

GREEN'S RELATIONS AND REGULARITY IN CENTRALIZERS OF PERMUTATIONS

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Abstract. For any permutation σ on $N = \{1, 2, \dots, n\}$: (i) Green's relations are characterized in the centralizer $C(\sigma)$ of σ (relative to the semigroup PT_n of partial transformations on N); and (ii) a criterion is given for $C(\sigma)$ to be a regular semigroup (inverse semigroup, union of groups).

1. Introduction. Let PT_n denote the semigroup of partial transformations on $N = \{1, 2, \dots, n\}$, and let S_n denote the symmetric group of permutations on N , the group of units of PT_n . For $\gamma \in PT_n$, the set

$$C(\gamma) = \{\alpha \in PT_n : \alpha\gamma = \gamma\alpha\}$$

is a subsemigroup of PT_n , called the *centralizer* of γ .

Centralizers of partial transformations are studied in [3], where the elements of $C(\gamma)$ are characterized. It is shown in [7] that for a permutation $\sigma \in S_n$, $C(\sigma)$ can be embedded into a wreath product of two semigroups determined by the number and length of cycles in σ . Centralizers in some subsemigroups of PT_n have also been studied. A structure of centralizers in the symmetric group S_n is presented in [8]. A representation and order of centralizers in the symmetric inverse semigroup I_n are given in [4] and [5]. A construction of centralizers in the full transformation semigroup T_n is presented in [1]. Many results from the above references are collected in [6].

In this paper, we study centralizers of permutations in PT_n . Section 2 introduces notation, definitions, and some preliminary results. In Section 3, Green's relations in $C(\sigma)$ (for any $\sigma \in S_n$) are determined. Section 4 characterizes the permutations $\sigma \in S_n$ whose centralizer $C(\sigma)$ is a regular semigroup (inverse semigroup, union of groups). In particular, we find that $C(\sigma)$ is an inverse semigroup if and only if it is a union of groups. As an illustration, the egg-box structure of a specific centralizer is presented (Section 5).

2. Preliminary results. For $\alpha \in PT_n$, the domain and range of α will be denoted by $\text{dom } \alpha$ and $\text{ran } \alpha$, respectively. The *kernel* of α , denoted by $\ker \alpha$, is the equivalence relation on $\text{dom } \alpha$ defined by $x (\ker \alpha) y \iff x\alpha = y\alpha$. Denoting by $\text{dom } \alpha / \ker \alpha$ the partition of $\text{dom } \alpha$ induced by $\ker \alpha$, we have $|\text{dom } \alpha / \ker \alpha| = |\text{ran } \alpha|$. This common cardinality of $\text{dom } \alpha / \ker \alpha$ and $\text{ran } \alpha$ is called the *rank* of α and denoted $\text{rank } \alpha$. For example, for

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & 3 & 4 & - & 4 & 3 & 1 \end{pmatrix} \in PT_7,$$

$\text{dom } \alpha = \{1,2,3,5,6,7\}$, $\text{ran } \alpha = \{1,3,4\}$, $\text{ker } \alpha = | 1 \ 3 \ 5 | 2 \ 6 | 7 |$ (we identify $\text{ker } \alpha$ with $\text{dom } \alpha/\text{ker } \alpha$), and $\text{rank } \alpha = 3$.

Throughout the paper, we shall use the following characterization of the elements of $C(\sigma)$ ($\sigma \in S_n$), which is a special case of [3, Theorem 5].

THEOREM 2.1. *Let $\sigma \in S_n$ and $\alpha \in PT_n$. Then $\alpha \in C(\sigma)$ if and only if for every cycle $(x_0x_1 \dots x_{k-1})$ in σ such that some $x_i \in \text{dom } \alpha$, the following conditions are satisfied:*

- (i) $\{x_0, x_1, \dots, x_{k-1}\} \subseteq \text{dom } \alpha$;
- (ii) *there is a cycle $(y_0y_1 \dots y_{m-1})$ in σ such that m divides k and for some index j ,*

$$x_0\alpha = y_j, \quad x_1\alpha = y_{j+1}, \quad x_2\alpha = y_{j+2}, \dots,$$

where the subscripts on y s are calculated modulo m . \square

Let $\sigma \in S_n$ be a permutation with cycle decomposition $\sigma = a_1 \dots a_t$ (1-cycles included). For $\alpha \in C(\sigma)$, define a partial transformation t_α on the set $A = \{a_1, \dots, a_t\}$ of the cycles in σ by:

- (1) $\text{dom } t_\alpha$ consists of all cycles $a = (x_0x_1 \dots x_{k-1}) \in A$ such that some x_i is in $\text{dom } \alpha$;
- (2) for each $a = (x_0x_1 \dots x_{k-1}) \in \text{dom } t_\alpha$ and each $b = (y_0y_1 \dots y_{m-1}) \in A$

$$at_\alpha = b \iff x_i\alpha = y_j \text{ for some } x_i \text{ and some } y_j.$$

By Theorem 2.1, t_α is well defined. Speaking informally, $at_\alpha = b$ if α wraps cycle a around cycle b one or more times. As an example, consider the permutation $\sigma = abc = (1 \ 2)(3 \ 4 \ 5)(6 \ 7 \ 8 \ 9)$ in S_9 and $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ - & - & 4 & 5 & 3 & 1 & 2 & 1 & 2 \end{pmatrix} \in C(\sigma)$. Then $t_\alpha = \begin{pmatrix} a & b & c \\ - & b & a \end{pmatrix}$.

For a cycle a , $\ell(a)$ will denote the length of a . For example, if $a = (1 \ 2 \ 3)$, then $\ell(a) = 3$.

We shall frequently use the following lemma.

LEMMA 2.2. *If $\sigma \in S_n$, $a = (x_0x_1 \dots x_{k-1})$ and $b = (y_0y_1 \dots y_{m-1})$ are cycles in σ , and $\alpha, \beta \in C(\sigma)$, then:*

- (1) $t_{\alpha\beta} = t_\alpha t_\beta$;
- (2) if $at_\alpha = b$ then $\ell(b)$ divides $\ell(a)$;
- (3) $b \in \text{ran } t_\alpha$ if and only if $\{y_0, y_1, \dots, y_{m-1}\} \subseteq \text{ran } \alpha$;
- (4) $at_\alpha = bt_\alpha$ if and only if $x_i\alpha = y_j\alpha$ for some x_i and some y_j .

