Canad. Math. Bull. Vol. **32** (1), 1989

ON THE DETERMINISTIC AND ASYMPTOTIC σ-ALGEBRAS OF A MARKOV OPERATOR

BY

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ABSTRACT. Let P be a Markov operator on $L_\infty(X, \Sigma, m)$ which does not disappear (i.e., $P1_A \equiv 0 \Rightarrow 1_A \equiv 0$). We study the relationship between the σ-algebras

$$\Sigma_n(P) = \{A \in \Sigma : \exists B_n \text{ with } P^n 1_A = 1_{B_n}\}, \Sigma_d(P) = \bigcap_{n=1}^{\infty} \Sigma_n(P)$$

(the *deterministic* σ-algebra), and the *asymptotic* σ-algebra

$$\Sigma_t(P) = \{A \in \Sigma : \forall n \exists 0 \leqq f_n \leqq 1 \text{ with } P^n f_n = 1_A\}.$$

When m is a σ-finite invariant measure, $f \in L_p(m)(1 \leqq p < \infty)$ is $\Sigma_n(P)$ measurable iff $P^{*n}P^n f = f$, and also iff $P^n f$ has the same distribution as f. The case of a convolution operator on a locally compact group is considered.

0. **Introduction.** Let (X, Σ, m) be a σ-finite measure space, and P a Markov-operator in $L_\infty(X, \Sigma, m)$, i.e., a linear operator in L_∞ of norm $\leqq 1$ (called a contraction), which satisfies:

(i) $0 \leqq f \in L_\infty \Rightarrow 0 \leqq Pf$;

(ii) $P1 = 1$

(iii) $0 \leqq f_n \leqq 1$ in L_∞ and $f_n \downarrow 0 \Rightarrow Pf_n \downarrow 0$.

The measure m is called *invariant* if $\int Pf\, dm = \int f\, dm$ holds for all f. In that case, P is also a contraction in $L_1(m)$, and therefore in all spaces $L_p(m)$, $1 \leqq p \leqq \infty$; see e.g. [K, p.65].

If f is any function and we write $f = 1_B$, we assert the existence of a set B with $f = 1_B$. We do not distinguish measurable functions or sets from their equivalence classes mod nullsets.

The deterministic σ-algebra $\Sigma_d = \{A : P^n 1_A = 1_{B_n} \forall n\}$ was introduced for the study of limit theorems of $P^n f$, when m is invariant for P. We quote the general results, proved in [F1]:

THEOREM A. $I \equiv \{f \in L_2(m) : \|P^n f\|_2 = \|f\|_2 \forall n\} = \{f \in L_2(m) : P^{*n}P^n = f \forall n\} = L_2(X, \Sigma_d(P), m)$

THEOREM B. (i) *I is invariant for P, and $P_{|I}$ is an isometry.*

Received by the editors February 17, 1987 and, in revised form, April 6, 1988
AMS Subject Classification (1980): Primary 60J05, 60J15: Secondary 47A35

(ii) *If* $f \perp I$, *then* $P^n f \longrightarrow 0$ *weakly in* $L_2(m)$.

Using this approach, Foguel [F1] (p.96–98) succeeded in obtaining a proof of the Jamison-Orey theorem: If Σ_d is trivial for an aperiodic Harris operator with *finite* invariant measure, and $f \in L_1$ satifies $\int f\, dm = 0$, then $\|P^n f\|_1 \longrightarrow 0$.

M. Rosenblatt [R, p.113–115] showed that *in general* we cannot have strong convergence in Theorem B(ii), even if the invariant measure is finite, and Σ_d is trivial. A different example was recently given in [AB].

In [L1] it is shown that for the predual T of P, acting in $L_1(m)$, we have (without requiring an invariant measure)

$$\|T^n u\|_1 \longrightarrow 0 \Leftrightarrow \int ugdm = 0 \quad \forall g \in \bigcap_{n=1}^{\infty} P^n\{f \in L_\infty : 0 \leqq f \leqq 1\}.$$

(see also [D] for more discussion).

Hence, it is a natural question to ask if it is enough to check only against $g \epsilon \Sigma_t = \Sigma_t(P)$, the set of indicator functions in the above intersection, (as is suggested by the result for P obtained from a non-singular point transformation).

If m is a σ-finite invariant measure for T, theorem 2.1 below asserts $\Sigma_d(T) = \Sigma_t(P)$. Together with Rosenblatt's example, this implies that the answer to the above question is negative even for P with an invariant probability.

