{U}}$, or*

ii) *whenever G is a V-regular group and $H \times G$ is irregular, then $K \times G$ is already irregular for some proper section K of H.*

Condition ii) can be split into the condition that either $H \times G$ is regular for every V-regular group G or some V-regular section of H has irregular direct product with a V-regular group. As we shall see, the latter occurrence is comparatively rare; and any group satisfying the former would be of a particularly pathological nature. So it is reasonable to restrict our attention to groups satisfying condition i). Observe that, if

$$\{1\} \to R \to F \to G \to \{1\}$$

gives a presentation of a group G and if ρ is an endomorphism of F which preserves the $\underline{\underline{V}}$-relators of G for some variety $\underline{\underline{V}}$, then ρ naturally induces an endomorphism of $\underline{\underline{V}}(G)$. If G is irregular, but every proper section of G is regular, we call G minimal irregular.

THEOREM B. *Let H be a* RMVI *group satisfying condition* i) *of Lemma* B1 *and suppose that $H \times H$ is regular. Then there exist*

a) *a minimal irregular group D belonging to* $\operatorname{var}(C_p \operatorname{wr} C_p)$,

b) *a monolithic, 2-generator, V-regular group G,*

c) *a monolithic, 2-generator, group B of exponent p, such that*

i) *H, D, G and B have 2-generator presentations with respect to which, if Ω is the set of automorphisms of X_2 which preserve the $\underline{\underline{B}}_p\underline{\underline{A}}$-relators of G, then*

a) *Ω consists of all automorphisms of X_2 which preserve the $\underline{\underline{B}}_p\underline{\underline{A}}$ – relators of H,*

b) *Ω preserves the $\underline{\underline{N}}_{p-1}$-relators of D,*

c) *Ω preserves the $\underline{\underline{A}}$-relators of B, and, furthermore, if H has exponent p^2 then Ω contains all the automorphisms of X_2 which preserve the $\underline{\underline{A}}$-relators of B;*

ii) *the monoliths of H, D, G, B are isomorphic as Ω-groups;*

iii) *H is a quotient of a subdirect product of D and G, B is a quotient of G and, if $H \in \underline{\underline{B}}_{p^{n-1}}\underline{\underline{N}}_c \wedge \underline{\underline{B}}_{p^n}$, then $B \in \underline{\underline{N}}_c$.*

The theorem is, of course, somewhat technical. It does, however, enable the answer to question 2 i) to be determined by means of relatively straightforward calculations involving the structure of V-regular groups or even of groups of exponent p. The introduction of the group B of exponent p is only of real use in the case when H has exponent p^2, since only then does it determine Ω and the determination of Ω is usually the major part of a calculation involving the theorem. In this case, however, it does reduce the study of these RMVI-groups to the determination of the structure of groups of exponent p.

1.4. We now pick up a question left open from the previous discussion:

if G and H are V-regular groups, is $G \times H$ regular?

or, rephrasing to obtain a slightly stronger version,

if $\underline{\underline{V}}$ and $\underline{\underline{W}}$ are regular varieties, is $\underline{\underline{U}} \vee \underline{\underline{V}}$ regular?

We begin with a technical result relating this question to the question of minimal irregular varieties (that is, irregular varieties in which every proper subvariety is regular) of groups. It was shown, in Groves (1973a), that $\operatorname{var}(C_p \, wr \, C_p)$ is the only minimal irregular variety under a number of different restrictions; for example, if the nilpotency class is less that $3p-2$. The question of whether other minimal irregular varieties exist appears to be open; it was shown, however, that this is equivalent to the existence of a verbal subgroup in a relatively free group of exponent p which is operator isomorphic to the minimal verbal subgroup of $F_2(\operatorname{var}(C_p \, wr \, C_p))$. Observe that any two generator relatively free group may be considered as an operator group for the set of operators induced from the automorphism group of an absolutely free group. It is in this sense that the term operator isomorphism is meant.

THEOREM C. *Let $\underline{\underline{U}}$ be an irregular variety of p-groups. There exist regular subvarieties $\underline{\underline{V}}$ and $\underline{\underline{W}}$ of $\underline{\underline{U}}$ having irregular join $\underline{\underline{V}} \vee \underline{\underline{W}}$ if and only if $\underline{\underline{U}}$ contains a regular subvariety $\underline{\underline{V}}_1$ such that $F_2(\underline{\underline{V}}_1)$ has a verbal subgroup, contained in $\underline{\underline{B}}_p\underline{\underline{A}}(F_2(\underline{\underline{V}}_1))$, which is operator isomorphic to the unique minimal verbal subgroup of $F_2(\operatorname{var}(C_p \, wr \, C_p))$.*

By "lifting" the verbal subgroup found in this theorem, we can obtain a similar subgroup of a relatively free group of exponent p. However, this now guarantees the existence of a minimal irregular variety other than $\operatorname{var}(C_p \, wr \, C_p)$.

