J. Austral. Math. Soc. **29** (*Series A*) (1980), 393–398

A FIXED POINT THEOREM FOR NON-EXPANSIVE, CONDENSING MAPPINGS

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(Received 27 February; revised 23 July 1979)

Communicated by J. B. Miller

Abstract

A lemma is obtained which guarantees that non-expansive mappings on contractive spaces have fixed points. An example shows that Schauder's fixed point theorem cannot be extended to contractive spaces, but a theorem for contractive spaces, analogous to a result of B. N. Sadovskii on convex spaces, is derived from the lemma. Finally, some local results for ε-chainable contractive spaces are given.

1980 Mathematics subject classification (*Amer. Math. Soc.*): 47 H 10.

Keywords and phrases: Fixed points, non-expansive mappings, condensing mappings, contractive metric spaces.

Dotson (1972) proved that non-expansive self-mappings of compact star-shaped subsets of Banach spaces have fixed points. Göhde (1974) showed that such self-mappings of a closed, bounded star-shaped subset of a Hilbert space also have fixed points, and Reinermann and Stallbohm (1974) announced a fixed point theorem for condensing maps on a shrinkable subset of a Banach space. A number of other results on star-shaped sets are given by Reinermann and Stallbohm (1974) and by Müller and Reinermann (1977).

A more general class of sets containing the star-shaped sets (and possessing an important feature of those sets) may be called *contractive*; that is a subset S of a metric space is contractive if there exists a sequence $\{\varphi_n\}$ of contraction mappings of S into itself such that $\varphi_n x \to x$ for each x in S. S will be called *uniformly contractive* if, given $\varepsilon > 0$, there exists an N such that $n \geqslant N$ implies that $d(\varphi_n x, x) < \varepsilon$ for all x in S.

It is known (Smart (1974), p. 36) that non-expansive self-mappings of compact, uniformly contractive spaces have fixed points. In fact, the condition that the space be uniformly contractive can be relaxed.

LEMMA 1. *Let C be a compact subset of a metric space X, and T a non-expansive self-map of C. If there exist contraction maps φ_n: $TC \to C$ such that $\varphi_n x \to x$ for all x in TC, then T has a fixed point in C.*

PROOF. For each n, $\varphi_n T$ is a contraction map on C and so has a unique fixed point x_n in C. We may assume that $x_n \to x$ for some x in C. Given any $\eta > 0$ there exists an m such that $d(x_m, x) < \frac{1}{2}\eta$ and $d(\varphi_m Tx, Tx) < \frac{1}{2}\eta$. Now

$$d(\varphi_m Tx_m, \varphi_m Tx) \leqslant d(x_m, x) < \tfrac{1}{2}\eta$$

as well, and hence

$$d(\varphi_m Tx_m, Tx) \leqslant (\varphi_m Tx_m, \varphi_m Tx) + d(\varphi_m Tx, Tx) < \tfrac{1}{2}\eta + \tfrac{1}{2}\eta = \eta.$$

Thus there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\varphi_{n_j} Tx_{n_j} \to Tx$. But $\varphi_{n_j} Tx_{n_j} = x_{n_j} \to x$ and so $Tx = x$.

Observe that if C is a compact, contractive subset of X, the conditions of the lemma are satisfied.

A natural question is whether or not Schauder's fixed-point theorem for compact, convex subsets of normed spaces can be extended to contractive spaces. Counterexamples by Knill (1967) and Müller and Reinermann (1977) show that the Schauder theorem cannot be so extended, even to compact star-shaped subsets of $\mathbf{R}^3$. Both of these counterexamples are rather difficult; we therefore furnish a counterexample that will be easier for those familiar with the classic example of Kinoshita (1953).

EXAMPLE 1. Using the spherical coordinate system (r, θ, φ) in $\mathbf{R}^3$ define:

$$A_1 = \{(r, \theta, \varphi) \mid 0 \leqslant r < 1,\ 0 \leqslant \theta \leqslant 2\pi,\ 0 \leqslant \varphi \leqslant \tfrac{1}{4}\pi\},$$
$$A_2 = \{(r, \theta, \varphi) \mid r = 1,\ 0 \leqslant \theta \leqslant 2\pi,\ 0 \leqslant \varphi \leqslant \tfrac{1}{4}\pi\},$$
$$A_3 = \{(r, \theta, \varphi) \mid 1 < r \leqslant 2,\ 0 \leqslant \theta < \infty,\ \varphi = \tfrac{1}{2}\tan^{-1}\theta\},$$
$$A_4 = \{(\mathrm{r}, \theta, \varphi) \mid 1 < r \leqslant 2,\ 0 \leqslant \theta \leqslant 2\pi,\ \varphi = \tfrac{1}{4}\pi\}.$$

Let $B_1 = A_2 \cup A_3 \cup A_4$ and $A = A_1 \cup B_1$. Then A is compact and star-shaped. B_1 is homeomorphic to Kinoshita's example and so does not have the fixed-point property. But B_1 is a retract of A and hence A lacks the fixed-point property as well. See Figure 1.