Proof. Immediate by the definition of t_α and Theorem 2.1. \square

3. Green's relations. If S is a semigroup and $a, b \in S$, we say that $a \mathcal{L} b$ if $S^1a = S^1b$, $a \mathcal{R} b$ if $aS^1 = bS^1$, and $a \mathcal{J} b$ if $S^1aS^1 = S^1bS^1$, where S^1 is the semigroup S with an identity adjoined. We define \mathcal{H} as the intersection of \mathcal{L} and \mathcal{R} , and \mathcal{D} as the join of \mathcal{L} and \mathcal{R} , i.e., the smallest equivalence containing both \mathcal{L} and \mathcal{R} . These five equivalences are known as *Green's relations* [2, p. 45]. The relations \mathcal{L} and \mathcal{R}

commute, and consequently $\mathcal{D} = \mathcal{L} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{L}$. If S is finite then $\mathcal{D} = \mathcal{J}$. For $a \in S$, we denote the equivalence classes of a with respect to $\mathcal{L}, \mathcal{R}, \mathcal{J}, \mathcal{H}$, and \mathcal{D} by L_a, R_a, J_a, H_a , and D_a , respectively.

Green's relations in the semigroup PT_n are well known [2, Exercise 17, p. 63].

LEMMA 3.1. *If $\alpha, \beta \in PT_n$, then the following hold.*

- (1) $\alpha \mathcal{L} \beta \iff \text{ran } \alpha = \text{ran } \beta$.
- (2) $\alpha \mathcal{R} \beta \iff \text{ker } \alpha = \text{ker } \beta$.
- (3) $\alpha \mathcal{H} \beta \iff \text{ran } \alpha = \text{ran } \beta \text{ and } \text{ker } \alpha = \text{ker } \beta$.
- (4) $\alpha \mathcal{D} \beta \iff \text{rank } \alpha = \text{rank } \beta$. \square

A description of Green's relations in $C(\sigma)$ ($\sigma \in S_n$) will involve t_α ($\alpha \in C(\sigma)$). The following lemma clarifies the relation between the range and kernel of α and t_α .

LEMMA 3.2. *If $\sigma \in S_n$ and $\alpha, \beta \in C(\sigma)$, then*

- (1) $\text{ran } \alpha = \text{ran } \beta \iff \text{ran } t_\alpha = \text{ran } t_\beta$,
- (2) $\text{ker } \alpha = \text{ker } \beta \implies \text{ker } t_\alpha = \text{ker } t_\beta$.

Proof. Statement (1) follows from (3) of Lemma 2.2 and Theorem 2.1. To show (2), suppose $\text{ker } \alpha = \text{ker } \beta$. Let $a = (x_0x_1 \dots x_{k-1})$ and $b = (y_0y_1 \dots y_{m-1})$ be cycles in σ . Then,

$$\begin{aligned} (a, b) \in \text{ker } t_\alpha &\iff at_\alpha = bt_\alpha \\ &\iff x_i\alpha = y_j\alpha \text{ for some } x_i \text{ and some } y_j \text{ (by (4) of Lemma 2.2)} \\ &\iff x_i\beta = y_j\beta \text{ (since } \text{ker } \alpha = \text{ker } \beta) \\ &\iff at_\beta = bt_\beta \text{ (by (4) of Lemma 2.2)} \\ &\iff (a, b) \in \text{ker } t_\beta. \quad \square \end{aligned}$$

The implication in (2) cannot be reversed. For example, if $\sigma = ab = (1\ 2)(3) \in S_3$, then for $\alpha = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & - \end{pmatrix}$ and $\beta = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 3 & - \end{pmatrix}$ in $C(\sigma)$, we have $t_\alpha = \begin{pmatrix} a & b \\ a & - \end{pmatrix}$ and $t_\beta = \begin{pmatrix} a & b \\ b & - \end{pmatrix}$. Thus $\text{ker } t_\alpha = \text{ker } t_\beta = |a|$, but $\text{ker } \alpha = |1\ 2|$ is different from $\text{ker } \beta = |1\ 2|$.

There is no corresponding result for ranks. It is possible to have $\alpha, \beta \in C(\sigma)$ with $\text{rank } \alpha = \text{rank } \beta$ but $\text{rank } t_\alpha \neq \text{rank } t_\beta$ as well as with $\text{rank } t_\alpha = \text{rank } t_\beta$ but $\text{rank } \alpha \neq \text{rank } \beta$.

For $\sigma \in S_n$, $\alpha \in C(\sigma)$, and $b \in \text{ran } t_\alpha$, we denote by $t_\alpha^{-1}(b)$ the set of all cycles $a \in \text{dom } t_\alpha$ such that $at_\alpha = b$.

The following theorem characterizes Green's \mathcal{L} relation in $C(\sigma)$.

THEOREM 3.3. *Let $\sigma \in S_n$ and let $\alpha, \beta \in C(\sigma)$. Then, $\alpha \mathcal{L} \beta$ (in $C(\sigma)$) if and only if the following conditions are satisfied:*

- (1) $\text{ran } t_\alpha = \text{ran } t_\beta$;
- (2) for every $c \in \text{ran } t_\alpha = \text{ran } t_\beta$:

- (a) if $a \in t_\alpha^{-1}(c)$, then there exists $b \in t_\beta^{-1}(c)$ such that $\ell(b)$ divides $\ell(a)$;
- (b) if $a \in t_\beta^{-1}(c)$, then there exists $b \in t_\alpha^{-1}(c)$ such that $\ell(b)$ divides $\ell(a)$.

Proof. Suppose $\alpha \mathcal{L} \beta$. Then $\alpha \mathcal{L} \beta$ in PT_n and so (1) holds by (1) of Lemma 3.1 and (1) of Lemma 3.2. To show (2)(a), suppose $c \in \text{ran } t_\alpha = \text{ran } t_\beta$ and let $a \in t_\alpha^{-1}(c)$. Since $\alpha \mathcal{L} \beta$, we have $\alpha = \gamma\beta$ for some $\gamma \in C(\sigma)$ and so, by (1) of Lemma 2.2, $t_\alpha = t_\gamma t_\beta$. Since $at_\alpha = c$, there is a cycle b in σ such that $at_\gamma = b$ and $bt_\beta = c$. Thus $\ell(b)$ divides $\ell(a)$ (by (2) of Lemma 2.2) and $b \in t_\beta^{-1}(c)$. The condition 2(b) follows by symmetry.

