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# EIGENVALUES OF FINITE BAND-WIDTH HILBERT SPACE OPERATORS AND THEIR APPLICATION TO ORTHOGONAL POLYNOMIALS

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**1. Introduction. Main results.** The main result of this paper concerns the eigenvalues of an operator in the Hilbert space $l^2$ that is represented by a matrix having zeros everywhere except in a neighborhood of the main diagonal. Write $(c)^+$ for the positive part of a real number $c$, i.e., put $(c)^+ = c$ if $c \geqq 0$ and $(c)^+ = 0$ otherwise. Then this result can be formulated as follows.

THEOREM 1.1. *Let $k \geqq 1$ be an integer, and consider the operator* $\mathbf{S}$ *on $l^2$ such that*

$$(1.1) \qquad \mathbf{S}\langle \sigma_n \rangle_{n=0}^{\infty} = \Bigg\langle \left(\frac{1}{2} + c_{n-k,n}\right) \theta_{n-k,n}\sigma_{n-k} + \left(\frac{1}{2} + c_{n+k,n}\right) \theta_{n+k,n}\sigma_{n+k} + \sum_{j=-k+1}^{k-1} c_{n+j,n}\sigma_{n+j} \Bigg\rangle_{n=0}^{\infty}$$

*(here one has to take $\sigma_n = 0$ for $n < 0$), where $c_{mn}$ and $\theta_{mn}$ are complex numbers such that $c_{n-k,n}, c_{n+k,n} \geqq -1/2$ are real and $|\theta_{mn}| = 1$. Assume that the numbers $c_{mn}$ satisfy*

$$(1.2) \qquad \sum_{l=n}^{\infty} \left( (c_{l-k,l})^+ + (c_{l+k,l})^+ + \sum_{j=-k+1}^{k-1} |c_{l+j,l}| \right) < \frac{k}{36(n+2k)}$$

*for all $n \geqq 0$. Then* $\mathbf{S}$ *has no eigenvalues of absolute value $\geqq 1$.*

As we will point out at the end of Section 3, if the operator $\mathbf{S}$ satisfying the assumptions of this theorem is self adjoint, then it is easy to see by a theorem of H. Weyl (see e.g. [**15**, the first theorem in §134, p. 367]) that the spectrum of $\mathbf{S}$ has no limit points outside the interval $[-1, 1]$. Thus it follows that in this case the spectrum of $\mathbf{S}$ is entirely included in the interval $[-1, 1]$. If $\mathbf{S}$ is not self adjoint then the above result does not imply that its spectrum is included in

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the disc $|z| \leqq 1$, even though this seems to be the case. The connection between resuts such as Theorem 1.1 (or Theorem 1.2 below) and Bargmann's result in [**2**] about the number of bound states in a central field of force is discussed in [**8**, **9**].

The constant 1/36 on the right-hand side of (1.2) is not the best possible, but our methods seem only to be able to give a slight improvement. By doing the calculations in (2.6) in the proof of Lemma 2.1 somewhat more precisely, we can show that the conclusion of the above theorem remains valid if the constant 1/36 in (1.2) is replaced by $1/(24\sqrt{2}+\epsilon)$ for an arbitrary $\epsilon > 0$; this value is approximately $1/(33.942+\epsilon)$.

In the rest of the paper we will discuss the consequences of this result for orthogonal polynomials on the real line. To set the framework for our discussion, let $\alpha$ be a positive measure on the real line whose moments are finite and whose support is an infinite set. Then, as is known, there is a unique system of polynomials $p_n = p_n(d\alpha, x)$ that are orthonormal on the real line with respect to the measure $\alpha$, i.e., are such that

$$(1.3) \qquad \int_{-\infty}^{\infty} p_m(x)p_n(x)d\alpha(x) = \delta_{mn} \quad (m, n \geqq 0),$$

where $\delta_{mn} = 1$ if $m = n$, otherwise $\delta_{mn} = 0$. These polynomials satisfy a recurrence equation

$$(1.4) \qquad xp_n(x) = a_{n+1}p_{n+1}(x) + b_np_n(x) + a_np_{n-1}(x) \quad (n \geqq 0),$$

where $p_{-1}(x) = 0$ and $p_0(x) = \gamma_0 > 0$. Here $a_n(d\alpha) = a_n = \gamma_{n-1}/\gamma_n$, with $\gamma_n(d\alpha) = \gamma_n > 0$ (for $n \geqq 0; \gamma_{-1} = 0$) being the leading coefficient of $p_n$ (cf. e.g. [**7**, Formula (I.2.4), p. 17] or [**17**, Formula (3.2.1), p. 42]).

Fix a measure $\alpha$ as described, and let $p_n$ be a system of orthogonal polynomials associated with the measure $\alpha$ on the real line, i.e., put $p_n = p_n(d\alpha, x)$. Let $t_n$ denote the system of orthonormal Chebyshev polynomials, associated with the measure $\alpha_t$; that is,

$$d\alpha_t(x) = (1-x^2)^{-1/2}dx$$

for $-1 < x < 1$ and $d\alpha_t(x) = 0$ otherwise. In an earlier paper with V. Totik [**12**], we discussed the properties of the measure in the interval $[-1, 1]$ under the assumption that the quantity

$$(1.5) \qquad \epsilon_{kmn} = \sqrt{\pi/2}\left(\int_{-\infty}^{\infty} t_kp_mp_nd\alpha - \int_{-\infty}^{\infty} t_kt_mt_nd\alpha_t\right)$$

is small. In particular, we proved that if, for a fixed integer $k > 0$ we have

$$(1.6) \qquad \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} n|\epsilon_{kmn}| < \infty,$$

then $\alpha$ is absolutely continuous in the interval $[-1, 1]$ (cf. Theorem 6.2 of the quoted paper). Equation (1.6) appears to contain a doubly infinite sum. We may, however, observe that

$$\epsilon_{kmn} = 0 \quad \text{for } |m - n| > k. \tag{1.7}$$

Indeed, if e.g. $n > m + k$, then $t_k p_m$ is a polynomial of degree $<n$, hence it is expressible as a linear combination of the $p_l$'s for $l < n$. Therefore, the first integral on the right-hand side of (1.5) is zero in view of the orthogonality (cf. (1.3)) of the $p_l$'s; the second integral is also zero, for similar reasons. In the present paper we will use Theorem 1.1 to study the measure $\alpha$ outside the interval $[-1, 1]$. Our main result about orthogonal polynomials is the following.