We also study $\Sigma_n(T) = \{A \epsilon \Sigma \; : \; \exists B_n \text{ with } T^n 1_A = 1_{B_n}\}$. E.g., we show that f is $\Sigma_n(T)$-measurable iff $f, Tf, \ldots, T^n f$ have the same distribution with respect to the σ-finite invariant measure m.

In the particular case of irreducible convolution operators on a locally compact group we identify $\Sigma_d = \Sigma_t$.

P is called *non-disappearing* if $P1_A = 0$ implies $1_A = 0$. (Equivalently, $f \geqq 0, Pf = 0 \Rightarrow f = 0$). Clearly, Markovian operators having a σ-finite invariant measure and conservative operators are non-disappearing.

The following lemma is included here since the reference may not be readily accessible:

LEMMA 0 [F2]. (i) *If* P *is Markovian,* $P1_{B_1} = 1_{A_1}$ *and* $P1_{B_2} = 1_{A_2}$, *then* $P(1_{B_1 \cup B_2}) = 1_{A_1 \cup A_2}$.

(ii) *If, in addition,* P *is non-disappearing, then* $Pg = 1_A$ *with* $0 \leqq g \leqq 1$ *implies the existence of a unique* $B \epsilon \Sigma$ *with* $g = 1_B$.

PROOF. (i) $P(1_{B_1 \cup B_2}) = P(1_{B_1} + 1_{B_1^c \cap B_2}) = P1_{B_1} + P1_{B_1^c \cap B_2} = P1_{B_1} \vee P1_{B_1^c \cup B_2}$ (since $P1_{B_1^c} = 1_{A_1^c}) \leqq P1_{B_1} \vee P1_{B_2}$. The reverse inequality is clear.

(ii) $P(1-g) = 1_{A^{c\prime}}$ and hence $P(g \wedge (1-g)) \leqq 1_A \wedge 1_{A^c} = 0$. Hence $g \wedge (1-g) = 0$ and $g = 1_B$. If also $P1_C = 1_A$, then $P(1_B \wedge 1_{C^c}) \leqq P1_B \wedge P1_{C^c} = 1_A \wedge 1_{A^c} = 0$. Hence $B \subset C$, and by symmetry $B = C$. □

1. **The deterministic and asymptotic σ-algebras.** Let $\Sigma_n = \{A \in \Sigma : P^n 1_A = 1_B\}$. Then, since P^n is a Markov operator, Lemma 0 easily yields that Σ_n is a σ-

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algebra, and that $\Sigma_{n+1} \subset \Sigma_n$ if P is non-disappearing. We shall assume throughout that P is non-disappearing. Then

$$\Sigma_d := \bigcap_{n=1}^{\infty} \Sigma_n$$

is the *determinisitic σ-algebra.*

THEOREM 1.1. $f \in L_\infty(\Sigma_n) \Leftrightarrow P^n(fg) = (P^n f)(P^n g) \forall g \in L_\infty$.

PROOF. We may assume $n = 1$.

Let $L = \{f \in L_\infty : P(fg) = (Pf)(Pg) \forall g \in L_\infty\}$. It is easy to check that L is an algebra, and w^*-closed. Let $S = \{A \in \Sigma : 1_A \in L\}$. It was proved in [L3] that S is a σ-algebra, and that $L = L_\infty(X, S, m)$. For $A \in S$ we have $P(1_A) = (P1_A)^2$, so $A \in \Sigma_1$. Thus $S \subset \Sigma_1$.

Let $A \in \Sigma_1$. Then $P1_A = 1_B$. For $0 \leqq g \in L_\infty$ we have $P(1_A g) \leqq \|g\|_\infty P1_A = \|g\|_\infty 1_B$. Hence $P(1_A g) = 0$ a.e. on B^c. Hence, applying the argument to $A^c, P(1_{A^c} g) = 0$ a.e. on B. Hence $1_B Pg = [P(1_A g) + P(1_{A^c} g)]1_B = 1_B P(1_A g)$.

Since $P(1_A g) = 0$ on $B^c, P(1_A g) = 1_B Pg = (P1_A)(Pg)$. It follows easily that $A \in S$. Hence $S = \Sigma_1$. □

COROLLARY 1.2. $f \in L_\infty(\Sigma_d) \Leftrightarrow P^n(fg) = (P^n f)(P^n g) \forall g \in L_\infty, \forall n$.