COROLLARY C. *If a variety $\underline{\underline{U}}$ of p-groups contains two regular subvarieties $\underline{\underline{V}}$ and $\underline{\underline{W}}$ with irregular join $\underline{\underline{V}} \vee \underline{\underline{W}}$, then $\underline{\underline{U}}$ contains a minimal irregular variety other than* $\operatorname{var}(C_p \, wr \, C_p)$.

As we have remarked, $\operatorname{var}(C_p \, wr \, C_p)$ is the only minimal irregular variety under a number of restrictions and so the question with which we began the section has been answered under these restrictions. We give precise details in the next section.

1.5. In this section, we apply the results of the preceeding three sections to obtain some specific results about the regularity of direct products.

THEOREM D1. *Let G and H be V-regular p-groups. Then $G \times H$ is regular if*

i) *p is 3 or 5,*

ii) *G and H have class at most $3p-3$,*

iii) *G and H have derived groups of class at most 2,*

iv) *in G and H, commutators of weight 3 or more have order dividing p.*

Parts i)–iii) are an immediate consequence of Corollary C and of Theorem 3.6 of Groves (1973a). Part iv) follows from observing that the derived group of any 2-generator group satisfying this condition is cyclic and so can have no verbal subgroup isomorphic to the minimal verbal subgroup of $F_2(\mathrm{var}(C_p \, wr \, C_p))$.

We now apply Theorem B to some special classes of groups.

THEOREM D2. *Let G be a regular p-group and suppose that either*

i) *$p = 3$, or*

ii) *G satisfies the law $[x_1, x_2, x_3]^p = 1$, or*

iii) *$p = 5$ and G is metabelian of exponent 25.*

If $G \times G$ is regular, then G is V-regular.

The first two cases above are quite restrictive; we shall have more to say about them in the next theorem. The third case has been included because it appears to exhibit at least some of the complexity of more general situations. It is not meant to be, in any sense, best possible. On the contrary it appears likely that lengthier calculations based on Theorem B would give rise to a corresponding weakening of the restrictions.

Finally, we collect all the technical results so far to obtain definite criteria for the regularity of a direct product in two special cases.

THEOREM D3. *Let G and H be regular p-groups and suppose that either*

i) *$p = 3$, or*

ii) *G and H satisfy the law $[x_1, x_2, x_3]^p = 1$. Then $G \times H$ is regular if and only if G (say) is V-regular and, if G' has exponent p^m, then every two generator subgroup K of H satisfies:*

$$K / \underline{\underline{B}}_{p^m} \underline{\underline{A}}(K) \text{ is } V\text{-regular.}$$

Observe that the condition on H above is a natural one; regularity may be defined by this condition with $m = 1$.

It is again not intended to imply that the restrictions in Theorem D3 are in any way necessary. However, relaxing these restrictions does, in this case, lead to a significantly more complicated situation — as the following example shows.

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EXAMPLE D4. Let G and H be two generator 5-groups, of exponent 25, class 5, with commutators of weight 5 or more having order dividing 5, and with the following extra relations (with $p = 5$);

$$\underline{G}: \quad [x, y, x] = 1, \quad [x, 3y]^p = [x, 4y]$$

$$\underline{H}: \quad [x, y, x]^p [x, 2y, x]^{-1} = [x, 2y]^p [x, 3y]^{-1}$$

$$[x, y, 2x][x, 2y, x] = [x, 3y]$$

$$[x, 3y]^p = [x, 4y]$$

$$[x, 2y, x] = [x, 3y]^p, \quad [x, y, 2x]^p = 1$$

$$[x, 3y, x] = [x, y, 3x] = [x, 4y], \quad [x, 2y, 2x] = 1.$$

Then a routine verification will show that both G and H are RMVI-groups. However, $G \times H$ is regular because (for example), G and H have different $\underline{\underline{B}}_p\underline{\underline{A}}$-relators with respect to any generating set. To see this, observe that $|\underline{\underline{B}}_p\underline{\underline{N}}_2(G)/\underline{\underline{B}}_p\underline{\underline{N}}_3(G)| = 5$ whereas $|\underline{\underline{B}}_p\underline{\underline{N}}_2(H)/\underline{\underline{B}}_p\underline{\underline{N}}_3(H)| = 25$.

2. Proof of Theorems A

2.1. We begin with a preliminary lemma which will also be useful in the proof of Lemma 3.1.

LEMMA A. *Let A, B and C be groups such that C is isomorphic to a section of $A \times B$ but that, if A or B is replaced by a proper section, the resulting direct product has no section isomorphic to C. Then any one of A, B, C is isomorphic to a quotient of a subdirect product of the other two. Furthermore, if C is monolithic, then so are A and B.*

This lemma is an easy consequence of the results and proofs following 53.21 in Neumann (1967).

2.2. PROOF OF THEOREM A1. Because $G_1 \times G_2$ is irregular, we can choose sections $H_1 \times H_2$ of G_1 and G_2 such that $H_1 \times H_2$ is irregular, but the direct product of any proper section of H_1 with H_2, or vice versa, is regular. Then $H_1 \times H_2$ contains a minimal irregular section D, but the direct product of any proper section of H_1 with H_2, or vice versa, does not. Hence Lemma A applies.