THEOREM 1.2. *Let $k \geqq 1$. Assume we have*

$$\sum_{l=n}^{\infty} \left( (\epsilon_{k,l,l-k})^{+} + (\epsilon_{k,l,l+k})^{+} + \sum_{j=-k+1}^{k-1} |\epsilon_{k,l,l+j}| \right) < \frac{k}{36(n + 2k)} \tag{1.8}$$

*for every large enough integer $n$. Then the support of $\alpha$ contains only finitely many points outside the interval* $[-1, 1]$.

For $k = 1$, a similar result was established by Nikishin [**14**, Theorem 1, §2, p. 24]; we will comment about his result in more detail at the end of Section 5. Previously, the conclusion was derived under the stronger condition (1.6) (with $k = 1$) by Geronimo and Case [**10**, Theorem 1(c), (d), p. 473]. Chihara-Nevai [**4**] gives a simplified proof of this weaker result, and Guseĭnov [**11**, Theorem 1, p. 596] states it without proof. Under the assumption of (1.6), bounds are derived for the number of eigenvalues outside the interval $[-1, 1]$ in terms of the size of the sum on the left-hand side of (1.6) in [**8**, Theorem 1, p. 917] and [**9**, Theorem (III.1)]; Geronimo also obtained estimates, as yet unpublished, for the moments of these eigenvalues.

Our present result is nearly the best possible. More precisely, according to [**3**] (cf. Theorem 2 on p. 359, and the example given by Formula (2.8) on pp. 360–361) or [**13**, Theorem 3, pp. 133–134], the result in Theorem 1.2 becomes false if the constant 1/36 on the right-hand side of (1.8) is replaced by a certain larger constant, e.g. by 1/8 according to the latter paper. It would be of some interest to know the best constant on the right-hand side of (1.8); by a modification of our arguments we can improve the constant given there to $1/(24 + \epsilon)$ for any $\epsilon > 0$. This improvement is somewhat greater than the improvement described after Theorem 1.1 in the constant in (1.2). The reason is that while the conclusion of Theorem 1.1 states the nonexistence of eigenvalues of absolute value $\geqq 1$, Theorem 1.2 allows finitely many such eigenvalues, and to show this we may strengthen the assumptions in Lemma 2.1 below by requiring that $x_n = 0$ for "small" $n$.

It is a consequence of the classical theorem of H. Weyl mentioned above that a condition weaker than (1.6) guarantees that the part of the support of $\alpha$ outside the interval $[-1, 1]$ consists only of countably many points. More precisely, we have

THEOREM 1.3. *Let $k \geqq 1$. Assume that*

$$(1.9) \qquad \lim_{m,n\to\infty} \epsilon_{kmn} = 0.$$

*Then the part of the support of $\alpha$ outside the interval $[-1, 1]$ is a bounded countable set that has no limit points other than possibly $-1$ or $1$.*

In the proofs of these theorems, we will need the following lemma on the recurrence coefficients $a_n$ and $b_n$ of the polynomials $p_n$ (cf. Equation (1.4)).

LEMMA 1.4. *Let $k \geqq 1$, and assume that the numbers $\epsilon_{kmn}$ are bounded. Then the recurrence coefficients $a_n$ and $b_n$ are also bounded.*

Under an assumption seemingly weaker (but not in fact; cf. the remark after the theorem below) than (1.9) we can derive a stronger conclusion. To make sense of (1.10)(ii), note that $\epsilon_{kmn}$ is clearly real according to its definition (1.5).

THEOREM 1.5. *Let $k \geqq 2$, and assume*

$$(1.10) \qquad \text{(i)} \lim_{n\to\infty} \epsilon_{k,n,n+k} = 0 \quad \textit{and} \quad \text{(ii)} \limsup_{n\to\infty} \epsilon_{k,n,n+k-2} \leqq 0.$$

*Then we have*

$$(1.11) \qquad \text{(i)} \lim_{n\to\infty} a_n = 1/2 \quad \textit{and} \quad \text{(ii)} \lim_{n\to\infty} b_n = 0$$

*for the recurrence coefficients $a_n, b_n$ in (1.4) of the polynomials $p_n$.*

For $k = 1$ this is not true (in fact, if $k = 1$ then (1.10)(i) implies (1.10)(ii)), but, as we will point out at the beginning of Section 5, the conclusion in (1.11) is still valid in this case if we assume (1.9) instead of (1.10). It easily follows from this theorem that (1.10) actually implies (1.9) for $k \geqq 2$, as one can see by noting that the numbers $\epsilon_{kmn}$ are expressible as polynomials of the recurrence coefficients $a_l, b_l, a_l(d\alpha_t)$, and $b_l(d\alpha_t)$ according to (5.6) below.

**2. Difference inequalities. Proof of the main theorem.** The key role in the proof of Theorem 1.1 is played by a lemma on difference inequalities. In order to formulate this lemma let $\langle x_n \rangle_{n=-\infty}^{\infty}$ be a sequence of reals. We define the forward shift operator $\Delta$ by putting

$$\Delta^0 x_n = x_n \quad \text{and} \quad \Delta^{l+1} x_n = \Delta^l x_{n+1} - \Delta^l x_n \quad (l \geqq 0).$$

The following simple lemma plays a key role in our considerations below.

