THE LOTOTSKY TRANSFORM AND BERNSTEIN POLYNOMIALS

J. P. KING

The Bernstein polynomials

$$B_n(f;x) = \sum_{k=0}^{n} f(k/n)\binom{n}{k} x^k (1-x)^{n-k} \tag{1}$$

associated with a function f defined on $[0, 1]$ have been the subject of much recent research and have been generalized in several directions (**1**; **2**; **5**). The generalized Lototsky or $[\mathrm{F}, d_n]$ matrix (**3**) has also been the subject of extensive research. The elements a_{nk} of this matrix are defined by

$$\begin{gathered} a_{00} = 1, \qquad a_{0k} = 0 \ (k \neq 0), \\ \prod_{i=1}^{n} \frac{y + d_i}{1 + d_i} = \sum_{k=0}^{n} a_{nk}\, y^k, \end{gathered} \tag{2}$$

where $\{d_i\}$ is a sequence of complex numbers with $d_i \neq -1$ $(i = 1, 2, \ldots)$. It is the purpose of this note to point out a connection between the Lototsky matrix and the Bernstein polynomials which gives yet another extension of the latter.

It is convenient to make a change of notation. If we let $h_i = 1/(1 + d_i)$, equation (2) has the form

$$\prod_{i=1}^{n} (h_i y + 1 - h_i) = \sum_{k=0}^{n} a_{nk}\, y^k. \tag{3}$$

Now let $\{h_i(x)\}$ be a sequence of functions defined on $[0, 1]$. Let $a_{nk} = a_{nk}(x)$ be the elements of the Lototsky matrix given by (3) corresponding to the sequence $\{h_i(x)\}$. For each f defined on $[0, 1]$ let

$$L_n(f;x) = \sum_{k=0}^{n} f(k/n) a_{nk}(x). \tag{4}$$

It is easy to see that if $h_i(x) = x$ $(i = 1, 2, \ldots)$, then $L_n(f;x) = B_n(f;x)$. Therefore, in this sense, the functions $L_n(f;x)$ provide an extension of the Bernstein polynomials. The following theorem gives sufficient conditions on the sequence $\{h_i(x)\}$ to insure that $L_n(f;x) \to f(x)$.

THEOREM. *For $f \in C[0, 1]$ let $L_n(f;x)$ be defined by* (4) *and let $\{s_i(x)\}$ denote the $(C, 1)$ transform of the sequence $\{h_i(x)\}$. If $0 \leqslant h_i(x) \leqslant 1$ $(i = 1, 2, \ldots)$*

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and if $\{s_i(x)\}$ *converges uniformly to* x *on* $[0, 1]$, *then*

$$\lim_{n\to\infty} L_n(f; x) = f(x)$$

uniformly on $[0, 1]$.

Proof. According to a theorem of Korovkin (**4**, p. 14) it is sufficient to show that

$$L_n(1; x) \to 1, \qquad L_n(t; x) \to x, \qquad L_n(t^2; x) \to x^2,$$

uniformly on $[0, 1]$ and that L_n is a positive linear operator. It is clear that L_n is linear. Furthermore, $f \geqslant 0$ implies that $L_n \geqslant 0$ since $a_{nk}(x) \geqslant 0$ whenever $0 \leqslant h_i(x) \leqslant 1$.

We have

$$L_n(1; x) = 1 \qquad (n = 1, 2, \ldots),$$

$$L_n(t; x) = \sum_{k=0}^{n} (k/n) a_{nk}(x),$$

and

$$L_n(t^2; x) = \sum_{k=0}^{n} (k/n)^2 a_{nk}(x).$$

If we let

$$P_n = \prod_{i=1}^{n} (y h_i(x) + 1 - h_i(x))$$

and

$$r_i(x, y) = \frac{h_i(x)}{y h_i(x) + 1 - h_i(x)},$$

we have

$$P_n' = \sum_{i=1}^{n} r_i(x, y) \cdot P_n, \tag{5}$$

and

$$P_n'' = \left\{ \left[\sum_{i=1}^{n} r_i(x, y) \right]^2 - \sum_{i=1}^{n} r_i^2(x, y) \right\} \cdot P_n, \tag{6}$$

where the differentiation is with respect to y. Also

$$P_n' = \sum_{k=0}^{n} k a_{nk}(x) y^{k-1} \tag{7}$$

and

$$P_n'' = \sum_{k=0}^{n} k(k-1) a_{nk}(x) y^{k-2}. \tag{8}$$

If we set $y = 1$ in (5) and (7), we obtain

$$\frac{1}{n} \sum_{k=0}^{n} k a_{nk}(x) = s_n(x). \tag{9}$$

Similarly, it follows from (6), (8), and (9) that

$$\frac{1}{n^2}\sum_{k=0}^{n} k^2 a_{nk}(x) = \frac{1}{n}\{s_n(x) - t_n(x)\} + s_n^2(x), \tag{10}$$

where $\{t_n(x)\}$ is the $(C, 1)$ transform of the sequence $\{h_n^2(x)\}$.

It is easy to see that $0 \leqslant h_i(x) \leqslant 1$ implies $t_n(x) = O(1)$ so that $t_n(x)/n \to 0$ uniformly on $[0, 1]$. This proves the theorem.

COROLLARY. *If* $0 \leqslant h_i \leqslant 1$ *and if* $\{h_i(x)\}$ *converges uniformly to* x *on* $[0, 1]$, *then*

$$\lim_{n\to\infty} L_n(f; x) = f(x)$$

uniformly on $[0, 1]$.

Proof. The $(C, 1)$ transform is a regular summability method and preserves uniform convergence so that $s_n(x) \to x$ uniformly on $[0, 1]$.

It seems worth while to give an example of a sequence $\{h_i(x)\}$ that is not convergent to x while its $(C, 1)$ transform is. It is not difficult to see that the following example suffices:

$$h_i(x) = \begin{cases} \frac{x}{2}\ (0 \leqslant x \leqslant \frac{1}{2}), & \frac{3x}{2} - \frac{1}{2}\ (\frac{1}{2} \leqslant x \leqslant 1), \quad i \text{ odd}, \\ \frac{3x}{2}\ (0 \leqslant x \leqslant \frac{1}{2}), & \frac{x}{2} + \frac{1}{2}\ (\frac{1}{2} \leqslant x \leqslant 1), \quad i \text{ even}. \end{cases}$$

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Lehigh University