Choose L to be a 2-generator irregular subgroup of $H_1 \times H_2$ so that L is a subdirect product of $H_1 \times H_2$ and, by Lemma A, D is a quotient of L. Let

$$\{1\} \to R \to X_2 \to L \to \{1\}$$

be a presentation of L. Since H_1, H_2 and D are all quotients of L, the above presentation induces presentations of these groups; denote the subgroups of relators by S_1, S_2 and T respectively. Then, by Lemma A, $S_1 \cap T = S_2 \cap T = R$. But, by Theorem 2 of Mann (1971) $T \geqq \underline{\underline{B}}_p\underline{\underline{A}}(X_2)$ and so $\underline{\underline{B}}_p\underline{\underline{A}}(X_2) \cap S_1 = \underline{\underline{B}}_p\underline{\underline{A}}(X_2) \cap T \cap S_1 = \underline{\underline{B}}_p\underline{\underline{A}}(X_2) \cap T \cap S_2 = \underline{\underline{B}}_p\underline{\underline{A}}(X_2) \cap S_2$. Thus, with respect to the given presentations, the $\underline{\underline{B}}_p\underline{\underline{A}}$-relators of H_1 and H_2 are the same. This completes the proof of the theorem.

2.3. Proof of Theorem A2. Let

$$\{1\} \to R_i \to X_2 \xrightarrow{\rho_i} H_i \to \{1\} \qquad (i = 1, 2)$$

be the given 2-generator presentations of H_1 and H_2. Let $a_1, b_1 \in H_1$, choose $x, y \in X_2$ such that $x\rho_1 = a_1$, $y\rho_1 = b_1$ and denote $x\rho_2$ and $y\rho_2$ by a_2, b_2 respectively. We claim that, if $w(x_1, x_2) \in \underline{\underline{B}}_p\underline{\underline{A}}(X_2)$, then $w(a_1, b_1) = 1$ if and only if $w(a_2, b_2) = 1$. For,

$$\begin{aligned} w(a_1, b_1) = 1 &\leftrightarrow w(x\rho_1, y\rho_1) = 1 \leftrightarrow w(x, y)\rho_1 = 1 \\ &\leftrightarrow w'(x_1, x_2)\rho_1 = w(x(x_1, x_2), y(x_1, x_2))\rho = 1 \\ &\leftrightarrow w' \in \underline{\underline{B}}_p\underline{\underline{A}}(X_2) \cap R_1 \leftrightarrow w' \in \underline{\underline{B}}_p\underline{\underline{A}}(X_2) \cap R_2 \leftrightarrow w'\rho_2 = 1 \\ &\leftrightarrow w(x\rho_2, y\rho_2) = 1 \leftrightarrow w(a_2, b_2) = 1. \end{aligned}$$

Let $v(x_1, x_2)$ denote $(x_1x_2)^p x_2^{-p} x_1^{-p}$. As $H_1 \times H_2$ is regular, there exists $w(x_1, x_2) \in \underline{\underline{B}}_p\underline{\underline{A}}(X_2)$ such that

$$v((a_1, a_2), (b_1, b_2)) = w((a_1, a_2), (b_1, b_2))$$

or,

$$v(a_i, b_i) = w(a_i, b_i) \qquad (i = 1, 2).$$

Similarly, if $c, d \in H_1$, there exists $\bar{w}(x_1, x_2) \in \underline{\underline{B}}_p\underline{\underline{A}}(X_2)$ such that

$$v(c, d) = \bar{w}(c, d)$$

and $v(a_2, b_2) = \bar{w}(a_2, b_2)$. Hence $\bar{w}(a_2, b_2) = w(a_2, b_2)$ and so, by the previous part of the proof, $\bar{w}(a_1, b_1) = w(a_1, b_1)$.

Hence, if $a_1, b_1, c, d \in H_1$, then there exists $\bar{w}(x_1, x_2) \in \underline{\underline{B}}_p\underline{\underline{A}}(X_2)$ such that

$$v(a_1, b_1) = w(a_1, b_1) = \bar{w}(a_1, b_1); \quad v(c, d) = \bar{w}(c, d).$$

That is, $H_1 \times H_1$ is regular; similarly $H_2 \times H_2$ is regular.

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3. Proof of Theorem B

We begin with the preliminary Lemma B1.

3.1. PROOF OF LEMMA B1. Suppose that there is a V-regular group G such that $H \times G$ is irregular but $K \times G$ is regular for every proper section K of G; further, choose G such that $H \times L$ is regular for every proper section L of G. Then $H \times G$ contains a minimal irregular section D and, by Lemma A, H is a subdirect product of G and H. Thus $H \in \operatorname{var}(D) \vee \operatorname{var}(G)$ and so, by Theorem 2.2 of Groves (1973a), $H \in \operatorname{var}(C_p \, wr \, C_p) \vee \operatorname{var}(G) \vee \underline{\underline{B}}$, where $\underline{\underline{B}}$ is some p-abelian variety. But $\operatorname{var}(G) \vee \underline{\underline{B}}$ is regular — Grün (1954) — and so the proof of the lemma is complete.