Conversely, suppose (1) and (2) hold. We shall construct $\gamma \in C(\sigma)$ such that $\alpha = \gamma\beta$. First, we set $\text{dom } \gamma = \text{dom } \alpha$. To define the values of γ , let $a = (x_0x_1 \dots x_{k-1})$ be a cycle in σ with $a \in \text{dom } t_\alpha$, and let $c = (y_0y_1 \dots y_{m-1}) = at_\alpha$. By Theorem 2.1, m divides k and for some index j ,

$$x_0\alpha = y_j, \quad x_1\alpha = y_{j+1}, \quad x_2\alpha = y_{j+2}, \dots,$$

where the subscripts on y s are calculated modulo m . By (1) and (2)(a), $c \in \text{ran } t_\beta$ and there is $b = (w_0w_1 \dots w_{p-1}) \in \text{dom } t_\beta$ such that $bt_\beta = c$ and p divides k . By Theorem 2.1, m divides p and for some index i ,

$$w_0\beta = y_i, \quad w_1\beta = y_{i+1}, \quad w_2\beta = y_{i+2}, \dots,$$

where the subscripts on y s are calculated modulo m . Let $u \in \{0, 1, \dots, p-1\}$ be an index such that $w_u\beta = y_j$. Since p divides k , we may define

$$x_0\gamma = w_u, \quad x_1\gamma = w_{u+1}, \quad x_2\gamma = w_{u+2}, \dots,$$

where the subscripts on w s are calculated modulo p . By the construction of γ and Theorem 2.1, we have $\alpha = \gamma\beta$ and $\gamma \in C(\sigma)$. By symmetry, there is $\delta \in C(\sigma)$ such that $\beta = \delta\alpha$, which concludes the proof. \square

To illustrate Theorem 3.3, let $\sigma = abcd = (1\ 2)(3\ 4\ 5)(6)(7) \in S_7$ and consider $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 6 & 6 & 7 & 7 & 7 & 6 & 6 \end{pmatrix}$ and $\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 7 & 7 & 6 & 6 & 6 & 7 & - \end{pmatrix}$ in $C(\sigma)$. Calculating $t_\alpha = \begin{pmatrix} a & b & c & d \\ c & d & c & c \end{pmatrix}$ and $t_\beta = \begin{pmatrix} a & b & c & d \\ c & c & d & - \end{pmatrix}$, we see that (1) of Theorem 3.3 holds, but (2) does not hold. Indeed, $a \in t_\alpha^{-1}(c)$ and $\ell(a) = 2$, but the only element of $t_\beta^{-1}(c)$ is b , for which $\ell(b) = 3$. Hence, α and β are not in the same \mathcal{L} -class in $C(\sigma)$. Note, however, that $\alpha \mathcal{L} \beta$ in PT_n since $\text{ran } \alpha = \text{ran } \beta$.

For any integers i and m , $m \geq 1$, we denote by $(i)_m$ the unique integer j such that $i \equiv j \pmod{m}$ and $0 \leq j \leq m-1$.

Unlike the \mathcal{L} relation, Green's \mathcal{R} relation in $C(\sigma)$ is simply the restriction of the \mathcal{R} relation in PT_n to $C(\sigma) \times C(\sigma)$.

THEOREM 3.4. *Let $\sigma \in S_n$ and let $\alpha, \beta \in C(\sigma)$. Then $\alpha \mathcal{R} \beta$ (in $C(\sigma)$) if and only if $\ker \alpha = \ker \beta$.*

Proof. If $\alpha \mathcal{R} \beta$ in $C(\sigma)$, then $\alpha \mathcal{R} \beta$ in PT_n and so $\ker \alpha = \ker \beta$ by (2) of Lemma 3.1. Conversely, suppose $\ker \alpha = \ker \beta$. We shall construct $\gamma \in C(\sigma)$ such that $\alpha\gamma = \beta$. First, we set $\text{dom } \gamma = \text{ran } \alpha$. To define the values of γ , let

$b = (y_0y_1 \dots y_{m-1}) \in \text{ran } t_\alpha$ and let $a = (x_0x_1 \dots x_{k-1})$ be a cycle in $\text{dom } t_\alpha$ such that $at_\alpha = b$. By Theorem 2.1, m divides k and for some index j ,

$$x_0\alpha = y_j, \quad x_1\alpha = y_{j+1}, \quad x_2\alpha = y_{j+2}, \dots,$$

where the subscripts on y s are calculated modulo m . Since $\ker \alpha = \ker \beta$, we have $\ker t_\alpha = \ker t_\beta$ by (2) of Lemma 3.2, which implies $\text{dom } t_\alpha = \text{dom } t_\beta$. Thus $a \in \text{dom } t_\beta$ and let $c = (z_0z_1 \dots z_{p-1}) = at_\beta$. By Theorem 2.1, p divides k and for some index i ,

$$x_0\beta = z_i, \quad x_1\beta = z_{i+1}, \quad x_2\beta = z_{i+2}, \dots,$$

where the subscripts on z s are calculated modulo p . Note that $\ker \alpha = \ker \beta$ implies $m = p$. (Indeed, if, say, $m < p$, then $x_0\alpha = x_m\alpha = y_j$, implying $z_i = x_0\beta = x_m\beta = z_{(i+m)_p}$, which is a contradiction since for $m < p$, $z_i \neq z_{(i+m)_p}$.) Thus we may define

$$y_j\gamma = z_i, \quad y_{j+1}\gamma = z_{i+1}, \quad y_{j+2}\gamma = z_{i+2}, \dots,$$

where the subscripts on y s and on z s are calculated modulo $m (= p)$. By the construction of γ and Theorem 2.1, $\gamma \in C(\sigma)$. It remains to show that $\alpha\gamma = \beta$. Since $\text{dom } \gamma = \text{ran } \alpha$ and $\text{dom } \alpha = \text{dom } \beta$, we have $\text{dom } (\alpha\gamma) = \text{dom } \beta$. Let $w \in \text{dom } (\alpha\gamma) = \text{dom } \beta$. Then there is $d = (w_0w_1 \dots w_{q-1}) \in \text{dom } t_\alpha$ such that $w = w_s$ for some index s . Let $b = (y_0y_1 \dots y_{m-1}) = dt_\alpha$, $a = (x_0x_1 \dots x_{k-1})$, and $c = (z_0z_1 \dots z_{p-1})$ be the cycles used in the construction of γ . Let $y_v = w_s\alpha$ ($v \in \{0, 1, \dots, m-1\}$) and let u be the unique number in $\{0, 1, \dots, m-1\}$ such that $v = (j+u)_m$. Then $w_s(\alpha\gamma) = y_v\gamma = z_{(j+u)_m}$. Note that $x_u\alpha = y_{(j+u)_m} = y_v = w_s\alpha$. This and the fact that $\ker \alpha = \ker \beta$ give $w_s\beta = x_u\beta = z_{(j+u)_m}$, which shows that $\alpha\gamma = \beta$. By a similar construction, we obtain $\delta \in C(\sigma)$ such that $\beta\delta = \alpha$, which concludes the proof. \square

