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SOME RELATIONS BETWEEN DIFFERENTIAL GEOMETRIC INVARIANTS AND TOPOLOGICAL INVARIANTS OF SUBMANIFOLDS[1)]

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§ 1. Introduction.

Let M be an n-dimensional manifold immersed in an m-dimensional euclidean space E^m and let ∇ and $\tilde{\nabla}$ be the covariant differentiations of M and E^m, respectively. Let X and Y be two tangent vector fields on M. Then the second fundamental form h is given by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y) . \tag{1.1}$$

It is well-known that $h(X, Y)$ is a normal vector field on M and it is symmetric on X and Y. Let ξ be a normal vector field on M, we write

$$\tilde{\nabla}_X \xi = -A_\xi(X) + D_X \xi , \tag{1.2}$$

where $-A_\xi(X)$ and $D_X\xi$ denote the tangential and normal components of $\tilde{\nabla}_X\xi$. Then we have

$$\langle A_\xi(X), Y\rangle = \langle h(X, Y), \xi\rangle , \tag{1.3}$$

where $\langle\ ,\ \rangle$ denotes the scalar product in E^m. The mean curvature vector H is defined by $H = (1/n)\,\mathrm{trace}\, h$. Let S denote the length of h and α the length of H.

In this paper we shall obtain some relations between differential geometric invariants and a topological invariants of M. In particular, we shall prove that, for any closed n-dimensional submanifold M in E^m, the geometric invariant given by the integral of S^n depends on a topol-

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ogical structure of M. Moreover, if the submanifold is δ-pinching in E^m (for the definition, see § 4), then the total mean curvature, i.e., the geometric invariant given by the integral of α^n, also depends on the same topological structure of M. In particular, we see that among all δ-pinching submanifolds in E^m with a fixed $\delta > -1$, the submanifolds with large homology groups must have large total mean curvature.

§ 2. Basic formulas.

Let ξ be a unit normal vector field on M. We define the i-th mean curvature $K_i(\xi)$ at ξ by

$$\det(I + tA_\xi) = 1 + \sum_{i=1}^{n} \binom{n}{i} K_i(\xi) t^i \,, \tag{2.1}$$

where I is the identity transformation of the tangent spaces of M, t a parameter and $\binom{n}{i} = n!/i!(n-i)!$. Let R be the curvature tensor of M, i.e.,

$$R(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]} \,.$$

Then the Gauss equation is given by

$$\langle R(X, Y)Z, W\rangle = \langle h(Y, Z), h(X, W)\rangle - \langle h(X, Z), h(Y, W)\rangle \,. \tag{2.2}$$

Let $E_1, \cdots, E_n$ be local orthonormal tangent vector fields of M. Then the scalar curvature ρ is defined by

$$\rho = \sum_{j=1}^{n} \left(\sum_{i=1}^{n} \langle R(E_i, E_j)E_j, E_i\rangle \right) . \tag{2.3}$$

From (2.2) and (2.3) we have

$$\rho = n^2\alpha^2 - S^2 \,. \tag{2.4}$$

§ 3. Integral inequality for S^n.

Let $\mathscr{F}$ be a field and let $H_i(M; \mathscr{F})$ be the i-th homology group of M over the field $\mathscr{F}$. Let $\beta_i(M; \mathscr{F})$ be the dimension of the i-th homology group $H_i(M; \mathscr{F})$. We define a topological invariant $\beta(M)$ by

$$\beta(M) = \max \left\{ \sum_{i=0}^{n} \beta_i(M; \mathscr{F}) : \mathscr{F} \text{ fields} \right\} .$$

The main aim of this section is to prove the following.

THEOREM 1. *Let M be an n-dimensional closed manifold immersed in a euclidean m-space E^m. Then we have*

$$\int_M S^n dV \geqq \left\{\left(\frac{n}{2}\right)^{n/2} c_n\right\}\beta(M) , \tag{3.1}$$

where c_n is the area of a unit n-sphere. The equality sign of (3.1) *holds when and only when M is diffeomorphic to an n-sphere and M is imbedded as a hypersphere of an $(n+1)$-dimensional linear subspace of E^m.*

Proof. Let M be an n-dimensional closed manifold immersed in E^m and ξ be any unit normal vector field on M. We denote by $S(\xi)$ the length of the second fundamental tensor A_ξ at ξ. Let $\xi_1, \cdots, \xi_{m-n}$ be local orthonormal normal vector fields of M in E^m and $\xi = \sum_{r=1}^{m-n} \cos \gamma_r \xi_r$. Then we have

$$A_\xi = \sum \cos \gamma_r A_r , \qquad A_r = A_{\xi_r} . \tag{3.2}$$

Hence we have

$$S(\xi)^2 = \operatorname{trace}(A_\xi^2) = \sum_{r,s} \cos \gamma_r \cos \gamma_s \operatorname{trace}(A_r A_s) . \tag{3.3}$$