LEMMA 2.1. *Let $\langle x_n\rangle_{n=1}^{\infty}$ be a sequence of reals, and consider the difference inequality*

$$\triangle^2 x_n \geqq -\sum_{j=n}^{n+2} \delta_j x_j \quad (n \geqq 1), \tag{2.1}$$

*where $\delta_j \geqq 0$. Assume that*

$$\sum_{j=n}^{\infty} \delta_j < \frac{1}{18n} \tag{2.2}$$

*holds for every $n \geqq 2$. Then the only solution of* (2.1) *satisfying*

$$x_1 = 0 \quad \text{and} \quad x_n \geqq 0 \quad \text{for} \quad n \geqq 1 \tag{2.3}$$

*and*

$$\lim_{n\to\infty} x_n = 0 \tag{2.4}$$

*is the trivial solution, i.e., the one for which $x_n = 0$ for all $n \geqq 1$.*

*Proof.* Assume $\langle x_n\rangle_{n=1}^{\infty}$ is a nontrivial solution of (2.1) satisfying (2.3) and (2.4), and let $N > 0$ be the least integer for which $\triangle x_N < 0$. Put

$$\eta = \max_{1\leqq j<N} (\sqrt{j+1}\,\triangle x_j),$$

and let $m\,(1 \leqq m < N)$ be an integer for which this maximum is attained. Clearly $\eta > 0$. Writing

$$s_n = \sum_{j=n}^{\infty} \delta_j,$$

by (2.1) we have

$$\begin{aligned}
\triangle x_N - \frac{\eta}{\sqrt{m+1}} &= \triangle x_N - \triangle x_m \\
&= \sum_{n=M}^{N-1} \triangle^2 x_n \geqq -\sum_{n=m}^{N-1}\sum_{j=n}^{n+2} \delta_j x_j \\
&\geqq -3\sum_{n=m}^{N+1} \delta_n x_n = -3\sum_{n=m}^{N+1}(s_n - s_{n+1})x_n \\
&= -3\left(\sum_{n=m+1}^{N+1} s_n(x_n - x_{n-1}) + s_m x_m - s_{N+2}x_{N+1}\right) \\
&= -3\left(\sum_{n=m}^{N} s_{n+1}\triangle x_n + s_m x_m - s_{N+2}x_{N+1}\right) \\
&\geqq -3\left(\sum_{n=m}^{N} s_{n+1}\triangle x_n + s_m x_m\right).
\end{aligned} \tag{2.5}$$

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Now we have

$$\triangle x_n \leqq \eta/\sqrt{n+1} \quad \text{for} \quad 1 \leqq n \leqq N;$$

hence we also have

$$x_m = \sum_{n=1}^{m-1} \triangle x_n \leqq \sum_{n=1}^{m-1} \frac{\eta}{\sqrt{n+1}} < \eta \int_0^{m-1} \frac{dt}{\sqrt{t+1}} < 2\sqrt{m}\eta.$$

By using these and (2.2) (if $m = 1$ then we need this latter also for $n = 1$, whereas it was only assumed for $n \geqq 2$; note, however, that in (2.1) we can take $\delta_1 = 0$ because $x_1 = 0$ by (2.3), and this choice makes (2.2) valid for $n = 1$ as well), we can estimate the right-hand side of (2.5). We obtain that

$$\text{(2.6)} \qquad \triangle x_N - \frac{\eta}{\sqrt{m+1}} > -\frac{\eta}{6}\left(\sum_{n=m}^{\infty} \frac{1}{(n+1)\sqrt{n+1}} + \frac{2}{\sqrt{m}}\right)$$
$$> -\frac{\eta}{6}\left(\int_m^{\infty} \frac{dt}{t^{3/2}} + \frac{2}{\sqrt{m}}\right) = -\frac{2\eta}{3\sqrt{m}} > -\frac{\eta}{\sqrt{m+1}}.$$

Thus $\triangle x_N > 0$, a contradiction, completing the proof.

Next we turn to the

*Proof of Theorem* 1.1. Assume, on the contrary, that $\lambda$ is an eigenvalue of $\mathbf{S}$ with $|\lambda| \geqq 1$, and let $\langle \sigma_n \rangle_{n=0}^{\infty}$ be the corresponding eigenvector. Then

$$2\lambda\sigma_n = (1 + 2c_{n-k,n})\theta_{n-k,n}\sigma_{n-k} + (1 + 2c_{n+k,n})\theta_{n+k,n}\sigma_{n+k}$$
$$+ \sum_{j=-k+1}^{k-1} 2c_{n+j,n}\sigma_{n+j} \quad (\sigma_n = 0 \quad \text{for} \quad n < 0).$$

Using the assumptions that $|\theta_{mn}| = 1$ and $c_{n-k,n}, c_{n+k,n} \geqq -1/2$, we obtain that

$$2|\sigma_n| \leqq (1 + 2(c_{n-k,n})^+)|\sigma_{n-k}| + (1 + 2(c_{n+k,n})^+)|\sigma_{n+k}|$$
$$+ \sum_{j=-k}^{k} 2|c_{n+j,n}||\sigma_{n+j}|$$

for all $n \geqq 0$. Adding this inequality for $l = nk, nk+1, \ldots, (n+1)k-1$ replacing $n$, and writing

$$\text{(2.7)} \qquad x_n = \sum_{l=nk}^{(n+1)k-1} |\sigma_l|,$$

we obtain

$$2x_n \leqq x_{n-1} + x_{n+1} + \sum_{j=n-1}^{n+1} \delta_j x_j \quad (n \geqq 0),$$

where

$$\delta_m = 2\max\{(c_{l-k,l})^+, (c_{l+k,l})^+, |c_{\nu l}| : mk \leqq l < (m+1)k, |l-\nu| \leqq k-1\} \qquad (n \geqq 0; \delta_{-1} = 0).$$

That is,

$$\Delta^2 x_n \geqq -\sum_{j=n}^{n+2} \delta_j x_j \quad (n \geqq -1).$$

Now $x_{-1} = 0$ in view of (2.7), as $\sigma_n = 0$ for $n < 0$. Moreover, (2.2) holds (with $\delta_{j-2}$ replacing $\delta_j$) for the $\delta_n$'s just defined. Therefore $\lim_{n\to\infty} x_n = 0$ cannot hold according to Lemma 2.1. This is a contradiction, since $\langle \sigma_n \rangle_{n=0}^{\infty} \in l^2$, i.e.,

$$\sum_{n=0}^{\infty} |\sigma_n|^2 < \infty.$$

The proof is complete.