COROLLARY 1.3. *P maps $L_\infty(\Sigma_d)$ into $L_\infty(\Sigma_d)$. The restriction of P to $L_\infty(\Sigma_d)$ is multiplicative and induces a homomorphism of Σ_d.*

This result corresponds to theorem A in the introduction, without assuming the existence of an invariant measure.

DEFINITION $\Sigma_t = \{A \in \Sigma$ *: for $\forall n$ there is $0 \leqq f_n \leqq 1$ with $P^n f_n = 1_A$*$\}$.

PROPOSITION 1.4. *Let P be non-disappearing. If $A \in \Sigma_t$, then each $0 \leqq f_n \leqq 1$ satisfying $P^n f_n = 1_A$ is uniquely determined, $f_n = 1_{A_n}, A_n \in \Sigma_t$, and $P1_{A_{n+1}} = 1_{A_n}$.*

PROOF. As P^n is non-disappearing, the uniqueness and $f_n = 1_{A_n}$ follow from Lemma 0. Moreover, $1_A = P^n P^m 1_{A_{n+m}}$ and $1_A = P^n 1_{A_n}$ yield $P^m 1_{A_{n+m}} = 1_{A_n}$. As m was arbitrary $A_n \in \Sigma_t$. □

THEOREM 1.5. *Σ_t is a σ-algebra.*

PROOF. Let $A, B \in \Sigma_t$. Then $P^n 1_{A_n} = 1_A, P^n 1_{B_n} = 1_B$, with $A_n, B_n \in \Sigma_t$. Hence, adding

$$P^n(1_{A_n \cap B_n^c}) \leqq P^n 1_{A_n} \wedge P^n 1_{B_n^c} = 1_A \wedge 1_{B^c} = 1_{A \cap B^c}$$

and

$$P^n(1_{A_n \cap B_n}) \leqq 1_{A \cap B},$$

we have $P^n 1_{A_n} \leqq 1_A$. Since $P^n 1_{A_n} = 1_A, P^n(1_{A_n \cap B_n}) = 1_{A \cap B}$.Hence Σ_t is closed under intersections and complements.

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The above also shows that $A \subset B \Rightarrow A_n \subset B_n$ for every n. Hence, if $B_k \uparrow A$, $B_k \in \Sigma_t$, then the $B_{n,k}$ which satisfy $P^n 1_{b_{n,k}} = 1_{B_k}$ will satisfy $B_{n,k} \subset B_{n,k+1}$. Let

$$A_n = \bigcup_{k=1}^{\infty} B_{n,k}.$$

Then $P^n 1_{A_n} = \lim_k P^n 1_{B_{n,k}} = \lim_k 1_{B_k} = 1_A$ and $A \in \Sigma_t$. Thus Σ_t is an σ-algebra. □

REMARK. Σ_t is called the *asymptotic* σ-algebra. When $Pf(x) = f(\theta x)$ for some nonsingular θ,

$$\Sigma_t = \bigcap_{n=1}^{\infty} \theta^{-n}\Sigma.$$

In that case Σ_t is also called *tail-σ*-algebra.

DEFINITION *For $A \in \Sigma_t$, define $\Psi(A) = A_1$, which is well-defined by proposition 1.4, and maps Σ_t into Σ_t. The proof of theorem 1.5 shows that Ψ is a homomorphism of the σ-algebra. We have $\Psi^n(A) = A_n$ (when $P^n 1_{A_n} - 1_A$), and it is easily verified that $\Psi^n(\Sigma_t) = \Sigma_t \cap \Sigma_n$.*

Remember that P induces a homomorphism of Σ_d, and denote $P(A) = B$ when $P1_A = 1_B$. Then $P^n(A) = B_n$. $P^n(\Sigma_d)$ is a σ–algebra, and $P^{n+1}(\Sigma_d) \subset P^n(\Sigma_d)$.

THEOREM 1.6.

$$\bigcap_{n=0}^{\infty} \Psi^n(\Sigma_t) = \Sigma_t \cap \Sigma_d = \bigcap_{n=0}^{\infty} P^n(\Sigma_d).$$

PROOF. The first equality follows from the above relations $\Psi^n(\Sigma_t) = \Sigma_t \cap \Sigma_n$. Denote $\bigcap_{n=0}^{\infty} P^n(\Sigma_d)$ by Σ_a, so $\Sigma_a \subset \Sigma_d$. Let $A \in \Sigma_a$. Then there exist $A_n \in \Sigma_d$ with $P^n 1_{A_n} = 1_A$. Hence $\Sigma_a \subset \Sigma_t \cap \Sigma_d$.