3.2. PROOF OF THEOREM B. By assumption, $H \in \operatorname{var}(C_p \, wr \, C_p) \vee \underline{\underline{U}}$ for some regular variety $\underline{\underline{U}}$. Hence, by Lemma A, we can choose monolithic groups $D \in \operatorname{var}(C_p \, wr \, C_p)$ and $G \in \underline{\underline{U}}$ such that any one of H, D and G is a quotient of a subdirect product of the other two. That we can choose G to be 2-generator, and D to be minimal irregular, is easily verified. Because of the inter-relationship between these groups, we can choose presentations

$$\{1\} \to R \to X_2 \to H \to \{1\}$$

$$\{1\} \to S \to X_2 \to D \to \{1\}$$

$$\{1\} \to T \to X_2 \to G \to \{1\},$$

such that $R \cap S = S \cap T = T \cap R$ (see the proof of Theorem A1).

We now investigate the consequences of the regularity conditions on H and G. Since G is V-regular, it satisfies a law $v = v(x_1, x_2) = (x_1x_2)^p x_2^{-p} x_1^{-p} w(x_1, x_2)$, where $w(x_1, x_2) \in \underline{\underline{B}}_p \underline{\underline{A}}(X_2)$; let V be the corresponding verbal subgroup of X_2. Since $H \times H$ is regular, then, for any elements $a, b, c, d \in H$, there exists $\bar{w}(x_1, x_2) \in \underline{\underline{B}}_p \underline{\underline{A}}(X_2)$ such that

$$v(a, b) = \bar{w}(a, b); \qquad v(c, d) = \bar{w}(c, d).$$

Alternatively, and more conveniently, the regularity of $H \times H$ implies that, if ρ, σ are endomorphisms of X_2, then there exists $w_{\rho\sigma} \in \underline{\underline{B}}_p \underline{\underline{A}}(X_2)$ such that

$$v\rho R = w_{\rho\sigma}\rho R; \qquad v\sigma R = w_{\rho\sigma}\sigma R.$$

It is easily verified that any proper quotient of H is a quotient of a subdirect product of G with a proper section of D, and so any proper quotient of H satisfies the law v. Hence vR generates the monolith of X_2/R. Since this monolith is clearly a subgroup of $\underline{\underline{B}}_p \underline{\underline{A}}(X_2)R/R$, there exists $u \in \underline{\underline{B}}_p \underline{\underline{A}}(X_2)$ such that $vR = uR$. (There is some degree of freedom in the choice of u which can be exercised when applying the theorem to ease the calculations.)

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We now transfer this information about H over into G. Now, since D is minimal irregular, VD/D has order p and so, for all endomorphisms ρ of X_2,

$$v\rho S = v^{f(\rho)}S$$

for some integer $f(\rho)$ satisfying $0 \leqq f(\rho) \leqq p$. But $v, v\rho \in T$ and so, as $S \cap T = R$,

$$v\rho R = v^{f(\rho)}R.$$

Hence, as $vR = uR$, for all endomorphisms ρ, σ of X_2, there exists $w_{\rho\sigma} \in \underline{\underline{B}}_p\underline{\underline{A}}(X_2)$ such that

$$u^{f(\rho)}R = w_{\rho\sigma}\rho R; \qquad u^{f(\sigma)}R = w_{\rho\sigma}\sigma R.$$

Thus, as $u, w_{\rho\sigma} \in \underline{\underline{B}}_p\underline{\underline{A}}(X_2)$, and $R \cap \underline{\underline{B}}_p\underline{\underline{A}}(X_2) = T \cap \underline{\underline{B}}_p\underline{\underline{A}}(X_2) = \bar{T}$ say,

$$u^{f(\rho)}\bar{T} = w_{\rho\sigma}\rho\bar{T}; \qquad u^{f(\sigma)}\bar{T} = w_{\rho\sigma}\sigma\bar{T}.$$

We now look at the effect of choosing ρ from Ω; suppose $\sigma = 1$. Then $\bar{T}\rho = \bar{T}$ and

$$u^{f(\rho)}\rho^{-1}\bar{T} = w_{\rho 1}\bar{T} = u\bar{T}$$

and so $u^{f(\rho)}\bar{T} = u\rho\bar{T}$ whenever $\rho \in \Omega$. Suppose now that $\rho \in \Omega$ and that σ is an arbitrary endomorphism of X_2. Then

$$u^{f(\sigma\rho)}\bar{T} = (w_{\sigma\rho,\sigma})\sigma\rho\bar{T}; \qquad u^{f(\sigma)}\bar{T} = (w_{\sigma\rho,\sigma})\sigma\bar{T}.$$