There is, however, a consequence of this lemma which provides an analogue, for contractive sets, of Sadovskii's theorem (1967) for convex sets. Recall that a map T is condensing if, for bounded sets $D \subseteq X$ with $\alpha[D] > 0$, $\alpha[TD] < \alpha[D]$ where α, the measure of non-compactness, may be defined:

$$\alpha[D] = \inf\{\varepsilon > 0 \mid D \text{ is covered by a finite number of closed balls, centered at points of } X, \text{ of radius} \leqslant \varepsilon\}.$$

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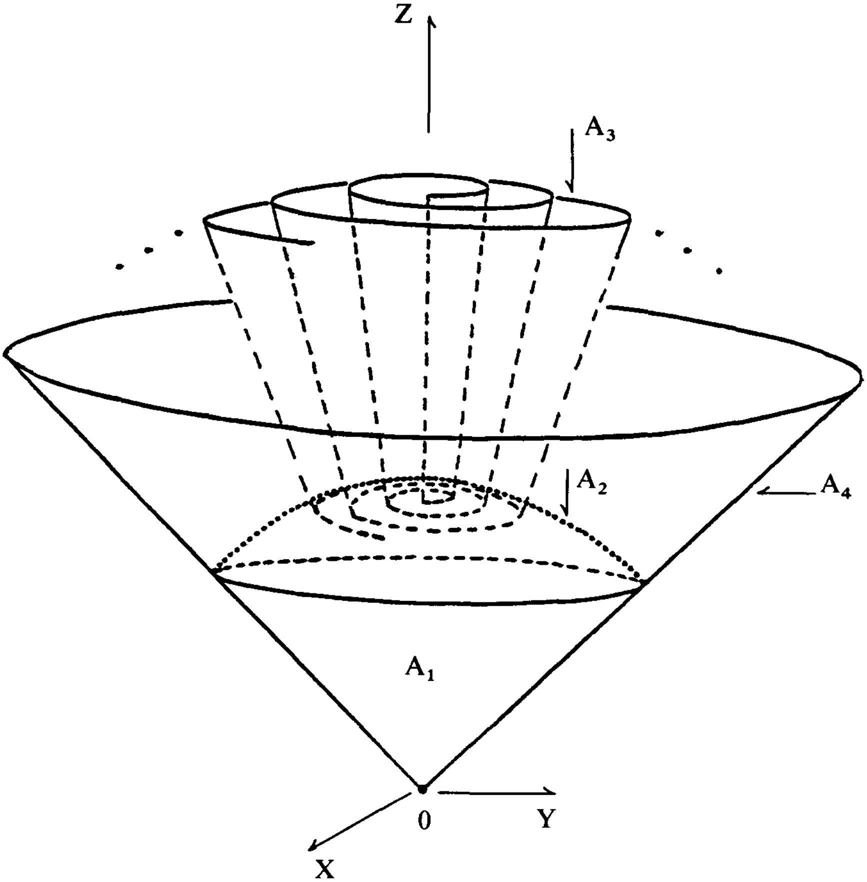

FIGURE 1

THEOREM 1. *Let X be a complete, contractive metric space with contractions $\{\varphi_n\}$. Let C be a closed, bounded subset of X and φ_n: $C \to C$. If $T: C \to C$ is nonexpansive and condensing, then T has a fixed point in C.*

PROOF. Let $y \in C$. Define $\Omega_0 = y$ and $\Omega_n = \bigcup_{i=0}^{\infty} \varphi_i T\Omega_{n-1}$ for $n \geqslant 1$ where we agree that φ_0 is the identity map on X. Let $\Omega = \bigcup_{n=0}^{\infty} \Omega_n$. We claim that $\alpha[\Omega] = 0$ for, if not, $\alpha[T\Omega] < \alpha[\Omega]$. But $T\Omega = \bigcup_{n=0}^{\infty} T\Omega_n$ and we may write

$$\Omega = \Omega_0 \cup \left\{ \bigcup_{i=1}^{\infty} \varphi_i T\Omega \right\} \cup T\Omega.$$