Let $A \in \Sigma_t \cap \Sigma_d$. $A \in \Sigma_d \Rightarrow P^n 1_A = 1_{B_n}$. $A \in \Sigma_t$ implies that there are $A_k \in \Sigma_t$, with $P1_{A_{k+1}} = 1_{A_k}, A_0 = A$. Then, for $k > n$ we have $P^k 1_{A_n} = P^{k-n} 1_A = 1_{B_{k-n}}$. Since $P^k 1_{A_n} = 1_{A_{n-k}}$ for $k \leqq n$, we have that $A_n \in \Sigma_d$, and $1_A \in P^n(\Sigma_d)$ for every n. Hence $\Sigma_t \cap \Sigma_d \subset \Sigma_a$, and equality holds. □

REMARK. P and Ψ are automorphisms of Σ_a, with $P^{-1} = \Psi$. Σ_a is called the *automorphic* σ-algebra [F2].

It was proved in [L3, lemma C] that if P is conservative and ergodic, the eigenfunctions corresponding to unimodular eigenvalues are Σ_a-measurable.

2. **Results for P having a σ-finite invariant measure.** If m is invariant for P, then P is also a contraction of $L_1(m)$ which preserves integrals. Hence P^* is also a Markov operator in $L_\infty(m)$, $P^*1 = 1$ (since P preserves integrals), and m is invariant for P^*. (See [F1] or [F2] for more details on the dual Markov operator.) We denote $\int fg\, dm$ by $\langle f, g \rangle$, for $|fg| \in L_1(m)$.

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THEOREM 2.1. *Let m be a σ-finite invariant measure for P, and P^* the dual Markov operator. Then $\Sigma_d(P) = \Sigma_t(P^*)$.*

PROOF. Let $A \in \Sigma_d(P)$. Then $P^n 1_A = 1_{B_n}$. Fix n, and let $E_k \uparrow B_n^c$ with $m(E_k) < \infty$. Then

$$\langle 1_A, P^{*n} 1_{E_k} \rangle = \langle P^n 1_A, 1_{E_k} \rangle = \langle 1_{B_n}, 1_{E_k} \rangle = 0,$$

and $P^{*n} 1_{E_k} \leqq 1_{A^c}$. Letting $k \longrightarrow \infty$ we obtain $P^{*n} 1_{B_n^c} \leqq 1_{A^c}$. Hence also $P^{*n} 1_{B_n} \leqq 1_A$, and equality must hold. Hence $A \in \Sigma_t(P^*)$.

For the converse, let $A \in \Sigma_t(P^*)$. Then there are $A_n \in \Sigma_t(P^*)$ with $P^{*n} 1_{A_n} = 1_A$. We prove $P^n 1_A = 1_{A_n}$ using the technique above. □

REMARK. $\Sigma_d(P)$ may be different from $\Sigma_d(P^*) = \Sigma_t(P)$.

COROLLARY 2.2. *Under the above assumptions* (i) $A \in \Sigma_n(P) \Leftrightarrow P^{*n} P^n 1_A = 1_A$.
(ii) $P 1_A = 1_A \Leftrightarrow P^{*n} 1_A = 1_A$. *(Note that $m(A)$ may be infinite.)*

PROOF. (i) If $P^{*n} P^n 1_A = 1_A$, then $P^n 1_A$ is an indicator function by lemma 0. If $A \in \Sigma_n(P), P^n 1_A = 1_A = 1_{B_n}$ implies by the previous proof $P^{*n} P^n 1_A = P^{*n} 1_{B_n} = 1_A$.
(ii) $B_n = A$ in the above shows $P_A^* = 1_A$ if $P 1_A = 1_A$. □

REMARKS. 1. If m is finite, then $P^{*n} P^n$ has m as a finite invariant measure, and for $f \in L_1(X, \Sigma, m)$ we have $P^{*n} P^n f = f \Leftrightarrow f \in L_1(\Sigma_n(P), m)$, becasue $P^{*n} P^n$ is conservative [K, lemma 3.3.3].