**3. Self adjoint operators and orthogonal polynomials.** In this section we present the proofs of Lemma 1.4 of Theorem 1.3 and present some observations helpful for the proof of Theorem 1.2. We start out with describing the connection between the support of the measure $\alpha$ and the spectrum of a certain self adjoint operator in a Hilbert space. To this end, let $\alpha$ be a positive measure on the real line whose moments are finite and whose support is an infinite set, and $p_n = p_n(d\alpha)$ be the corresponding system of orthonormal polynomials, and let $a_n = a_n(d\alpha)$ and $b_n = b_n(d\alpha)$ be the associated recurrence coefficients (cf. Equation (1.4)). Consider the operator $\mathbf{A} = \mathbf{A}(d\alpha)$ on the Hilbert space $l^2$ defined as

$$\text{(3.1)} \qquad \mathbf{A}\langle \sigma_n \rangle_{n=0}^{\infty} = \langle a_{n+1}\sigma_{n+1} + b_n\sigma_n + a_n\sigma_{n-1} \rangle_{n=0}^{\infty},$$

where $\sigma_n = 0$ is to be taken for $n = -1$.

The connection between this operator and the support of the measure $\alpha$ is worked out in detail in [**16**, §X.4, pp. 530–614]. This connection is particularly simple when the operator $\mathbf{A}$ is bounded, i.e., when the coefficients $a_n$, $b_n$ form bounded sequences. In this case $\mathbf{A}$ is self adjoint, and its spectrum equals the support of the measure $\alpha$.

This is actually not hard to see. Namely, there is a canonical isomorphism $\iota$ between the Hilbert spaces $l^2$ and $L^2_\alpha(-\infty,\infty)$. Under this isomorphism the sequence $\langle\sigma_n\rangle_{n=0}^{\infty}$ in $l^2$ corresponds to the function

$$f = \sum_{n=0}^{\infty} \sigma_n p_n \in L^2_\alpha(-\infty,\infty)$$

in $L^2_\alpha(-\infty,\infty)$, where the convergence on the right is meant in the sense of $L^2_\alpha(-\infty,\infty)$. It is easy to see from the recurrence equation (1.4) that, under this isomorphism, the operator $\mathbf{A}$ on $l^2$ corresponds to the operator

$$\mathbf{M} = \mathbf{M}(d\alpha) : f \mapsto xf$$

on $L^2_\alpha(-\infty,\infty)$, and it is easy to relate the spectrum of $\mathbf{M}$ to the support of $\alpha$.

The connection is more complicated if $\mathbf{A}$ is unbounded. In this case one can extend $\mathbf{A}$ to a self adjoint operator, but this extension may not be unique (see e.g. [**16**, Theorem 10.27, pp. 545–547], or, for a very brief summary, see [**6**, §XII.10, pp. 1275–77]). We will not need to know more about the details for our purposes, since by Lemma 1.4, whose proof we are about to present, we can exclude this case.

*Proof of Lemma* 1.4. The $n$th Chebyshev polynomial $T_n(x)$ can be written as $\cos n\theta$, where $\theta = \arccos x (n \geqq 0)$, and we have

$$t_0(x) = \sqrt{1/\pi}\,T_0(x) \quad \text{and} \quad t_n(x) = \sqrt{2/\pi}\,T_n(x) \quad \text{for } n \geqq 1.$$

Thus it is easy to see that

$$t_{n+k} - 2T_k t_n + t_{n-k} \equiv 0 \quad (n > k) \tag{3.2}$$

holds. Now, writing $p_n = p_n(d\alpha)$, we have

$$p_{n+k}(x) - 2T_k(x)p_n(x) + p_{n-k}(x) = -\sum_{j=-k}^{k} 2c_{n,n+j}p_{n+j}(x) \quad (n \geqq 0) \tag{3.3}$$

in view of the recurrence equation (1.4), where one has to take $p_n(x) = 0$ for $n < 0$. Here

$$\begin{aligned} c_{n,n+j} &= -\frac{1}{2}\int_{-\infty}^{\infty} (p_{n+k} - 2T_k p_n + p_{n-k})p_{n+j}\,d\alpha \\ &= -\frac{1}{2}\int_{-\infty}^{\infty} (t_{n+k} - 2T_k t_n + t_{n-k})t_{n+j}\,d\alpha_t + \epsilon_{k,n,n+j} \quad (n > k); \\ &= \epsilon_{k,n,n+j} \end{aligned} \tag{3.4}$$

the second equality in the latter formula follows from (1.5) and orthogonality relations (1.3), and the third equality follows from (3.2). Let $\mathbf{A} = \mathbf{A}(d\alpha)$ be as in (3.1), and define the operator $\mathbf{B}$ on $l^2$ by the equation

$$\text{(3.5)} \qquad \mathbf{B}\langle\sigma_n\rangle_{n=0}^{\infty} = \left\langle \frac{1}{2}\sigma_{n-k} + \frac{1}{2}\sigma_{n+k} + \sum_{j=-k}^{k} c_{n+j,n}\sigma_{n+j} \right\rangle_{n=0}^{\infty};$$

here one has to take $\sigma_n = 0$ for $n < 0$. (Of course, $c_{mn} = c_{nm}$ for all $m, n \geqq 0$ by (1.3) and the first equality in (3.4); this equality is valid for all $n \geqq 0$ if one takes $p_l = 0$ for $l < 0$). As the numbers $\epsilon_{kmn}$ are bounded by our assumptions, so are the numbers $c_{kmn}$ according to (3.4); hence $\mathbf{B}$ is a bounded operator defined everywhere on $l^2$. By (3.3) we have

$$\text{(3.6)} \qquad T_k(\mathbf{A})\langle\sigma_n\rangle_{n=0}^{\infty} = \mathbf{B}\langle\sigma_n\rangle_{n=0}^{\infty}$$

if, say, all but finitely many of the $\sigma_n$'s are zero. Now let $\mathbf{A}'$ be a self adjoint extension of A; such an extension exists, as pointed out just before this proof. The spectrum of $\mathbf{A}'$ lies on the real line; therefore, its resolvent set is not empty. Hence the polynomial $T_k(\mathbf{A}')$ is a closed operator (cf. [**5**, §VII.9, Theorem 7, p. 602]). So, by (3.5) and (3.6), $T_k(\mathbf{A}') = B$, and so the spectrum of $T_k(\mathbf{A}')$ is bounded. Therefore, by the Spectral Mapping Theorem for polynomials of unbounded operators (see [**5**, §VII.9, Theorem 10, p. 604]), the spectrum of $\mathbf{A}'$ is bounded. $\mathbf{A}'$ being self adjoint, this means that $\mathbf{A}'$ is bounded (instead of the last two quoted sources, we can use the spectral theory of unbounded self adjoint operators; cf. e.g. [**6**, §XII.2.9, Corollary 8 and Theorem 9, p. 1200] or [**15**, §§127–128]). Therefore $\mathbf{A}$ is also a bounded operator, showing that the recurrence coefficients $a_n$ and $b_n$ are bounded. The proof of Lemma 1.4 is complete.