COROLLARY 3.5. *Let $\sigma \in S_n$ and let $\alpha, \beta \in C(\sigma)$. Then, $\alpha \mathcal{H} \beta$ (in $C(\sigma)$) if and only if $\text{ran } t_\alpha = \text{ran } t_\beta$, $\ker \alpha = \ker \beta$, and (2) of Theorem 3.3 is satisfied.*

Proof. Follows from Theorem 3.3, Theorem 3.4, and the fact that $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$. \square

The next theorem characterizes Green's \mathcal{D} relation in $C(\sigma)$.

THEOREM 3.6. *Let $\sigma \in S_n$ and let $\alpha, \beta \in C(\sigma)$. Then, $\alpha \mathcal{D} \beta$ (in $C(\sigma)$) if and only if the following conditions are satisfied.*

- (1) $\text{rank } t_\alpha = \text{rank } t_\beta$.
- (2) *The sets $\text{ran } t_\alpha$ and $\text{ran } t_\beta$ can be ordered, say,*

$$\text{ran } t_\alpha : c_1, c_2, \dots, c_u,$$

$$\text{ran } t_\beta : d_1, d_2, \dots, d_u,$$

in such a way that for each i , $1 \leq i \leq u$, $\ell(c_i) = \ell(d_i)$ and:

- (a) *if $a \in t_\alpha^{-1}(c_i)$, then there exists $b \in t_\beta^{-1}(d_i)$ such that $\ell(b)$ divides $\ell(a)$;*
- (b) *if $a \in t_\beta^{-1}(d_i)$, then there exists $b \in t_\alpha^{-1}(c_i)$ such that $\ell(b)$ divides $\ell(a)$.*

Proof. Suppose $\alpha \mathcal{D} \beta$. Since $\mathcal{D} = \mathcal{R} \circ \mathcal{L}$, there is $\delta \in C(\sigma)$ such that $\alpha \mathcal{R} \delta$ and $\delta \mathcal{L} \beta$. Then $\ker t_\alpha = \ker t_\delta$ (by Theorem 3.4 and (2) of Lemma 3.2) and $\text{ran } t_\delta = \text{ran } t_\beta$ (by

Theorem 3.3), which implies $\text{rank } t_\alpha = \text{rank } t_\delta = \text{rank } t_\beta$. Select an ordering of

$$\text{ran } t_\alpha : c_1, c_2, \dots, c_u.$$

Since $\alpha\mathcal{R}\delta$, $\alpha\gamma = \delta$ for some $\gamma \in C(\sigma)$, which gives $t_\alpha t_\gamma = t_\delta$ by (1) of Lemma 2.2. Moreover, by the proof of Theorem 3.4, γ can be selected in such a way that $\text{dom } t_\gamma = \text{ran } t_\alpha$, $\text{ran } t_\gamma = \text{ran } t_\delta$, and for each $c_i \in \text{dom } t_\gamma = \text{ran } t_\alpha$, the cycle $c_i t_\gamma$ has the same length as c_i . Since t_γ maps $\text{ran } t_\alpha$ onto $\text{ran } t_\delta$ and $|\text{ran } t_\alpha| = |\text{ran } t_\delta|$, we also have that t_γ is one-one. Therefore, setting $d_i = c_i t_\gamma$ ($1 \leq i \leq u$), we obtain the corresponding ordering of

$$\text{ran } t_\beta = \text{ran } t_\delta = \text{ran } t_\gamma : d_1, d_2, \dots, d_u,$$

with $\ell(c_i) = \ell(d_i)$ for each i . Let $i \in \{1, \dots, u\}$. Then, for every cycle a in σ ,

$$\begin{aligned} a \in t_\alpha^{-1}(c_i) &\iff at_\alpha = c_i \\ &\iff (at_\alpha)t_\gamma = d_i \text{ (since } c_i t_\gamma = d_i \text{ and } t_\gamma \text{ is one-one)} \\ &\iff at_\delta = d_i \text{ (since } t_\delta = t_\alpha t_\gamma) \\ &\iff a \in t_\delta^{-1}(d_i). \end{aligned}$$

Thus $t_\alpha^{-1}(c_i) = t_\delta^{-1}(d_i)$ and so (2) is satisfied by the fact that $\delta\mathcal{L}\beta$ and Theorem 3.3.

Conversely, suppose that (1) and (2) are satisfied. For $i \in \{1, \dots, u\}$, let $c_i = (x_{i0}x_{i1} \dots x_{i,r_i-1})$ and $d_i = (y_{i0}y_{i1} \dots y_{i,r_i-1})$. Let $\gamma, \gamma' \in PT_n$ be transformations with $\text{dom } \gamma = \text{ran } \alpha$, $\text{dom } \gamma' = \text{ran } \beta$, and values determined by $x_{ij}\gamma = y_{ij}$ and $y_{ij}\gamma' = x_{ij}$ ($1 \leq i \leq u, 0 \leq j \leq r_i-1$). Then, by Theorem 2.1, $\gamma, \gamma' \in C(\sigma)$. Setting $\delta = \alpha\gamma$, we have $\delta\gamma' = \alpha\gamma\gamma' = \alpha$, which gives $\alpha\mathcal{R}\delta$. By the definitions of γ and δ , we have that $\text{ran } t_\delta = \{d_1, \dots, d_u\} = \text{ran } t_\beta$ and that for each $i, 1 \leq i \leq u, t_\alpha^{-1}(c_i) = t_\delta^{-1}(d_i)$. This, (2), and Theorem 3.3 imply $\delta\mathcal{L}\beta$, which, coupled with $\alpha\mathcal{R}\delta$, gives $\alpha\mathcal{D}\beta$. \square

Recall the example given after Theorem 3.3: $\sigma = abcd = (1\ 2)(3\ 4\ 5)(6)(7) \in S_7$, $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 6 & 6 & 7 & 7 & 7 & 6 & 6 \end{pmatrix}$ and $\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 7 & 7 & 6 & 6 & 6 & 7 & - \end{pmatrix}$ in $C(\sigma)$. Calculating $t_\alpha = \begin{pmatrix} a & b & c & d \\ c & d & c & c \end{pmatrix}$ and $t_\beta = \begin{pmatrix} a & b & c & d \\ d & c & d & - \end{pmatrix}$, we have $\text{rank } t_\alpha = \text{rank } t_\beta = 2$. Moreover, ordering $\text{ran } t_\alpha : c, d$ and $\text{ran } t_\beta : d, c$ we see that (2) of Theorem 3.6 is also satisfied. Hence $\alpha\mathcal{D}\beta$ in $C(\sigma)$.