Thus,

$$u^{f(\sigma\rho)}\rho^{-1}\bar{T} = (w_{\sigma\rho,\sigma})\sigma\bar{T} = u^{f(\sigma)}\bar{T}$$

and so,

$$u^{f(\sigma\rho)}\bar{T} = (u^{f(\sigma)})\rho\bar{T} = (u\rho)^{f(\sigma)}\bar{T} = u^{f(\rho)f(\sigma)}\bar{T},$$

by the result just proved. Hence, if $\rho \in \Omega$,

$$f(\sigma\rho) = f(\sigma)f(\rho) \text{ (modulo } p).$$

We now consider the effects of this information in D, retaining the notation for ρ and σ. By assumption, $v\sigma S = v^{f(\sigma)}S$ – say $v\sigma = v^{f(\sigma)}s_\sigma$ for some $s_\sigma \in S$. Then,

$$(v\sigma\rho)S = (v^{f(\sigma)}\rho)(s_\sigma\rho)S = (v^{f(\sigma)f(\rho)})(s_\sigma\rho)S = (v^{f(\sigma\rho)})(s_\sigma\rho)S.$$

Hence, as $(v\sigma\rho)S = v^{f(\sigma\rho)}S$, $s_\sigma\rho \in S$. Thus, for all $\rho \in \Omega$ and all endomorphisms σ of X_2,

$$s_\sigma = (v\sigma)(v^{-f(\sigma)}) \in S(\rho)^{-1}.$$

We proceed to show that Ω preserves the $\underline{\underline{N}}_{p-1}$-relators of D. Let V_1 be the subgroup generated by all the s_σ, let S_1 be the verbal subgroup of X_2 corresponding to $\operatorname{var}(D)$ and let N denote $\underline{\underline{N}}_{p-1}(X_2)$. Now V_1S_1 clearly has index p in VS_1. But, as VS/S — the monolith of X_2/S — has order p, so also does $VS_1/VS_1 \cap S$. But, $V_1 \leqq S$ and so $V_1S_1 \leqq VS_1 \cap S$; thus $V_1S_1 = VS_1 \cap S$. Now, $VS_1 = NS_1$ (see, for example, the discussion in Groves (1973a; page 80)). Hence,

$$(NS_1 \cap S)S_1 = V_1S_1 \leqq (\bar{S}\rho^{-1})S_1$$

and so

$$N \cap S = (NS_1 \cap S)S_1 \cap N \leqq (S\rho^{-1})S_1 \cap N = S\rho^{-1} \cap N = (N \cap S)\rho^{-1}.$$

Thus $\rho \in \Omega$ preserves the $\underline{\underline{N}}_{p-1}$-relators — $N \cap S$ — of D.

To complete this section of the proof, observe that

$$\phi\colon u^kR \to v^kS; \quad \psi\colon u^kR \to u^kT$$

are Ω-isomorphisms from the monolith of H to the monoliths of D and G respectively. This completes the proof of all those parts of the theorem which do not involve the group B. We introduce B by means of a separate lemma.

3.3. LEMMA B2. *Let G be a regular group which is also an operator group for some set of operators Λ. Suppose that G has a unique minimal normal Λ-subgroup M. Then G has a Λ-chief section K/L such that*

i) *K/L is the unique minimal normal Λ-subgroup of G/L and is Λ-isomorphic to M,*

ii) *G/L has exponent p.*

PROOF. Suppose that G has exponent p^n. Let $L_1 = \{g \in G\colon g^{p^{n-1}} = 1\}$ and let $K_1 = \{g \in G\colon g^{p^{n-1}} \in M\}$. Then, since G is regular, K_1 and L_1 are normal Λ-subgroups of G.

Define $\phi\colon K_1/L_1 \to M$ by

$$\phi\colon aL_1 \mapsto a^{p^{n-1}}.$$

i) *ϕ is well-defined.* For, if $a_1L_1 = a_2L_1$, then $a_2^{-1}a_1 \in L_1$ and so $(a_2^{-1}a_1)^{p^{n-1}} = 1$. However, as M is a unique minimal Λ-subgroup of G, it must lie in the last non-trivial term of the lower central series of G and so be central. Thus, if $a \in K_1$, then $[a^{p^{n-1}}, b] = 1$ for all $b \in G$ and so, as G is regular, $[a, b]^{p^{n-1}} = 1$. Thus $(a_2^{-1}a_1)^{p^{n-1}} = a_2^{-p^{n-1}}a_1^{p^{n-1}}$ and so $(a_1L_1)\phi = a_1^{p^{n-1}} = a_2^{p^{n-1}} = (a_2L_1)\phi$.

ii) *ϕ is monomorphic.* This is clear from the choice of L_1.

iii) *ϕ is epimorphic.* As $\underline{\underline{B}}_{p^{n-1}}(G)$ is a non-trivial normal Λ-subgroup of G, $M \leqq \underline{\underline{B}}_{p^{n-1}}(G)$ and so ϕ is epimorphic.

iv) *ϕ is a Λ-homomorphism.* If $\lambda \in \Lambda$, then

$$((aL_1)\lambda)\phi = ((a\lambda)L_1)\phi = (a\lambda)^{p^{n-1}} = (a^{p^{n-1}})\lambda = ((aL_1)\phi)\lambda.$$

Thus ϕ is a Λ-isomorphism and K/L is Λ-isomorphic to M. Finally, as $\underline{\underline{B}}_{p^{n-1}}(\underline{\underline{B}}_p(G)) = \{1\}$, $\underline{\underline{B}}_p(G) \leqq L$ and so G/L has exponent p.