We are now able to present the

*Proof of Theorem* 1.3. Let $\mathbf{A}$ be as before. Then $\mathbf{A}$ is bounded according to the lemma just proved, and $T_k(\mathbf{A}) = \mathbf{B}$, where $\mathbf{B}$ is the operator defined by (3.5). The self adjoint operator $\mathbf{Q}$ defined as

$$\mathbf{Q}\langle\sigma_n\rangle_{n=0}^{\infty} = \left\langle \frac{1}{2}\sigma_{n-k} + \frac{1}{2}\sigma_{n+k} \right\rangle_{n=0}^{\infty}$$

has norm 1, and so its spectrum is included in the interval $[-1, 1]$. Since the operator $\mathbf{B} - \mathbf{Q}$ is compact according to (1.9), (3.4), and (3.5), by H. Weyl's theorem (see e.g. [**15**, the first theorem in §134, p. 367]) the spectra of $\mathbf{B}$ and $\mathbf{Q}$ have the same limit points. Thus the spectrum of $T_k(\mathbf{A}) = \mathbf{B}$ has no limit points outside the interval $[-1, 1]$. By the Spectral Mapping Theorem (see e.g. [**15**, §150]), the same is true about the spectrum of $\mathbf{A}$. Hence the assertion of Theorem 1.3 is verified.

It is now easy to justify the remark made after Theorem 1.1. Write $(c)^-$ for the negative part of $c$, i.e., put $(c)^- = c$ if $c \leqq 0$ and put $(c)^- = 0$ otherwise. Assume that $\mathbf{S}$ is self adjoint, and define the operator $\mathbf{Q}'$ on $l^2$ by putting

$$\mathbf{Q}'\langle\sigma_n\rangle_{n=0}^{\infty} = \left\langle \left(\frac{1}{2} + (c_{n-k,n})^-\right)\theta_{n-k,n}\sigma_{n-k} + \left(\frac{1}{2} + (c_{n+k,n})^-\right)\theta_{n+k,n}\sigma_{n+k}\right\rangle_{n=0}^{\infty} \quad (\sigma_n = 0 \text{ for } n < 0).$$

Then $\mathbf{Q}'$ is also self adjoint and $\|\mathbf{Q}'\| \leqq 1$; thus the spectrum of $\mathbf{Q}'$ is included in the interval $[-1, 1]$. As the operator $\mathbf{S} - \mathbf{Q}'$ is compact in virtue of (1.2), it follows from Weyl's theorem quoted in the preceding proof that the spectrum of $\mathbf{S}$ has no limit points outside the interval $[-1, 1]$. This is what we wanted to show.

**4. Proof of theorem 1.2.** The proof of Theorem 1.2 relies on Theorem 1.1 and the idea that the addition of a self adjoint operator of finite rank to another self adjoint operator has only a "small" effect on the spectrum of the latter. This idea was exploited in [**1**, §2.1, pp. 39–42] and [**10**, pp. 484–486]. The precise statement we need of this idea can be formulated as follows.

LEMMA 4.1. *Let* $\mathbf{B}$, $\mathbf{S}$, *and* $\mathbf{T}$ *be bounded self adjoint operators on the Hilbert space* $H$ *with* $\mathbf{B} = \mathbf{S} + \mathbf{T}$. *Assume that the range of* $\mathbf{T}$ *has dimension* $m < \infty$, *and that the spectrum of* $\mathbf{S}$ *is included in the interval* $(-\infty, b]$. *Then* $\mathbf{B}$ *has at most* $m$ *eigenvalues (counted with multiplicities) that are greater than* $b$.

Of course, by Weyl's theorem quoted above, the part of the spectrum of $\mathbf{B}$ in the interval $(b, \infty)$ consists purely of isolated eigenvalues of finite multiplicities. By applying this lemma to the operator $-\mathbf{A}$, an analogous assertion can be concluded when we assume that the spectrum of $\mathbf{S}$ is included in the interval $[a, \infty)$.

*Proof.* Assume, on the contrary, that $\lambda_0, \lambda_1, \ldots, \lambda_m$ are (not necessarily distinct) eigenvalues greater than $b$ of $\mathbf{B}$, and let $f_0, f_1, \ldots, f_m$ be the corresponding (pairwise orthogonal) eigenvectors. Let $f = \sum_{j=0}^{m} \alpha_j f_j$ be a nonzero linear combination of these eigenvectors that is orthogonal to the range of $\mathbf{T}$. Then

$$(\mathbf{T}f, f) = 0,$$

and

$$(\mathbf{B}f, f) = \sum_{j=0}^{m} \lambda_j \alpha_j^2 \|f_j\|^2 > b\|f\|^2.$$

On the other hand,

$$(\mathbf{S}f, f) \leqq b\|f\|^2$$

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follows from the Spectral Representation Theorem for self adjoint operators (see e.g. [**15**, §107]). These relations contradict the equation $\mathbf{B} = \mathbf{S} + \mathbf{T}$, completing the proof.