In a finite semigroup S , the \mathcal{D} -classes are partially ordered by the following relation:

$$D_a \leq D_b \iff S^1 a S^1 \subseteq S^1 b S^1,$$

where $a, b \in S$. The relation \leq is a partial ordering since in a finite semigroup $\mathcal{D} = \mathcal{J}$. When studying the structure of a finite semigroup, it is important to determine not only the $\mathcal{L}, \mathcal{R}, \mathcal{H}$, and \mathcal{D} -classes, but also the partial ordering of \mathcal{D} -classes.

The next theorem determines the partial ordering of \mathcal{D} -classes in $C(\sigma)$.

THEOREM 3.7. *Let $\sigma \in S_n$ and let $\alpha, \beta \in C(\sigma)$ with $\text{ran } t_\alpha = \{c_1, c_2, \dots, c_u\}$. Then, $D_\alpha \leq D_\beta$ if and only if to each sequence*

$$s : a_1 \in t_\alpha^{-1}(c_1), a_2 \in t_\alpha^{-1}(c_2), \dots, a_u \in t_\alpha^{-1}(c_u), \tag{1}$$

we can assign a sequence of elements of $\text{ran } t_\beta$:

$$d^s : d_1^s, d_2^s, \dots, d_u^s, \tag{2}$$

in such a way that for all sequences s and t as in (1) and for all $i, j \in \{1, \dots, u\}$:

- (i) $\ell(c_i)$ divides $\ell(d_i^s)$;
- (ii) there is $b_i \in t_\beta^{-1}(d_i^s)$ such that $\ell(b_i)$ divides $\ell(a_i)$;
- (iii) if $d_i^s = d_j^t$, then $i = j$.

Proof. Suppose $D_\alpha \leq D_\beta$, i.e., $\alpha = \delta\beta\gamma$ for some $\delta, \gamma \in C(\sigma)$. By (1) of Lemma 2.2, $t_\alpha = t_\delta t_\beta t_\gamma$. Consider a sequence s as in (1) and let $i \in \{1, \dots, u\}$. Since $a_i t_\alpha = c_i$ and $t_\alpha = t_\delta t_\beta t_\gamma$, there are cycles b_i and d_i^s in σ such that $a_i t_\delta = b_i$, $b_i t_\beta = d_i^s$, and $d_i^s t_\gamma = c_i$. Then $b_i \in t_\beta^{-1}(d_i^s)$ and, by (2) of Lemma 2.2, $\ell(c_i)$ divides $\ell(d_i^s)$ and $\ell(b_i)$ divides $\ell(a_i)$. Thus, assigning $d_1^s, d_2^s, \dots, d_u^s$ to s , we have that (i) and (ii) are satisfied. To show (iii), assume that s and t are sequences as in (1) and that $i, j \in \{1, \dots, u\}$. Then,

$$d_i^s = d_j^t \Rightarrow d_i^s t_\gamma = d_j^t t_\gamma \Rightarrow c_i = c_j \Rightarrow i = j.$$

Conversely, suppose that to each sequence (1) we can assign a sequence (2) in such a way that the conditions (i)–(iii) are satisfied. We shall construct $\delta, \gamma \in C(\sigma)$ such that $\alpha = \delta\beta\gamma$. First, we define $\text{dom } \gamma$ to be the set of all elements that occur in any cycle d in σ such that $d = d_v^s$ for some sequence s as in (1) and some $v \in \{1, \dots, u\}$. To define the values of γ , let $d = d_v^s = (w_0 w_1 \dots w_{q-1})$ and let $c_v = (z_0 z_1 \dots z_{p-1})$. By (i), p divides q , and so we may define

$$w_0 \gamma = z_0, \quad w_1 \gamma = z_1, \quad w_2 \gamma = z_2, \dots,$$

where the subscripts on z s are calculated modulo p . By (iii), γ is well-defined. Next, we set $\text{dom } \delta = \text{dom } \alpha$. To define the values of δ , let $a = (x_0 x_1 \dots x_{k-1}) \in \text{dom } t_\alpha$. Then $a \in t_\alpha^{-1}(c_v)$ for some $v \in \{1, \dots, u\}$. Select a sequence s as in (1) with $a_v = a$, and let $d_v^s = (w_0 w_1 \dots w_{q-1})$ and $c_v = (z_0 z_1 \dots z_{p-1})$ be as in the construction of γ . By (ii), there is $b_v = (y_0 y_1 \dots y_{m-1}) \in t_\beta^{-1}(d_v^s)$ such that m divides k . By Theorem 2.1, p divides q , q divides m , and for some indices $i \in \{0, 1, \dots, p-1\}$ and $j \in \{0, 1, \dots, q-1\}$,

$$x_0 \alpha = z_i, x_1 \alpha = z_{i+1}, x_2 \alpha = z_{i+2}, \dots, \quad \text{and} \quad y_0 \beta = w_j, y_1 \beta = w_{j+1}, y_2 \beta = w_{j+2}, \dots,$$

where the subscripts on z s are calculated modulo p and the subscripts on w s are calculated modulo q . Let $r \in \{0, 1, \dots, m-1\}$ be an index such that $x_r \beta = w_i$. Since m divides k , we may define

$$x_0 \delta = y_r, \quad x_1 \delta = y_{r+1}, \quad x_2 \delta = y_{r+2}, \dots,$$

where the subscripts on y s are calculated modulo m . By the constructions of γ and δ and Theorem 2.1, we have $\delta, \gamma \in C(\sigma)$ and $\alpha = \delta\beta\gamma$. This concludes the proof. \square

Note that taking $s = t$ in (iii), we get that $d_1^s, d_2^s, \dots, d_u^s$ are pairwise distinct. This, coupled with (i), shows that if $D_\alpha \leq D_\beta$, then $\text{rank } t_\alpha \leq \text{rank } t_\beta$ and $\text{rank } \alpha \leq \text{rank } \beta$.