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For any $\delta > \alpha[T\Omega]$ let $\{B(x_k, \delta)\}_{k=1}^m$ be a finite cover for $T\Omega$ by balls centered at points $x_k \in X$ of radius δ. Since φ_i is a contraction $\varphi_i T\Omega \subseteq \bigcup_{k=1}^m B(\varphi_i x_k, \delta)$ for $i \geqslant 1$. Now given any $\varepsilon > 0$, we may find an N such that $n \geqslant N$ implies

$$d(\varphi_n x_k, x_k) < \varepsilon \quad \text{for each } k = 1, 2, \ldots, m.$$

Thus if $z \in \varphi_n T\Omega$ for $n \geqslant N$ we shall have

$$d(z, \varphi_n x_k) \leqslant \delta \quad \text{for some } k,\ 1 \leqslant k \leqslant m.$$

So

$$d(z, x_k) \leqslant d(z, \varphi_n x_k) + d(\varphi_n x_k, x_k) \leqslant \delta + \varepsilon.$$

Hence $\{\bigcup_{i=N}^\infty \varphi_i T\Omega\} \cup T\Omega \subseteq \bigcup_{k=1}^m B(x_k, \delta+\varepsilon)$, and we have:

$$\begin{aligned} \Omega &= \Omega_0 \cup \left\{\bigcup_{i=1}^{N-1} \varphi_i T\Omega\right\} \cup \left\{\bigcup_{i=N}^{\infty} \varphi_i T\Omega\right\} \cup T\Omega \\ &\subseteq B(\Omega_0, \delta+\varepsilon) \cup \left\{\bigcup_{i=1}^{N-1} \bigcup_{k=1}^{m} B(\varphi_i x_k, \delta+\varepsilon)\right\} \cup \left\{\bigcup_{k=1}^{m} B(x_k, \delta+\varepsilon)\right\}. \end{aligned}$$

We conclude that $\alpha[\Omega] \leqslant \delta + \varepsilon$ and since $\delta > \alpha[T\Omega]$ and $\varepsilon > 0$ were arbitrary, $\alpha[\Omega] \leqslant \alpha[T\Omega]$. This is impossible and so $\alpha[\Omega] = 0$. Thus cl$[\Omega]$ is compact. Now it is easily checked that cl$[\Omega]$ is invariant under T and that φ_n: $T(\mathrm{cl}[\Omega]) \to \mathrm{cl}[\Omega]$. Then Lemma 1 assures us that T has a fixed point in cl$[\Omega]$.

Following Edelstein (1961) we define a mapping φ of a metric space X into itself to be (ε, λ) uniformly locally contractive if for any $x \in X$ and $p, q \in S(x, \varepsilon)$ we have $d(\varphi(p), \varphi(q)) < \lambda d(p, q)$. X is called an ε-chainable space if for $a, b \in X$ there exists a finite set of points $a = x_0, x_1, x_2, \ldots, x_n = b$ such that $d(x_{i-1}, x_i) < \varepsilon$ for $i = 1, 2, \ldots, n$. Edelstein has shown that if X is a complete ε-chainable space and φ a self-map of X which is (ε, λ)-uniformly locally contractive, then φ has a unique fixed point in X.

Using this result of Edelstein in the appropriate place we may obtain a version of Lemma 1 for ε-chainable spaces; the proof is quite similar to that of Lemma 1.

LEMMA 2. *Let X be a compact metric space which is uniformly locally contractive by $(\varepsilon_n, \lambda_n)$-uniformly locally contractive maps φ_n. If X is ε_n-chainable for each n and T: $X \to X$ is non-expansive, then T has a fixed point in X.*

It is sometimes possible to identify or produce spaces upon which may be defined (ε, λ)-uniformly local contractions by means of the following theorem.

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THEOREM 2. *Let X be a metric space and $\bar{\Phi}$ a $\bar{\lambda}$-contraction on X. Suppose Y is a metric space and there exists a bijective map $f: X \to Y$ so that $|f'| = K > 0$ 'biuniformly'; that is, given $\eta > 0$ there exists $\bar{\varepsilon} = \bar{\varepsilon}(\eta) > 0$ so that for $x_1 \neq x_2$ if either $d(x_1, x_2) < \bar{\varepsilon}$ or $d(f(x_1), f(x_2)) < \bar{\varepsilon}$ we shall have*

$$K - \eta < \frac{d(f(x_1), f(x_2))}{d(x_1, x_2)} < K + \eta.$$

Then there exists an (ε, λ)-uniformly local contraction φ on Y for some $\lambda \in (0, 1)$.