3.4. Proof of Theorem B (continued). Apply Lemma B2 (with Λ the trivial group of operators) to X_2/T to obtain a section K/L. Denote X_2/L by B; then B is a 2-generator group of exponent p. Also, if xL is central in X_2/L, then $[x, y] \in L$ for all $y \in X_2$. Hence $[x, y]^{p^{n-1}} \in T$ and so, as X_2/T is regular, $[x^{p^{n-1}}, y] \in T$, for all $y \in X_2$. Hence $x^{p^{n-1}}T$ is central, and has order p, in X_2. Thus, as X_2/T is monolithic, $x^{p^{n-1}}T$ belongs to the monolith. Hence $x \in K$; that is, K/L is the centre of X_2/L — which is therefore monolithic.

Suppose that $a \in L \cap X_2'$ and that $\rho \in \Omega$. Then $a^{p^{n-1}} \in T \cap \underline{\underline{B}}_p\underline{\underline{A}}(X_2)$ and so $(a^{p^{n-1}})\rho = (a\rho)^{p^{n-1}} \in T \cap \underline{\underline{B}}_p\underline{\underline{A}}(X_2)$. Thus $a\rho \in L \cap X_2'$ and Ω preserves the $\underline{\underline{A}}$-relators — $L \cap X_2'$ — of B. Suppose now, that H — and so G — has exponent p^2 and that the automorphism σ of X_2 satisfies

$$(L \cap X_2')\sigma = L \cap X_2'.$$

Then, if $a \in T \cap \underline{\underline{B}}_p\underline{\underline{A}}(X_2)$, $a = d^p$ where $d \in X_2'$. Hence $d \in L$ — as $d^p \in T$ — and so $d \in L \cap X_2'$. Thus $d\sigma \in L \cap X_2'$ and so $a\sigma = d^p\sigma = (d\sigma)^p \in T \cap \underline{\underline{B}}_p\underline{\underline{A}}(X_2)$; that is, $\sigma \in \Omega$. Hence Ω contains all the automorphisms of X_2 which preserve the $\underline{\underline{A}}$-relators of B.

Clearly the isomorphism between K/L and the monolith of X_2/T described in Lemma B2 is an Ω-isomorphism. Finally, if H — and so G — belongs to $\underline{\underline{B}}_{p^{n-1}}\underline{\underline{N}}_c \cap \underline{\underline{B}}_{p^n}$, then $\underline{\underline{N}}_c(X_2) \leqq L$ and so $B \in \underline{\underline{N}}_c$.

4. Proof of Theorems C

4.1. Proof of Theorem C. The proof is very similar to the proof of Proposition 3.3 of Groves (1973a), and so we shall omit some of the details.

Suppose that $\underline{\underline{V}}$ and $\underline{\underline{W}}$ are regular subvarieties of $\underline{\underline{U}}$ and that $\underline{\underline{V}} \vee \underline{\underline{W}}$ is irregular. Then $\underline{\underline{V}} \vee \underline{\underline{W}}$ contains a minimal irregular variety $\underline{\underline{D}}$. By applying Lemma A to the appropriate relatively free groups, it follows that $\underline{\underline{V}}$ and $\underline{\underline{W}}$ contain subvarieties $\underline{\underline{V}}_1$ and $\underline{\underline{W}}_1$ such that $\underline{\underline{V}}_1 \vee \underline{\underline{W}}_1 = \underline{\underline{V}}_1 \vee \underline{\underline{D}} = \underline{\underline{W}}_1 \vee \underline{\underline{D}}$. Furthermore, it is easily verified that $\underline{\underline{V}}_1$ and $\underline{\underline{W}}_1$ may be chosen in such a way that $F_2(\underline{\underline{V}}_1)$ and $F_2(\underline{\underline{W}}_1)$ contain unique minimal verbal subgroups.

It is now a straightforward exercise to verify that the minimal verbal subgroups of both $F_2(\underline{\underline{V}}_1)$ and $F_2(\underline{\underline{W}}_1)$ are operator isomorphic to the minimal verbal subgroup of $F_2(\underline{\underline{D}})$. Furthermore, from the proof of Proposition 3.3 of Groves (1973a), this is operator isomorphic to the minimal verbal subgroup of

$F_2(\operatorname{var}(C_p\, wr\, C_p))$. Finally, as $\underline{\underline{V}}_1$, for example, is not p-abelian, $\underline{\underline{B}}_p\underline{\underline{A}}(F_2(\underline{\underline{V}}_1))$ is nontrivial and so contains the minimal verbal subgroup of $F_2(\underline{\underline{V}}_1)$.