We are now in a position to present the

*Proof of Theorem* 1.2. Writing $\mathbf{A} = \mathbf{A}(d\alpha)$ as in (3.1), we know by Lemma 1.4 that $\mathbf{A}$ is a bounded operator. Thus, according to the comments following (3.1), the support of the measure $\alpha$ equals the spectrum of $\mathbf{A}$. Thus, by the Spectral Mapping Theorem (see e.g. [**15**, §150]), it is sufficient to show that the operator $\mathbf{B} = T_k(\mathbf{A})$ has only finitely many eigenvalues outside the interval $[-1, 1]$. To this end, it is sufficient to represent $\mathbf{B}$ as a sum described in Lemma 4.1. Now $\mathbf{B}$ can be written as in (3.5), where we have

$$(4.1) \qquad \sum_{l=n}^{\infty}\left((c_{l-k,l})^{+} + (c_{l+k,l})^{+} + \sum_{j=-k+1}^{k-1}|c_{l+j,l}|\right) < \frac{k}{36(n+2k)}$$

for every large enough $n$ according to (3.4) and (1.8) (note that $c_{mn} = c_{nm}$, as remarked immediately after (3.5)). Observe that here

$$(4.2) \qquad \text{(i)} \quad c_{n-k,n} > -1/2 \quad \text{and} \quad \text{(ii)} \quad c_{n+k,n} > -1/2$$

hold for $n > 2k$, say. To see e.g. (i), observe that, writing $\gamma_m$ and $\gamma_m(d\alpha_t)$ for the leading coefficients of $p_m$ and $t_m$, respectively, we have

$$\begin{aligned}(4.3) \qquad \int_{-\infty}^{\infty} t_k p_n p_{n-k}\, d\alpha &= \int_{-\infty}^{\infty} (\gamma_k(d\alpha_t)x^k) p_n(x)(\gamma_{n-k}x^{n-k})\, d\alpha(x) \\ &= \int_{-\infty}^{\infty} p_n(x)\gamma_k(d\alpha_t)(\gamma_{n-k}/\gamma_n)(\gamma_n x^n)\, d\alpha(x) \\ &= \gamma_k(d\alpha_t)(\gamma_{n-k}/\gamma_n)\int_{-\infty}^{\infty} p_n^2(x)\, d\alpha(x) \\ &= \gamma_k(d\alpha_t)(\gamma_{n-k}/\gamma_n) > 0.\end{aligned}$$

The first equality here holds since the lower order terms of $t_k$ and $p_{n-k}$ can be ignored in view of the orthogonality relations (1.3), and the last inequality holds since the leading coefficients of orthogonal polynomials are chosen to be positive, as mentioned right after (1.4). On the other hand, we have

$$(4.4) \qquad \int_{-\infty}^{\infty} t_k t_n t_{n-k}\, d\alpha_t = \sqrt{2/\pi}/2$$

according to e.g. (3.2), the equation $T_k = \sqrt{\pi/2}t_k$, and the orthogonality relations (the analogue of (1.3) for $\alpha_t$). Hence (4.2)(i) follows from (1.5) and (3.4). (4.2)(ii) follows similarly (or, in fact, it is the same statement with $n+k$ replacing $n$, since $c_{mn} = c_{nm}$, as remarked right after (3.5)).

Choose a nonnegative integer $N > 2k$ such that (4.1) holds for $n \geqq N$, and put $c'_{mn} = c_{mn}$ if $m, n \geqq N$ and $c'_{mn} = 0$ otherwise. Then the operator $\mathbf{S}$, defined as

$$\mathbf{S}\langle\sigma_n\rangle_{n=0}^{\infty} = \left\langle \frac{1}{2}\sigma_{n-k} + \frac{1}{2}\sigma_{n+k} + \sum_{j=-k}^{k} c'_{n+j,n}\sigma_{n+j} \right\rangle_{n=0}^{\infty}$$

$(\sigma_n = 0$ for $n < 0)$

has no eigenvalues outside the interval $[-1, 1]$, according to Theorem 1.1. Thus, putting $\mathbf{T} = \mathbf{B} - \mathbf{S}$, the assertion follows from Lemma 4.1. The proof is complete.

**5. Proof of theorem 1.5 and remarks on Nikishin's theorem.** Before we turn to the proof of Theorem 1.5, we will point out that in case $k = 1$ the conclusion in (1.11) follows from (1.9). In fact, noting that $t_1(x) = \sqrt{2/\pi}x$, it follows from (1.5), the recurrence formula (1.4) (for $\alpha$ and $\alpha_t$) and the orthogonality relations (1.3) that

$$\epsilon_{1nn} = (b_n - b_n(d\alpha_t)) \quad \text{and} \quad \epsilon_{1,n,n+1} = (a_{n+1} - a_{n+1}(d\alpha_t)),$$

hold for $n \geqq 0$, where $a_n = a_n(d\alpha)$ and $b_n = b_n(d\alpha)$. Now it is easy to see that

$$\text{(5.1)} \qquad \text{(i)}\ \ a_n(d\alpha_t) = 1/2 \quad \text{for} \quad n \geqq 2 \quad \text{and} \quad \text{(ii)}\ \ b_n(d\alpha_t) = 0 \quad \text{for } n \geqq 0$$

$(a_0(d\alpha_t) = 0$ and $a_1(d\alpha_t) = 1/\sqrt{2}$, the former by convention, since it never occurs in the recurrence formula with a nonzero coefficient), and so (1.11) follows from (1.9) in case $k = 1$.

Next we turn to the

*Proof of Theorem* 1.5. In order to see what (1.10) means in terms of the recurrence coefficients $a_n$ and $b_n$, consider formulas (4.3) and (4.4). According to these, with $n + k$ replacing $n$, we have

$$\text{(5.2)} \qquad \epsilon_{k,n,n+k} = \sqrt{\pi/2}\gamma_k(d\alpha_t)(\gamma_n/\gamma_{n+k}) - 1/2.$$