PROOF. For the given $\bar{\lambda} < 1$ pick η, $0 < \eta < K$, so that $[(K+\eta)/(K-\eta)]\bar{\lambda} < 1$ and suppose $\bar{\varepsilon} = \bar{\varepsilon}(\eta)$. Also pick $\bar{\bar{\varepsilon}} \leqslant \bar{\varepsilon}$ so that $[\bar{\lambda}/(K-\eta)]\bar{\bar{\varepsilon}} < \bar{\varepsilon}$. Then for $0 < d(y_1, y_2) < \bar{\bar{\varepsilon}}$ we shall have

$$\frac{d(f^{-1}(y_1), f^{-1}(y_2))}{d(y_1, y_2)} < \frac{1}{K-\eta}.$$

Thus

$$d(\bar{\Phi} f^{-1}(y_1), \bar{\Phi} f^{-1}(y_2)) \leqslant \bar{\lambda} d(f^{-1}(y_1), f^{-1}(y_2)) < \bar{\lambda}\left(\frac{1}{K-\eta}\right) d(y_1, y_2) < \bar{\varepsilon}.$$

Therefore if $\bar{\Phi} f^{-1} y_1 \neq \bar{\Phi} f^{-1} y_2$,

$$\frac{d(f\bar{\Phi} f^{-1}(y_1), f\bar{\Phi} f^{-1}(y_2))}{d(\bar{\Phi} f^{-1}(y_1), \bar{\Phi} f^{-1}(y_2))} < K + \eta$$

and so

$$d(f\bar{\Phi} f^{-1}(y_1), f\bar{\Phi} f^{-1}(y_2)) < (K+\eta)\, d(\bar{\Phi} f^{-1}(y_1), \bar{\Phi} f^{-1}(y_2)) < \left(\frac{K+\eta}{K-\eta}\right)\bar{\lambda} d(y_1, y_2)$$

for all such $y_1, y_2 \in Y$. Now if we define

$$\varphi(y) = f\bar{\Phi} f^{-1}(y) \quad \text{and} \quad \lambda = \left(\frac{K+\eta}{K-\eta}\right)\bar{\lambda} \quad \text{and} \quad \varepsilon = \tfrac{1}{2}\bar{\bar{\varepsilon}}$$

we have that φ is an (ε, λ)-uniformly local contraction on Y.

Using Lemma 2 and Theorem 2 above, we will exhibit a compact subset of $\mathbf{R}^3$ which is not star-shaped, does not have the fixed point property for continuous maps but does have the fixed point property for non-expansive maps.

EXAMPLE 2. Let Y be the set A in Example 1 to which is appended at the origin a regular Jordan arc Γ of length 2, the first half of which is straight and collinear with the vertical axis and the second half of which is not straight and has no point nearer than one unit to the set A. Let $\{\bar{\lambda}_n\}$ be a sequence of positive

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numbers, $\bar{\lambda}_n \nearrow 1$. Let L be the interval $[0,2]$ and let $f: L \to \Gamma$ so that the arc length along Γ from 0 to $f(t)$ is t. If $\bar{\phi}_n$ is the $\bar{\lambda}_n$-star contraction to 0 on L (that is $\bar{\phi}_n(x) = \overline{\lambda_n} x$), then the homeomorphism f induces a $(\lambda_n, \varepsilon_n)$-uniformly local contraction φ_n on Γ by Theorem 2, and we may assume $\varepsilon_n < \frac{1}{2}$ for all n. Because of the way λ_n was chosen in Theorem 2, $\lambda_n \geqslant \bar{\lambda}_n$. Thus $\lambda_n \to 1$. Finally, define φ_n on A to be the λ_n-star contraction to 0. It is easily seen that $\{\varphi_n\}$ is a sequence of $(\varepsilon_n, \lambda_n)$-uniformly local contractions on Y and that $\varphi_n x \to x$ uniformly on Y. Of course Y is ε_n-chainable for each n, and so Lemma 2 assures us that Y has the fixed point property for non-expansive maps. The set A of Example 1 is a retract of Y and so Y lacks the fixed point property for continuous maps.

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