To illustrate Theorem 3.7, consider $\sigma = abcde = (1\ 2)(3\ 4)(5\ 6\ 7)(8)(9) \in S_9$, and $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 8 & 8 & - & - & 8 & 8 & 8 & - & 9 \end{pmatrix}$ and $\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 4 & - & - & 7 & 5 & 6 & - & 8 \end{pmatrix}$ in $C(\sigma)$. Since $t_\alpha =$

$\begin{pmatrix} a & b & c & d & e \\ d & - & d & - & e \end{pmatrix}$, we have $\text{ran } t_\alpha = \{d, e\}$ and two sequences of type (1): $s : a, e$ and $t : c, e$. Since $t_\beta = \begin{pmatrix} a & b & c & d & e \\ b & - & c & - & d \end{pmatrix}$ with $\text{ran } t_\beta = \{b, c, d\}$, we can construct the corresponding sequences of type (2): $d^s : b, d$ and $d^t : c, d$ that satisfy (i)–(iii). Therefore, $D_\alpha \leq D_\beta$. Note that it is impossible to construct a sequence d_1, d_2 of elements of $\text{ran } t_\beta$ that would work for both s and t .

4. Regularity. An element a of a semigroup S is called *regular* if $a = axa$ for some x in S . If all elements of S are regular, we say that S is a *regular semigroup*. An element d' in S is called an *inverse* of a in S if $a = ad'a$ and $d' = d'ad'$. Since regular elements are precisely those that have inverses (if $a = axa$, then $d' = xax$ is an inverse of a), we may define a regular semigroup as a semigroup in which every element has an inverse.

If a \mathcal{D} -class D in S contains a regular element, then every element in D is regular, and we call D a *regular \mathcal{D} -class*. In a regular \mathcal{D} -class, every \mathcal{L} -class and every \mathcal{R} -class contains an idempotent (an element e with $e = ee$). If an \mathcal{H} -class H contains an idempotent, then H is a maximal subgroup of S .

If every element of a semigroup S has exactly one inverse, then S is called an *inverse semigroup*. An alternative definition is that S is an inverse semigroup if it is regular and its idempotents commute. If every element of S is in some subgroup of S , then S is called a *union of groups*. In other words, unions of groups are semigroups in which every \mathcal{H} -class is a group. (Unions of groups are also called completely regular semigroups [2, Proposition 4.1.1].) Both inverse semigroups and unions of groups are regular semigroups.

The following lemma describes regular elements in $C(\sigma)$.

LEMMA 4.1. *Let $\sigma \in S_n$. Then a transformation $\alpha \in C(\sigma)$ is regular if and only if for every $b \in \text{ran } t_\alpha$, there is $a \in t_\alpha^{-1}(b)$ such that $\ell(a) = \ell(b)$.*

Proof. Suppose $\alpha \in C(\sigma)$ is regular, i.e., $\alpha = \alpha\beta\alpha$ for some $\beta \in C(\sigma)$. Let $b \in \text{ran } t_\alpha$ and select $c \in t_\alpha^{-1}(b)$. Since $t_\alpha = t_\alpha t_\beta t_\alpha$ (by (1) of Lemma 2.2) and $ct_\alpha = b$, there is a cycle a in σ such that $ct_\alpha = b$, $bt_\beta = a$, and $at_\alpha = b$. Then $a \in t_\alpha^{-1}(b)$ and, by (2) of Lemma 2.2, $\ell(c) \geq \ell(b) \geq \ell(a) \geq \ell(b)$, implying $\ell(a) = \ell(b)$.

Conversely, suppose that the given condition is satisfied. We shall define $\beta \in C(\sigma)$ such that $\alpha = \alpha\beta\alpha$. First, set $\text{dom } \beta = \text{ran } \alpha$. To define the values of β , let $b = (y_0 y_1 \dots y_{m-1}) \in \text{ran } t_\alpha$. Then, by the assumption, we can find a cycle $a = (x_0 x_1 \dots x_{k-1})$ in $\text{dom } t_\alpha$ such that $at_\alpha = b$ and $k = m$. By Theorem 2.1, for some index j ,

$$x_0\alpha = y_j, \quad x_1\alpha = y_{j+1}, \quad x_2\alpha = y_{j+2}, \dots,$$

where the subscripts on y s are calculated modulo m . Since $k = m$, we may define

$$y_j\beta = x_0, \quad y_{j+1}\beta = x_1, \quad y_{j+2}\beta = x_2, \dots,$$

where the subscripts on y s and on x s are calculated modulo $m (= k)$. By the construction of β and Theorem 2.1, we have $\beta \in C(\sigma)$ and $\alpha = \alpha\beta\alpha$. This concludes the proof. \square

Using Lemma 4.1, we characterize the permutations $\sigma \in S_n$ for which $C(\sigma)$ is a regular semigroup.

THEOREM 4.2. *Let $\sigma \in S_n$. Then $C(\sigma)$ is a regular semigroup if and only if*

$$\text{for all cycles } a, b \in C(\sigma) : \ell(b) \text{ divides } \ell(a) \Rightarrow \ell(b) = \ell(a). \tag{3}$$

Proof. Suppose $C(\sigma)$ is a regular semigroup. Let $a = (x_0x_1 \dots x_{k-1})$ and $b = (y_0y_1 \dots y_{m-1})$ be cycles in σ such that m divides k . Consider $\alpha \in PT_n$ with $\text{dom } \alpha = \{x_0, x_1, \dots, x_{k-1}\}$ and with values defined by

$$x_0\alpha = y_0, \quad x_1\alpha = y_1, \quad x_2\alpha = y_2, \dots,$$

where the subscripts on y s are calculated modulo m . By Theorem 2.1, $\alpha \in C(\sigma)$. Since $\text{dom } t_\alpha = \{a\}$ and $\text{ran } t_\alpha = \{b\}$, we have $m = k$ by the fact that α is regular and Lemma 4.1.

Conversely, suppose (3) holds. Let $\alpha \in C(\sigma)$ and let $b \in \text{ran } t_\alpha$. Select an $a \in t_\alpha^{-1}(b)$. By (2) of Lemma 2.2 and (3), we have $\ell(b) = \ell(a)$. It follows by Lemma 4.1 that α is regular. \square

For example, for $\sigma = (1\ 2)(3\ 4\ 5)(6\ 7\ 8)$ and $\rho = (1\ 2)(3\ 4)(5\ 6\ 7\ 8)$ in S_8 , the centralizer $C(\sigma)$ is a regular semigroup whereas $C(\rho)$ is nonregular. Note that for any permutation $\sigma \in S_n$ (other than the identity) with at least one 1-cycle, $C(\sigma)$ is nonregular.