Writing $\theta = \arccos x$, we have

$$\text{(5.3)} \qquad \sqrt{\pi/2}t_k(x) = T_k(x) = \cos k\theta = 2^{k-1}x^k - k2^{k-3}x^{k-2} + \dots,$$

that is, $\gamma_k(d\alpha_t) = \sqrt{2/\pi}2^{k-1}$. Moreover,

$$\gamma_n/\gamma_{n+k} = (\gamma_n/\gamma_{n+1})(\gamma_{n+1}/\gamma_{n+2})\dots(\gamma_{n+k-1}/\gamma_{n+k}) = a_{n+1}a_{n+2}\dots a_{n+k};$$

for the last equality, see the text immediately following (1.4). Thus, by (5.2) and (1.10)(i),

$$\text{(5.4)} \qquad \lim_{n\to\infty} a_{n+1}a_{n+2}\dots a_{n+k} = 2^{-k}.$$

In order to discern the meaning of (1.10)(ii), we will show how to calculate

$$\int_{-\infty}^{\infty} x^l p_m(x) p_n(x) d\alpha(x) \quad (l \geqq 0)$$

by using the recurrence formula (1.4). Write (1.4) as

$$xp_n(x) = \sum_{j=0}^{\infty} a_{nj} p_j(x),$$

where, of course,

$$\text{(5.5)} \qquad \begin{aligned} &a_{j,j+1} = a_{j+1,j} = a_{j+1}, \quad a_{jj} = b_j, \quad \text{and} \\ &a_{mj} = 0 \quad \text{for} \quad |m-j| > 1 \quad (j, m \geqq 0). \end{aligned}$$

Then, writing $j_0 = n$, we have

$$x^l p_n(x) = \sum_{j_l=0}^{\infty} \sum_{j_1, j_2, \dots, j_{l-1}=0}^{\infty} \left( \prod_{i=0}^{l-1} a_{j_i j_{i+1}} \right) p_{j_l}(x).$$

Writing $j_0 = n$ and $j_l = m$, this implies

$$\text{(5.6)} \qquad \int_{-\infty}^{\infty} x^l p_n(x) p_m(x) d\alpha(x) = \sum_{j_1, j_2, \dots, j_{l-1}=0}^{\infty} \prod_{i=0}^{l-1} a_{j_i j_{i+1}} \quad (l \geqq 0).$$

Using this for $m = n + l$, we obtain

$$\text{(5.7)} \qquad \int_{-\infty}^{\infty} x^l p_n(x) p_{n+l}(x) d\alpha(x) = a_{n+1} a_{n+2} \dots a_{n+l},$$

since all the other terms on the right-hand side of (5.6) vanish in view of (5.5); we could have obtained this formula by using (4.3) (with $l$ replacing $k$) instead. It is not much harder to see that

$$\text{(5.8)} \qquad \begin{aligned} &\int_{-\infty}^{\infty} x^l p_n(x) p_{n+l-2}(x) d\alpha(x) \\ &\quad = a_{n+1} a_{n+2} \dots a_{n+l-2} \left( \sum_{j=0}^{l-1} a_{n+j}^2 + \sum_{0 \leqq j \leqq j' \leqq l-2} b_{n+j} b_{n+j'} \right) \quad (l \geqq 2). \end{aligned}$$

Indeed, using (5.6) with $m = n + l - 2$, and writing $j_0 = n$ and $j_l = m$ as in (5.6), we can see by (5.5) that every nonvanishing term on the right-hand side of (5.6) can be characterized in one of the following two ways. Either there is an $r$ with $0 \leqq r \leqq l - 1$ such that $j_{r+1} = j_r - 1$ and $j_{i+1} = j_i + 1$ for $i \neq r$

or there are $r$ and $s$ with $0 \leqq r < s \leqq l-1$ such that $j_{r+1} = j_r, j_{s+1} = j_s$, and $j_{i+1} = j_i + 1$ for $i \neq r, s (0 \leqq i \leqq l-1)$.

Hence, using (5.3), (5.7), and (5.8), and noting that

$$2 \sum_{0 \leqq j \leqq j' \leqq l-2} b_{n+j} b_{n+j'} = \sum_{j=0}^{l-2} b_{n+j}^2 + \left( \sum_{j=0}^{l-2} b_{n+j} \right)^2,$$

we obtain

$$\begin{aligned}(5.9)\quad \int_{-\infty}^{\infty} & T_k p_n p_{n+k-2} d\alpha \\ &= \int_{-\infty}^{\infty} (2^{k-1}x^k - k2^{k-3}x^{k-2}) p_n(x) p_{n+k-2}(x) d\alpha(x) \\ &= a_{n+1}a_{n+2}\dots a_{n+k-2} \left( 2^{k-1} \sum_{j=0}^{k-1} a_{n+j}^2 + 2^{k-2} \sum_{j=0}^{k-2} b_{n+j}^2 \right. \\ &\quad \left. + 2^{k-2} \left( \sum_{j=0}^{k-2} b_{n+j} \right)^2 - k2^{k-3} \right).\end{aligned}$$

An identical argument for the system $t_n = p_n(d\alpha_t)$ replacing $p_n = p_n(d\alpha)$ gives a similar formula for $\int_{-\infty}^{\infty} T_k t_n t_{n+k-2} d\alpha_t$. Substituting (5.1) into that formula, we obtain that

$$(5.10)\quad \int_{-\infty}^{\infty} T_k t_n t_{n+k-2} d\alpha_t = 0 \quad (n \geqq 2);$$

this formula could also have been obtained from (3.2), at least for $n > k$. Note that the numbers $a_\nu$ and $b_\nu$ are real, so the squares on right-hand side of (5.9) are positive; moreover, we have $a_\nu > 0$ for $\nu > 0$ (see the text immediately following (1.4)). Hence by (5.9), (5.10), (1.10)(ii), and (1.5) we obtain

$$(5.11)\quad \limsup_{n\to\infty} a_{n+1}a_{n+2}\dots a_{n+k-2} \left( 2 \sum_{j=0}^{k-1} a_{n+j}^2 + \sum_{j=0}^{k-2} b_{n+j}^2 - k/2 \right) \leqq 0.$$

Next we are going to use (5.4) and (5.11) to show that the numbers $a_n$ are bounded. By dropping the second sum from (5.11), we can see that there is a number $M$ such that

$$(5.12)\quad 4a_{\nu+1}a_{\nu+2}\dots a_{\nu+k-2} \sum_{j=0}^{k-1} a_{\nu+j}^2 \leqq k a_{\nu+1}a_{\nu+2}\dots a_{\nu+k-2} + M$$

holds for $\nu \geqq 1$, say. Now in case $k = 2$ the boundedness of the $a_n$'s is obvious from here, since in that case the product $a_{\nu+1}a_{\nu+2}\dots a_{\nu+k-2}$ is empty, and so it equals 1. The difficulty in case $k \geqq 3$ lies in the fact that it is not immediately obvious that the lim inf of this product is not zero.