Suppose now that $\underline{\underline{V}}$ contains a variety $\underline{\underline{V}}_1$ of the type described in the theorem. Let $\underline{\underline{D}}$ be a minimal irregular subvariety of $\underline{\underline{U}}$. Then the given verbal subgroup of $F_2(\underline{\underline{V}}_1)$ is also operator isomorphic to the unique minimal verbal subgroup of $F_2(\underline{\underline{D}})$. We may now use this isomorphism to form an 'amalgamated subdirect product' of the two free groups and so obtain a variety $\underline{\underline{W}}$—which has a law of the form $(x_1x_2)^p x_2^{-p} x_1^{-p} u(x_1, x_2)$. (The word $(x_1x_2)^p x_2^{-p} x_1^{-p}$ defines the maximal subvariety of $\underline{\underline{D}}$ and $u(x_1, x_2)$ defines the subvariety of $\underline{\underline{V}}_1$ corresponding to the given verbal subgroup of $F_2(\underline{\underline{V}}_1)$). The proof of this is almost identical to the second half of the proof of Proposition 3.3 of Groves (1973a), and so we omit it. Since the relevant verbal subgroup of $F_2(\underline{\underline{V}}_1)$ is less than $\underline{\underline{B}}_p\underline{\underline{A}}(F_2(\underline{\underline{V}}_1))$, we may suppose that $u(x_1, x_2) \in \underline{\underline{B}}_p\underline{\underline{A}}(X_2)$, and so $\underline{\underline{W}}$ is regular. Furthermore, it follows easily that $\underline{\underline{V}}_1 \vee \underline{\underline{D}} = \underline{\underline{D}} \vee \underline{\underline{W}} = \underline{\underline{W}} \vee \underline{\underline{V}}_1$ and so $\underline{\underline{V}}_1 \vee \underline{\underline{W}}$ is irregular. This completes the proof of the theorem.

4.2. Proof of Corollary C. By Theorem C, $\underline{\underline{U}}$ contains a regular subvariety $\underline{\underline{V}}_1$ such that $F = F_2(\underline{\underline{V}}_1)$ contains a verbal subgroup M which is operator isomorphic to the minimal verbal subgroup of $F_2(\operatorname{var}(C_p\, wr\, C_p))$. Clearly M is a minimal verbal subgroup of F and, without loss of generality, we may suppose it is the unique minimal verbal subgroup. Let Λ be the semigroup of endomorphisms of F. Then we can apply Lemma B2 to show the existence of a quotient F/L of F, with exponent p, which has a minimal verbal subgroup operator isomorphic to the previous ones. Hence, by Proposition 3.3 of Groves (1973a), there exists, in $\operatorname{var}(C_p\, wr\, C_p) \vee \operatorname{var}(F/L)$, and so in $\underline{\underline{U}}$, a minimal irregular variety other than $\operatorname{var}(C_p\, wr\, C_p)$.

5. Proof of Theorems D

5.1. Proof of Theorem D2. Observe, firstly, that because 2-generator V-regular 3-groups are metabelian and so have class 2 (Theorem 1.4 of Weichsel (1967)), a RMVI 3-group satisfies the law $[x_1, x_2, x_3]^3 = 1$. Hence case i) and case ii) can be dealt with together.

Let G be a RMVI-group satisfying condition ii) of the theorem. We show, firstly, that G satisfies condition i) of Lemma B (so that Theorem B is applicable). Now direct products of V-regular groups satisfying condition ii) are again V-regular, by Theorem D1. Hence it remains to show that $G \times H$ is irregular for some V-regular group H. Suppose that G' has exponent p^n.

Let H be the free 2-generator group of exponent p^n and class 2. Then H is V-regular and satisfies condition ii); also H' has exponent precisely p^n.

Also, as G is RMVI, $G/\underline{\underline{B}}_{p^{n-1}}\underline{\underline{A}}(G)$ is V-regular. Now, by Lemma 3 of Groves (1974), each V-regular group satisfying ii) has a law

$$(x_1x_2)^p = x_1^p x_2^p [x_1, x_2]^{kp}$$

where $k = -(p-1)/2$. Let $g, h \in G$ and let x, y be free generators of H. Then,

$$(xy)^p = x^p y^p [x, y]^{kp}; \quad (gh)^p = g^p h^p [g, h]^{kp} [g, h]^{lp^{n-1}}$$

for some l depending on g and h. Thus, if $G \times H$ is regular,

$$((x, g)(y, h))^p = (x, g)^p (y, h)^p [(x, g), (y, h)]^{mp}$$

for some m, and so $[x, y]^{mp} = [x, y]^{kp}$ and $[g, h]^{mp} = [g, h]^{kp + lp^{n-1}}$. Hence, as $[x, y]$ has order p^n, $p^n \mid (m-k)p$ and so, as $[g, h]$ also has order p^n, $p^n \mid lp^{n-1}$. Hence

$$(gh)^p = g^p h^p [g, h]^{kp},$$

for any elements $g, h \in G$; that is, G is V-regular—a contradiction.