In an inverse semigroup, only one \mathcal{H} -class in each \mathcal{L} -class (\mathcal{R} -class) is a group. In contrast, in a union of groups, every \mathcal{H} -class is a group. We note that in the class of centralizers of permutations, inverse semigroups and unions of groups coincide.

THEOREM 4.3. *For any $\sigma \in S_n$, the following conditions are equivalent:*

- (a) $C(\sigma)$ is an inverse semigroup;
- (b) $C(\sigma)$ is a union of groups;
- (c) for all cycles a, b in σ , if $\ell(b)$ divides $\ell(a)$ then $b = a$.

Proof. To show (a) \Rightarrow (c), suppose $C(\sigma)$ is an inverse semigroup and let $a = (x_0x_1 \dots x_{k-1})$ and $b = (y_0y_1 \dots y_{m-1})$ be cycles in σ such that m divides k . By Theorem 4.2, $m = k$. Suppose $a \neq b$. Define $\varepsilon, \xi \in PT_n$ by: $\text{dom } \varepsilon = \{x_0, \dots, x_{k-1}, y_0, \dots, y_{k-1}\}$, $\text{dom } \xi = \{y_0, \dots, y_{k-1}\}$, $x_i\varepsilon = y_i$, $y_i\varepsilon = y_i$, and $y_i\xi = y_i$ ($0 \leq i \leq k-1$). By the construction and Theorem 2.1, ε and ξ are idempotents in $C(\sigma)$ with $\varepsilon\xi = \varepsilon \neq \xi = \xi\varepsilon$, which is a contradiction (since idempotents commute in an inverse semigroup). Hence $b = a$.

To show (b) \Rightarrow (c), suppose $C(\sigma)$ is a union of groups and let a and b be cycles in σ as above. Again, $k = m$ and suppose $a \neq b$. Define $\alpha \in PT_n$ by: $\text{dom } \alpha = \{x_0, \dots, x_{k-1}\}$ and $x_i\alpha = y_i$ ($0 \leq i \leq k-1$). By the construction and Theorem 2.1, $\alpha \in C(\sigma)$ and $\alpha^2 = 0$, where 0 is the zero (empty) transformation. Since H_α is a group, we have $\alpha^2 \in H_\alpha$ and so $\alpha\mathcal{H}0$. This is a contradiction (by Corollary 3.5). Hence $b = a$.

Suppose (c) holds. Then, by Theorem 2.1, for every $\alpha \in C(\sigma)$, α is a permutation on its domain and t_α fixes each element of its domain. It follows that for some integer $p \geq 1$, $\alpha^p = \varepsilon$ is an idempotent such that $\text{dom } \varepsilon = \text{dom } \alpha$, $x\varepsilon = x$ for each $x \in \text{dom } \varepsilon$, and $t_\varepsilon = t_\alpha$. By Corollary 3.5, $\alpha\mathcal{H}\varepsilon$, which shows that $C(\sigma)$ is a union of groups. Further, the fact that elements of $C(\sigma)$ are permutations on their domains implies that idempotents in $C(\sigma)$ are one-one. Since one-one idempotents in PT_n commute, we have that $C(\sigma)$ is also an inverse semigroup. \square

5. Example. In this section, we shall use the results of Section 3 and Section 4 to present the structure of the centralizer $C(\sigma)$ for

$$\sigma = abc = (1\ 2)(3\ 4)(5\ 6\ 7\ 8). \tag{4}$$

We shall visualize each \mathcal{D} -class as an egg-box diagram, with each \mathcal{R} -class R_α (row) labelled by $\ker \alpha$ (see Theorem 3.4) and each \mathcal{L} -class L_α (column) labelled by $\text{ran } t_\alpha$ (see Theorem 3.3). In each \mathcal{H} -class H (cell), we shall place a representative α of H together with t_α , with α being an idempotent if H is a group. Idempotents will be indicated by asterisks.

To simplify notation, we shall write both $\alpha \in C(\sigma)$ and t_α as sequences of images. For example, for $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 2 & - & - & 3 & 4 & 3 & 4 \end{pmatrix}$ and $t_\alpha = \begin{pmatrix} a & b & c \\ a & - & b \end{pmatrix}$, we shall write $\alpha = 1\ 2\ -\ -\ 3\ 4\ 3\ 4$ and $t_\alpha = a - b$.

If $\alpha \in C(\sigma)$ with $\text{rank } \alpha = k$ and $\text{rank } t_\alpha = m$, we say that the \mathcal{D} -class D_α is of rank (k, m) . This definition is justified by Theorem 3.6, which implies that if $\alpha D \beta$, then $\text{rank } \alpha = \text{rank } \beta$ and $\text{rank } t_\alpha = \text{rank } t_\beta$.

By Theorem 2.1, the possible ranks of \mathcal{D} -classes in $C(\sigma)$ for the permutation (4) are: $(8,3)$, $(6,2)$, $(4,2)$, $(4,1)$, $(2, 1)$, and $(0,0)$.

Rank $(8, 3)$. There is one \mathcal{D} -class of this rank, say D_1 , namely the group of units of $C(\sigma)$ (see Fig. 1). Every member of D_1 maps either a onto a , b onto b , and c onto c or a onto b , b onto a , and c onto c . We have $2 \cdot 2 \cdot 4 = 16$ possibilities for the former case and the same number for the latter, giving the total of 32 elements in D_1 .

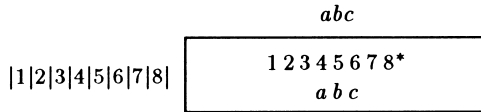


Figure 1. D_1 (group of units, 32 elements).

Rank $(6, 2)$. There is one \mathcal{D} -class of this rank, say D_2 (see Fig. 2). Look at the \mathcal{H} -class in the lower right-hand corner. Each member of this \mathcal{H} -class maps b onto b and c onto c . This can be done in $2 \cdot 4 = 8$ ways. Since all \mathcal{H} -classes in the same \mathcal{D} -class have the same cardinality, D_2 has $8 \cdot 8 = 64$ elements.