2. If m is infinite, we have for $1 \leqq p < \infty$ that $I_{n,p}(P) = \{f \in L_p(\Sigma, m) : P^{*n} P^n f = f\}$ satisfies:
(i) $f \in I_{n,p}(P) \Rightarrow |f| \in I_{n,p}(P)$.
(ii) $f, g \in I_{n,p}(P) \Rightarrow f \vee g, f \wedge g \in I_{n,p}(P)$
(iii) $f \in I_{n,p}, \alpha > 0 \Rightarrow f \wedge \alpha \in I_{n,p}(P)$.

For the proof of (iii) we proceed as in [F1]; ($p = 2$ was not used): Let $h = f \wedge \alpha$. Then $P^{*n} P^n h \leqq P^{*n} P^n f \wedge \alpha = h$. Hence $P^{*n} P^n (f - h) \geqq f - h \geqq 0$, and since $P^{*n} P^n$ is a contraction of L_p, equality holds, and $h \in I_{n,p}(P)$.

It follows that if $f \in L_p^+$ is in $I_{n,p}(P)$, then $1_{\{f > a\}} = \lim_k k(f - a)^+ \wedge 1 \in I_{n.p}(P)$. Thus, $I_{n,p}(P) = L_p(\Sigma_n(P), m)$ for $1 \leqq p < \infty$.

DEFINITION *The distribution of $f \in L_p(m), 1 \leqq p < \infty$, is defined (when m is σ-finite) by $m\{f > t\}$ for $t > 0$, $m\{f < t\}$ for $t < 0$ (which are finite since $f \in L_p$.)*

THEOREM 2.3. *Let m be a σ–finite invariant measure for P, and $f \in L_p(m), 1 \leqq p < \infty$. Then the following are equivalent:*
(i) $P^{*n} P^n f = f$
(ii) $f \in L_p(\Sigma(P), m)$
(iii) *$P^n f$ has the same distribution as f.*
(iv) *$f, Pf, \ldots, P^n f$ have the same distribution.*

PROOF. The equivalence of (i) and (ii) is discussed above.

(ii) $\Rightarrow$(iv). Since $\Sigma_n(P) \subset \Sigma_{n-1}(P) \ldots \subset \Sigma_1(P)$, it is enough to prove (iii) for $n = 1$, then apply it to $P^2, P^3, \ldots, P^n$.

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Let $f \in L_p(\Sigma_1(P), m)$ be a simple function: $f = \Sigma\, a_i 1_{A_i}$ with A_i disjoint in $\Sigma_1(P)$, and $m(A_i) < \infty$. Passing to complements in lemma 0 (i) yields $0 = P1_{A_i \cap A_j} = P1_{A_i} \wedge P1_{A_j}$ for $i \neq j$ and $A_i, A_j \in \Sigma_1$. Hence, if $1_{B_i} = P1_{A_i}$, we have $Pf = \Sigma\, a_i 1_{B_i}$ with disjoint sets B_i. Since $m(B_i) = m(A_i) < \infty$, Pf has the distribution of f.

Let now $f = f^+ - f^-$ be in $L_p(\Sigma_1(P), m), (1 \leqq p < \infty)$. Let $0 \leqq f_k, g_k$ be simple functions in $L_p(\Sigma_1(P))$ with $0 \leqq f_k \uparrow f^+, 0 \leqq g_k \uparrow f^-$. Then $Pf_k \uparrow Pf^+, Pg_k \uparrow Pf$. By theorem 1.1 we have $0 = P(f_k g_k) = (Pf_k)(Pg_k) \underset{k\to\infty}{\longrightarrow} (Pf^+)(Pf^-)$.

Hence $(Pf)^+ = Pf^+, (Pf)^- = Pf^-$. Thus, for $t > 0$, we obtain, by the beginning of the proof,

$$m\{Pf > t\} = m\{Pf^+ > t\} = \lim_k m\{Pf_k > t\} = \lim_k m\{f_k > t\} = m\{f > t\}.$$

Similarly, $m\{Pf < t\} = m\{f < t\}$ for $t < 0$, and Pf and f have the same distribution.

(iv) $\Rightarrow$ (iii) is obvious.