Thus $G \times H$ is irregular and so G satisfies condition i) of Lemma B. Now, the $\underline{\underline{B}}_p\underline{\underline{A}}$-relators of G (with respect to any presentation) consist of all laws derived from $[x_1, x_2, x_3]^p = 1$ together with $[x_1, x_2]^{p^n} = 1$. Thus the $\underline{\underline{B}}_p\underline{\underline{A}}$-relators are all laws and so are preserved by any substitution. This is clearly not true, however, for the $\underline{\underline{N}}_{p-1}$-relators of any minimal irregular group in $\operatorname{var}(C_p \, wr \, C_p)$ and so Theorem B applies to show that $G \times G$ is irregular. The proof of parts i) and ii) of the theorem now follows.

For groups satisfying condition iii), we begin with the observation that RMVI-groups of this type have class p (by Theorem 1.4 of Weichsel (1967)) and so it follows directly from the description of varieties of metabelian groups of class p given in Brisley (1971), that such groups satisfy condition i) of Lemma B.

The remainder of the proof is routine calculation (which we omit) based on Lemma B. Firstly, a general set of commutator relators for a 2-generator, monolithic, metabelian group of exponent 5 is written down. The set of substitutions which preserve these relators are then found. Then these substitutions are used to determine which minimal irregular groups in $\operatorname{var}(C_p \, wr \, C_p)$ have their $\underline{\underline{N}}_{p-1}$-relators preserved by them. Finally, it is shown that no isomorphism of the required type can exist between the respective monoliths.

5.2. Proof of Theorem D3. a) Suppose G and H are both V-irregular. Then, G and H contain RMVI-sections G_1 and H_1; we claim $G_1 \times H_1$ is irregular. As in the proof of Theorem D2, we can treat cases i) and ii) simultaneously. Suppose G_1 has exponent p^m, H_1 has exponent p^n and $n \geqq m$.

Then, as the $\underline{\underline{B}}_p\underline{\underline{A}}$-relators of G_1, H_1 or any quotient of these groups, follow from the law $[x_1, x_2, x_3]^p = 1$, and laws of the form $[x_1, x_2]^{p^k} = 1$, G_1 and $H_1/\underline{\underline{B}}_{p^m}\underline{\underline{A}}(H_1)$ have the same $\underline{\underline{B}}_p\underline{\underline{A}}$-relators. Hence, as $G_1 \times G_1$ is irregular (by Theorem D2), $G_1 \times H_1/\underline{\underline{B}}_{p^m}\underline{\underline{A}}(H_1)$ is irregular—by Theorem A2.

b) Suppose now that G is V-regular and that G' has exponent p^m. Then G has a 2-generator subgroup G_1 such that G_1' also has exponent p^m. Suppose that H has a 2-generator subgroup K with $K/\underline{\underline{B}}_{p^m}\underline{\underline{A}}(K)$ V-irregular. Then K has a 2-generator RMVI section L; let the exponent of L' be p^{m_1}. Let $G_2 = G_1/\underline{\underline{B}}_{p^{m_1}}\underline{\underline{A}}(G_1)$. Then it follows, as in Theorem D2, that L and G_2 have the same $\underline{\underline{B}}_p\underline{\underline{A}}$-relators (in both of cases i) and ii)—in case i) because G_2 has class 2). Because $L \times L$ is irregular (by Theorem D2), it follows from Theorem A2 that $G_2 \times L$ is irregular. Hence $G \times H$ is irregular and the condition given above for regularity is necessary.

c) Suppose now that $G \times H$ is irregular and that G is V-regular with the exponent of G' being p^m. Then, by Theorem A1, G and H contain 2-generator sections G_1 and H_1 such that $G_1 \times H_1$ is irregular and, with respect to suitable presentations, G_1 and H_1 have the same $\underline{\underline{B}}_p\underline{\underline{A}}$-relators. By Theorem D1, H_1 is V-irregular; also, H_1 has exponent p^m. Let K be a 2-generator subgroup of H such that $K/L = H_1$ for some L normal in H. $\underline{\underline{B}}_{p^m}\underline{\underline{A}}(H_1) = \{1\}$, then $\underline{\underline{B}}_{p^m}\underline{\underline{A}}(K) \leqq L$ and so $K/\underline{\underline{B}}_{p^m}\underline{\underline{A}}(K)$ is V-irregular. Thus the condition given above for regularity is also sufficient.

This completes the proof of Theorem D3.

Added in proof. C. Godsil (M.Sc. Thesis, University of Melbourne, 1976) has constructed a metabelian group G for which $G \times G$ is regular but G is not V-regular.

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