Assume now that $k \geqq 3$, and let $n \geqq 1$. Write

$$A_n = \min_{0 \leqq l \leqq k-1} a_{n+lk}.$$

Since the numbers $a_\nu$ are positive for $\nu \geqq 1$ (cf. the text right after (1.4)), it follows from (5.12) with $\nu = n + i(k-2)$ that

$$\begin{aligned} &4a_{n+i(k-2)+1}a_{n+i(k-2)+2}\dots a_{n+(i+1)(k-2)}\sum_{j=0}^{k-1} A_{n+j}^2 \\ &\leqq ka_{n+i(k-2)+1}a_{n+i(k-2)+2}\dots a_{n+(i+1)(k-2)} + M \quad (0 \leqq i \leqq k-1). \end{aligned}$$

Adding this for $0 \leqq i \leqq k-1$ and dividing by the coefficient of $4\sum_{j=0}^{k-1} A_{n+j}^2$, we obtain

$$\begin{aligned} 4\sum_{j=0}^{k-1} A_{n+j}^2 &\leqq k + \frac{M}{\sum_{i=0}^{k-1} a_{n+i(k-2)+1}a_{n+i(k-2)+2}\dots a_{n+(i+1)(k-2)}} \\ &\leqq k + (M/k)(a_{n+1}a_{n+2}\dots a_{n+k(k-2)})^{-1/k}, \end{aligned}$$

where the last inequality follows by an application of the inequality for arithmetic and geometric means. Since the limit of the product $a_{n+1}a_{n+2}\dots a_{n+k(k-2)}$ is $2^{-k(k-2)}$ as $n \longrightarrow \infty$ according to (5.4), it follows from here that the numbers $A_n$ are bounded. Now, since

$$\lim_{n\to\infty} \frac{a_{n+k}}{a_n} = \lim_{n\to\infty} \frac{a_{n+1}a_{n+2}\dots a_{n+k}}{a_n a_{n+1}\dots a_{n+k-1}} = 1 \tag{5.13}$$

according to (5.4), it also follows that the numbers $a_n$ are bounded.

We are now within easy reach of establishing (1.11). For this we will again consider the case $k = 2$ as well. That is, henceforth we again assume $k \geqq 2$ only. As the product $a_{n+1}a_{n+2}\dots a_{n+k-2}$ is bounded away from 0 in view of (5.4) and the boundedness of the $a_\nu$'s, and since the $a_\nu$'s are positive, by (5.11) we have

$$\limsup_{n\to\infty}\left(2\sum_{j=0}^{k-1} a_{n+j}^2 + \sum_{j=0}^{k-2} b_{n+j}^2\right) \leqq k/2. \tag{5.14}$$

Let $S$ be an arbitrary set of integers such that the limits

$$\alpha_j = \lim_{n\to\infty, n\in S} a_{j+kn} \quad (1 \leqq j \leqq k)$$

exist. For (1.11)(i) it is sufficient to show that each $\alpha_j$ must equal $1/2$. By (5.4) and (5.14) we have

$$1/4 = (\alpha_1^2\alpha_2^2 \dots \alpha_k^2)^{1/k} \geqq (\alpha_1^2 + \alpha_2^2 + \dots + \alpha_k^2)/k,$$

which, according to the case of equality in the inequality for arithmetic and geometric means is possible only if $\alpha_1 = \alpha_2 = \dots = \alpha_k = 1/2$. Thus (1.11)(i) follows. Now (1.11)(ii) follows from (5.14). The proof of Theorem 1.5 is complete.

We conclude with relating the case $k = 2$ of our Theorem 1.2 with Nikishin's Theorem 1 in [**14**, §2, p. 24]. This says the following:

Suppose the recurrence coefficients $a_l, b_l$ satisfy

$$\text{(5.15)} \quad \sum_{l=n}^{\infty} |a_l^2 - 1/4| = o(1/n)$$

and

$$\text{(5.16)} \quad \sum_{l=n}^{\infty} (|b_l + b_{l+1}| + |b_l b_{l+1}|) = o(1/n).$$

Then the support of $\alpha$ contains only finitely many points outside the interval $[-1, 1]$.

Now we have

$$\begin{aligned} \epsilon_{2nn} &= 2(a_n^2 + a_{n+1}^2 + b_n^2) - 1 \\ \epsilon_{2,n,n-1} &= 2a_{n-1}(b_{n-1} + b_n), \quad \epsilon_{2,n,n+1} = 2a_{n+1}(b_n + b_{n+1}), \\ \epsilon_{2,n,n-2} &= 2a_{n-1}a_n - 1/2, \quad \text{and} \quad \epsilon_{2,n,n+2} = 2a_{n+1}a_{n+2} - 1/2. \end{aligned}$$

Thus (1.8) with $k = 2$ can be rewritten as

$$\text{(5.17)} \quad \begin{aligned} \sum_{l=n}^{\infty} (&|2a_n^2 + 2a_{n+1}^2 + 2b_n^2 - 1| + a_n|b_{n-1} + b_n| \\ &+ a_{n+1}|b_n + b_{n+1}| + (2a_{n-1}a_n - 1/2)^+ \\ &+ (2a_{n+1}a_{n+2} - 1/2)^+ ) < \frac{1}{36(n+4)}. \end{aligned}$$

It is easy to see that this can be deduced from (5.15) and (5.16), i.e., that this is a condition weaker than (5.15)–(5.16), since

$$b_n^2 \leqq (b_n + b_{n+1})^2 + 2|b_n B_{n+1}|.$$

Actually, Nikishin needs only a weakened form of (5.15)–(5.16) in that he needs to require that the left-hand sides in these equations be smaller than $C_1/(n+4)$

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and $C_2/(n+5)$ with appropriate positive constants $C_1$ and $C_2$, similarly to our inequality (5.17) (the constants in Nikishin's paper are of the same order of magnitude as the constant in (5.17), but an exact comparison is not possible).

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