(iii) $\Rightarrow$ (ii) It is enoguh to prove only the case $n = 1$. We note that for any $g \in L_p$, we have $(Pg)^+ \leqq Pg^+$. This applies also to $g \in L_\infty$, and, more generally, to any g with $Pg^\pm$ well defined. Thus, if $a \geqq 0$ and $f \in L_p$, since $P1 = 1$, we have $(Pf - a)^+ = [P(f-a)]^+ \leqq P(f-a)^+$. Since $0 \leqq (f-a)^+ \leqq f^+$, we have $(f-a)^+ \in L_p$. We now assume that f and Pf have the same distribution, i.e., the measures on $\mathbf{R}$ $\mu_i(B) = m\{x : Pf(x) \in B\}$ and $\mu(B) = m\{x : f(x) \in B\}$ are equal. Since P is a contraction in L_p, using the change of variable formula we obtain

$$\begin{aligned}\int [P(f-a)^+]^p dm &\geqq \int \{[P(f-a)]^+\}^p dm = \int [(Pf-a)^+]^p dm \\ &= \int [(t-a)^+]^p d\mu_1(t) = \int [(t-a)^+]^p d\mu(t) = \int [(f-a)^+]^p dm \\ &= \|(f-a)^+\|_p^p \geqq \|P(f-a)^+\|_p^p.\end{aligned}$$

Hence $[P(f-a)]^+ = P(f-a)^+$, for $a \geqq 0$. Now

$$\begin{aligned}P1_{\{f>a\}} &= \lim_{k\to\infty} P[k(f-a)^+ \wedge 1] \\ &\leqq \lim_{k\to\infty} [kP(f-a)^+] \wedge 1 \\ &= \lim_{k\to\infty} [k(Pf-a)^+] \wedge 1 = 1_{\{Pf>a\}}.\end{aligned}$$

For $a > 0$, $m\{f > a\}$ and $m\{Pf > a\}$ are finite and equal. Hence

$$\int P1_{\{f>a\}} dm = \int 1_{\{f>a\}} dm = \int 1_{\{Pf>a\}} dm$$

shows that $P1_{\{f>a\}} = 1_{\{Pf>a\}}$, and $1_{\{f>a\}} \in \Sigma_1$.

For $a < 0$, we apply the above to $-f$. Hence f is Σ_1- measurable. □

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COROLLARY 2.4. *If* $f \in L_p(m), 1 \leqq p < \infty$, *then* $f \in L_p(\Sigma_d(P), m) \Leftrightarrow \{P^n f\}_{n=0}^{\infty}$ *is identically distributed.*

REMARKS. 1. The above corollary is another justification for the term "deterministic".

2. Although $\{P^n f\}$ converges in distribution ([AB], [KL, theorem 3.3]), the example in [R] has a finite invariant measure m, $\Sigma_d(P)$ trivial, and $f \in L_2$ such that $P^n f$ converges in distribution to a non-constant function. Thus, the limiting distribution need not be that of a Σ_d-measurable function.

3. For $p = 2$, the proof of (iii) $\Rightarrow$ (ii) is greatly simplified by the fact that $\|Pf\|_2 = \|f\|_2$, a property which is equivalent to $P^*Pf = f$. For $p = 1$ such a characterisation is false.

4. If m is *not* finite, m need not be σ-finite on Σ_d. We then define $X_1 = ess \sup\{A \in \Sigma_d : m(A) < \infty\}$, and m on $\Sigma_d \cap X_1$ is σ-finite. Our results then concern $f \in L_p(\Sigma_d \cap X_1)$. (since Σ_d is σ algebra, $X_1 \in \Sigma_d$).

3. **The deterministic σ-algebras of convolutions.** In this section we discuss convolution operators in locally compact σ-compact groups. We collect the known results in theorems 3.1 and 3.2. They were part of the motivation for this research. Let Σ be the Baire σ-algebra of a locally compact σ-compact topological group G, and let m be the right Haar measure. If μ is a regular probability on Σ, we define the transition probability $P(x, A) = \mu(x^{-1}A)$ and the Markov operator $Pf(x) = \int f(y)P(x, dy) = \int f(xy)d\mu(y) = \mu * f(x)$. Then m is a σ-finite invariant measure for P. It is finite if and only if G is compact. We denote by $T(x)$ the translation operator (by x).

THEOREM 3.1. *Let G be compact.*
(i) *If* $f \in L_2(m)$, *then* $\|P^n(f - E(f|\Sigma_d(P))\|_2 \longrightarrow 0$
(ii) Σ_d *is the σ-algebra generated by* $\{g \in C(G) : Pg = \lambda g, |\lambda| = 1\}$.