The right hand side of (3.3) is a quadratic form on $\cos \gamma_1, \cdots, \cos \gamma_{m-n}$. Hence, we may choose local orthonormal normal vector fields $\bar{\xi}_1, \cdots, \bar{\xi}_{m-n}$ such that with respect to this frame field, we have

$$S(\xi)^2 = \sum \rho_r \cos^2 \gamma_r , \qquad \rho_1 \geqq \rho_2 \geqq \cdots \geqq \rho_{m-n} \geqq 0 , \tag{3.4}$$

$$\rho_r = \operatorname{trace}(A_r^2) = S(\xi_r)^2 . \tag{3.5}$$

By the definition of S and ρ_r we have

$$S^2 = \rho_1 + \cdots + \rho_{m-n} . \tag{3.6}$$

In the following, let B_ν be the bundle of unit normal vectors of M in E^m so that a point of B_ν is a pair (x, ξ) where ξ is a unit normal vector at the point x in M. Then B_ν is a bundle of $(m-n-1)$-dimensional spheres over M and is a manifold of dimension $m-1$. Let Σ_x be the fibre of B_ν over x. Then there is a differential form $d\sigma$ of degree $m-n-1$ on B_ν such that its restriction to a fibre Σ_x is the volume element $d\Sigma_x$ of Σ_x. Hence $d\sigma \wedge dV$ is the volume element of the bundle B_ν. On the bundle B_ν we define a function f by

$$f(x, \xi) = S(\xi)^2 . \tag{3.7}$$

For $\xi = \sum \cos \gamma_r \bar{\xi}_r$ we have

$$f(x, \xi) = \sum \rho_r \cos^2 \gamma_r \ . \tag{3.8}$$

Since ρ_r, $r = 1, \cdots, m-n$, are nonnegative and $\sum_r \cos^2 \gamma_r = 1$, an inequality of Minkowski [1, p. 21] implies that

$$\begin{aligned} \left\{\int_{\Sigma_x} f^{n/2} d\Sigma_x\right\}^{2/n} &= \left\{\int_{\Sigma_x} (\sum \rho_r \cos^2 \gamma_r)^{n/2} d\Sigma_x\right\}^{2/n} \\ &\leqq \sum \left\{\rho_r \left(\int_{\Sigma_x} |\cos^n \gamma_r|\, d\Sigma_x\right)^{2/n}\right\} . \end{aligned} \tag{3.9}$$

Moreover, we have the following identity:

$$\int_{\Sigma_x} |\cos^n \gamma_r|\, d\Sigma_x = 2c_{n+p-1}/c_n \ . \tag{3.10}$$

Thus, by combining (3.6), (3.9) and (3.10), we find

$$S^n \geqq \frac{c_n}{2c_{m-1}} \int_{\Sigma_x} f^{n/2} d\Sigma_x \ . \tag{3.11}$$

On the other hand, from the definition of $K_n(\xi)$ and an elementary relation between elementary symmetric functions, we have $S(\xi)^n \geqq \sqrt{n^n}\,|K_n(\xi)|$. Hence, by using (3.11), we see that

$$\int_M S^n dV \geqq \sqrt{n^n}\, \frac{c_n}{2c_{m-1}} \int_{B_\nu} |K_n(\xi)|\, d\sigma \wedge dV \ . \tag{3.12}$$

By a well-known inequality of Chern-Lashof [4, II], we have

$$\int_{B_\nu} |K_n(\xi)|\, d\sigma \wedge dV \geqq c_{m-1}\beta(M) \ . \tag{3.13}$$

Thus, by combining (3.12) and (3.13), we obtain (3.1).

The remaining part of this theorem can be proved in a similar way as the corresponding results of Theorem 4.2 in [2, p. 229]. So we omit it.

Remark 1. Theorem 1 generalizes Theorem 4.1 of [3, II]. First, Theorem 1 drops the assumption of nonnegativeness of the scalar curvature of M. Second, if n is odd, the estimation is better than the one given in Theorem 4.1 of [3, II].

§ 4. Total mean curvature.

From Proposition 2.2 of [3, II] we see that the scalar curvature ρ is always bounded from above by $(n-1)S^2$ and bounded below by $-S^2$, i.e.,

$$-S^2 \leqq \rho \leqq (n-1)S^2 . \tag{4.1}$$

In the following, a submanifold M in E^m is said to satisfy a *δ-pinching in E^m* if we have

$$\delta S^2 \leqq \rho \leqq (n-1)S^2$$

for some $\delta \geqq -1$.

THEOREM 2. *Let M be an n-dimensional closed manifold immersed in a euclidean m-space E^m. If M satisfies a δ-pinching in E^m, then we have*

$$\int_M \alpha^n dV \geqq \left\{\frac{1}{2}\left(\frac{1+\delta}{n}\right)^{n/2} c_n\right\}\beta(M) . \tag{4.2}$$

The equality sign of (4.2) *holds when and only when M is $(n-1)$-pinching in E^m.*

Proof. If M is δ-pinching in E^m, then (2.4) implies

$$\alpha^2 \geqq \frac{1+\delta}{2} S^2 . \tag{4.3}$$

Hence, by combining Theorem 1 and (4.3) we obtain (4.2).

Now, if the equality sign of (4.2) holds, then the equality sign of (3.1) holds. Hence, Theorem 1 implies that M is imbedded as a hypersphere of an $(n+1)$-dimensional linear subspace of E^m. In this case we have $n^2\alpha^2 = nS^2$. Hence, by (2.4), we see that M is $(n-1)$-pinching in E^m. The remaining part of this Theorem is trivial.

Remark 2. If $\delta > -1$ and M is a minimal submanifold of a unit hypersphere of E^m, then M is δ-pinching in E^m when and only when the scalar curvature ρ of M satisfies the following inequality:

$$\rho \geqq \frac{\delta}{1+\delta} n^2 .$$

In this case, $\int_M \alpha^n dV$ is equal to the volume of M.

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