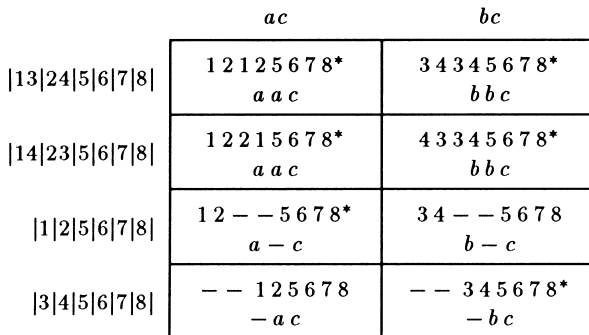


Figure 2. D_2 (regular, 64 elements).

Rank (4, 2). There are two \mathcal{D} -classes of this rank, say D_3 and D_4 , one regular and one nonregular (see Figs 3 and 4). Each \mathcal{H} -class in D_3 has 8 elements and each \mathcal{H} -class in D_4 has 4 elements.

| | |
|-------------|---|
| | ab |
| 157 268 3 4 | $1\ 2\ 3\ 4\ 1\ 2\ 1\ 2^*$ $a\ b\ a$ |
| 168 257 3 4 | $1\ 2\ 3\ 4\ 2\ 1\ 2\ 1^*$ $a\ b\ a$ |
| 1 2 357 468 | $1\ 2\ 3\ 4\ 3\ 4\ 3\ 4^*$ $a\ b\ b$ |
| 1 2 368 457 | $1\ 2\ 3\ 4\ 4\ 3\ 4\ 3^*$ $a\ b\ b$ |
| 1 2 3 4 | $1\ 2\ 3\ 4\ -\ -\ -\ -^*$ $a\ b-$ |

Figure 3. D_3 (regular, 40 elements).

| | | |
|-------------|---------------------------------------|---------------------------------------|
| | ab | ab |
| 13 24 57 68 | $1\ 2\ 1\ 2\ 3\ 4\ 3\ 4$ $a\ a\ b$ | $3\ 4\ 3\ 4\ 1\ 2\ 1\ 2$ $b\ b\ a$ |
| 14 23 57 68 | $1\ 2\ 2\ 1\ 4\ 3\ 4\ 3$ $a\ a\ b$ | $4\ 3\ 3\ 4\ 1\ 2\ 1\ 2$ $b\ b\ a$ |
| 1 2 57 68 | $1\ 2\ -\ -\ 3\ 4\ 3\ 4$ $a\ -\ b$ | $3\ 4\ -\ -\ 1\ 2\ 1\ 2$ $b\ -\ a$ |
| 3 4 57 68 | $-\ -\ 3\ 4\ 1\ 2\ 1\ 2$ $-\ b\ a$ | $-\ -\ 1\ 2\ 3\ 4\ 3\ 4$ $-\ a\ b$ |

Figure 4. D_4 (nonregular, 32 elements).

Rank (4, 1). There is one \mathcal{D} -class of this rank, say D_5 , with a single \mathcal{H} -class (see Fig. 5).

| | |
|---------|-------------------------------------|
| | c |
| 5 6 7 8 | $- - - -\ 5\ 6\ 7\ 8^*$ $- -\ c$ |

Figure 5. D_5 (regular, 4 elements).

| | <i>a</i> | <i>b</i> |
|-----------|------------------------------------|--------------------------------------|
| 1357 2468 | 1 2 1 2 1 2 1 2* <i>a a a</i> | 3 4 3 4 3 4 3 4* <i>b b b</i> |
| 1457 2368 | 1 2 2 1 1 2 1 2* <i>a a a</i> | 4 3 3 4 4 3 4 3* <i>b b b</i> |
| 1368 2457 | 1 2 1 2 2 1 2 1* <i>a a a</i> | 3 4 3 4 4 3 4 3* <i>b b b</i> |
| 1468 2357 | 1 2 2 1 2 1 2 1* <i>a a a</i> | 3 4 4 3 3 4 3 4* <i>b b b</i> |
| 157 268 | 1 2 -- -- 1 2 1 2* <i>a - a</i> | 3 4 -- -- 3 4 3 4 <i>b - b</i> |
| 168 257 | 1 2 -- -- 2 1 2 1* <i>a - a</i> | 3 4 -- -- 4 3 4 3 <i>b - b</i> |
| 357 468 | -- -- 1 2 1 2 1 2 <i>- a a</i> | -- -- 3 4 3 4 3 4* <i>- b b</i> |
| 368 457 | -- -- 1 2 2 1 2 1 <i>- a a</i> | -- -- 3 4 4 3 4 3* <i>- b b</i> |
| 13 24 | 1 2 1 2 -- -- -- * <i>a a -</i> | 3 4 3 4 -- -- -- * <i>b b -</i> |
| 14 23 | 1 2 2 1 -- -- -- * <i>a a -</i> | 4 3 3 4 -- -- -- * <i>b b -</i> |
| 1 2 | 1 2 -- -- -- -- * <i>a --</i> | 3 4 -- -- -- -- <i>b --</i> |
| 3 4 | -- -- 1 2 -- -- -- <i>- a -</i> | -- -- 3 4 -- -- -- * <i>- b -</i> |

Figure 6. D_6 (regular, 48 elements).

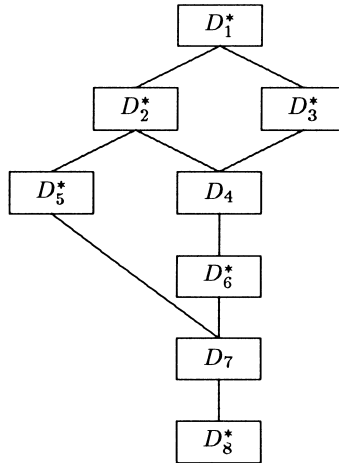
| | <i>a</i> | <i>b</i> |
|-------|----------------------------------|----------------------------------|
| 57 68 | -- -- -- 1 2 1 2 <i>- - a</i> | -- -- -- 3 4 3 4 <i>- - b</i> |

Figure 7. D_7 (nonregular, 4 elements).

Rank (2, 1). There are two \mathcal{D} -classes of this rank, say D_6 and D_7 , one regular and one nonregular (see Figs 6 and 7). Each \mathcal{H} -class in each of these two \mathcal{D} -classes has 2 elements.

Rank (0, 0). There is one \mathcal{D} -class of this rank, containing the zero transformation as the only element.

Thus the semigroup $C(\sigma)$ has 225 elements (189 regular and 36 nonregular) and 8 \mathcal{D} -classes (6 regular and 2 nonregular). Using Theorem 3.7, we can determine the partial ordering of \mathcal{D} -classes (see Fig. 8). Regular \mathcal{D} -classes are marked with asterisks.

Figure 8. Global structure of $C(\sigma)$.

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