PROOF. (i) The translation operators $[T(y)f](x) = f(xy)$ yield a strongly continuous representation of G in $L_2(m)$, i.e., $y \longrightarrow T(y)f$ is a continuous map from G to $L_2(m)$. Hence $\{T(y)f : y \in G\}$ is strongly compact. By a theorem of Mazur, $\overline{co}\{T(y)f : y \in G\}$ is also strongly compact. Since $P^n f \in \overline{co}\{T(y)f : y \in G\}$, $\{P^n f\}$ is strongly sequentially compact. By theorem B(ii), if $f \perp L_2(\Sigma_d(P), m), P^n f \longrightarrow 0$ weakly. Since it is strongly sequentially compact $\|P^n f\|_2 \longrightarrow 0$.

(ii) We managed to prove (i) without using the Jacobs-Deleeuw-Glicksberg decomposition [K]. We now use it in $C(G)$. The map $y \longrightarrow T(y)f$ is continuous from G into $C(G)$ when $f \in C(G)$. Hence, as above, $\{P^n f\}$ is strongly sequentially compact. By the decomposition theorem, $C(G) = C_0 \oplus C_1$, where C_1 is generated by $\{g \in C(G) : Pg = \lambda g, |\lambda| = 1\}$, and, for $f \in C_0(G)$, $\|p^n f\|_\infty \longrightarrow 0$. By (i) we have $C_1 \subset L_2(\Sigma_d(P))$ and $C_0 \perp L_2(\Sigma_d(P))$. Some approximation arguments yield the result. □

THEOREM 3.2. *Let G be Abelian.*

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(i) *If* $f \in L_1(m)$ *with* $\int_A f\,dm = 0$ *for* $\forall A \in \Sigma_d$ *then* $\|P^n f\|_1 \longrightarrow 0$.

(ii) Σ_d *is the* σ*-algebra generated by the continuous characters* $\{\Upsilon \in \hat{G} : |\hat{\mu}(\Upsilon)| = 1\}$.

This is the result of [DL]. (The details of the proof of (ii) appear in [L4]. It is also shown there that

$$\bigcap_{n=1}^{\infty} P^{*n}\{0 \leqq f \leqq 1\} = \{0 \leqq f \leqq 1 : P^*Pf = f\},$$

and that this set is contained in $L_\infty(\Sigma_d)$.)

We note that when G is Abelian, P and P^* commute. Hence $P^{*n}P^n = (P^*P)^n$ converges strongly (to a projection on the fixed points of P^*P). Thus, for $f \perp \{g \in L_2 : P^*Pg = g\}$ we have $P^{*n}P^n f \longrightarrow 0$, hence $\|P^n f\|_2 \longrightarrow 0$.

EXAMPLE. Bougerol [B] constructed an example of G (non-Abelian, of course), μ non-singular on G *adapted* (i.e., such that the support S of μ generates G as a topological group), S is not contained in a class of any compact normal subgroup, but for some $0 \leqq f$ continuous with compact support $\lim \|P^n f\|_\infty > 0$. It can be proved that necessarily $\lim_n \|P^n f\|_2 > 0$.

Inspecting the example, we find that the closed group H generated by $S^{-1}S$ is *normal*. Suppose $0 \neq g \in L_2(m)$ satisfies $P^*Pg = g$. Without loss of generality, $g \geqq 0$, and by regularization we may assume g continuous, vanishing at ∞, and $g(e) \neq 0$ (where e is the unit in G). Then $P^*Pg = g$ implies $g(xy) = g(x)$ for every $y \in S^{-1}S$ (P^* is given by $\check{\mu}(A) = \mu(A^{-1})$, and P^*P by $\check{\mu} * \mu$, whose support is $\overline{S^{-1}S}$). Hence $G_1 = \{y : g(xy) = g(x)\forall x\}$ is a closed subgroup containing $S^{-1}S$, so it is not compact. But $G_1 \subset \{y : g(y) = g(e) \neq 0\}$, which is compact – a contradiction. Hence $P^*Pg = g \in L_2$ implies $g \equiv 0$, and therefore the isometric part of P is trivial (Σ_d contains only sets of measure zero or infinity. It is *not* trivial in this example). Since $\lim \|P^n f\|_2 > 0$ this example shows that we do not necessarily have strong convergence in theorem B(ii) (quoted in the introduction) for convolution operators in general locally compact groups, although it holds in compact and Abelian groups.

In contrast to the above example (in which P is transient), we have the following.

THEOREM 3.3. *Let* μ *be adapted on* G *non-compact. If* P *is recurrent, then* $\|P^n f\|_2 \underset{n\to\infty}{\longrightarrow} 0$ *for every* $f \in L_2$.

PROOF. Derriennic [D] proved that $P^n f(x)$ converges to zero *everywhere*, for f continuous with compact support. Since P is recurrent, we apply [L2] to complete the proof. □

The main idea of [D] is to use the fact that a recurrent random walk is topologically *irreducible* (i.e., P has no closed sets which are absorbing). In terms of μ, this means that the closed *semigroup* generated by the support of μ is all of G.

PROPOSITION 3.4. *Let μ be a probability on a locally compact group G. If $P1_A = 1_B$ (in $L_\infty(m)$), then $T(y)1_A = 1_B$ for every y in the support of μ.*

PROOF. There is a set N with $m(N) = 0$ such that for $x \notin N$ we have $1_B(x) = P1_A(x) = \int 1_A(xy)du(y)$. Hence for $x \notin N$ we have $1_A(xy) = 1_B(x)$ for μ-a.e.y., or $\int |1_B(x) - 1_A(xy)|d\mu(y) = 0$ for $x \notin N$. Hence

$$\int \left[\int |T(y)1_A(x) - 1_B(x)|dm(x) \right] d\mu(y) = \int \left[\int |1_A(xy) - 1_B(y)|d\mu(y) \right] dm(x) = 0.$$

Hence $T(y)1_A = 1_B$ (in L_∞) for μ-a.e.y. Since the representation by translations in L_1 is continuous, the representation in L_∞ is weak* continuous. Hence $T(y)1_A = 1_B$ for every y in the support of μ. □

LEMMA 3.5.[W] *Let μ be irreducible with suppport S. Then H, the closed normal subgroup generated by SS^{-1}, equals the closed subgroup generated by $\bigcup_{n=1}^{\infty} S^nS^{-n}$, (and it also equals the closed subgroup generated by $\bigcup_{n=1}^{\infty} S^{-n}S^n$.)*

THEOREM 3.6. *Let P be irreducible. Then*
(i) $\Sigma_t(P) = \Sigma_d(P) = \{A : T(y)1_A = 1_A \forall y \in H\}$
(ii) *Σ_d is trivial $\Leftrightarrow H = G$.*

PROOF. (i) By the lemma, $T(y)1_A = 1_A \forall y \in H$ implies $P^{*n}P^n1_A = 1_A$ and $P^nP^{*n}1_A = 1_A$ for every n. Hence $\Sigma' \equiv \{A : T(y)1_A = 1_A \forall y \in H\} \subset \Sigma_d \cap \Sigma_t$.

If $A \in \Sigma_d$, proposition 3.4 implies $T(y)1_A = 1_{B_n}$ for y in S^n and $T(y)1_A = 1_A$ for $y \in S^{-n}S^n$, and, by lemma 3.5, $T(y)1_A = 1_A$ for $y \in H$. Hence $\Sigma_d \subset \Sigma'$, and $\Sigma_d = \Sigma'$.

If $A \in \Sigma_t$, then $P^n1_{A_n} = 1_A$ implies by proposition 3.4 that $T(y)1_A = 1_A$ for $y \in S^nS^{-n}$, hence, by lemma 3.5, for $y \in H$, and $\Sigma_t = \Sigma'$.

(ii) Let $H \neq G$. Since H is a normal subgroup, G/H is a locally compact group, with Haar measure $\hat{m}$. $G/H \neq \{e\}$, so there is $B \subset G/H$ open which is $\hat{m}$ non trivial. Let π be the canonial map of G onto G/H. Define $A = \pi^{-1}(B)$. Then $m(A) \neq 0$, $m(A^c) \neq 0$, so A is non trivial. By the definition, $x \in A \Rightarrow xH \subset A, x \in A^c \Rightarrow xH \subset A^c$, so $T(y)1_A = 1_A$ for $y \in H$, and $A \in \Sigma_d$. Hence Σ_d is not trivial.

Let $H = G$. If $A \in \Sigma_d$, then $T(y)1_A = 1_A$ for every $y \in G$. Hence A is trivial. □

COROLLARY 3.7. *If G is not compact and P is irreducible, then for $A \in \Sigma_d$ we have $m(A)$ zero or infinity.*

PROOF. Derriennic [D] showed that H cannot be compact.

We wish to thank the referee for his useful comments.

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