# 3

# Potential theory

The use of potential functions represents another approach to solving problems in magnetostatics. We first treat the vector potential $A$. The physical significance of the vector potential has been debated for many years and it plays an important role in time-dependent phenomena.[1] However in classical magnetostatics, the potentials are usually treated as auxiliary mathematical quantities that are used to simplify the calculation of the magnetic fields. A scalar potential $V_m$ can also be defined that satisfies Laplace's equation in current-free regions.

## 3.1 Vector potential

In our discussion of curl $B$ in Chapter 1, we saw that the magnetic field could be expressed in the form given by Equation 1.21.

$$\overrightarrow{B} = \frac{\mu_0}{4\pi} \nabla \times \int \frac{\overrightarrow{J}}{R} dV. \tag{3.1}$$

We can rewrite this equation as

$$\overrightarrow{B} = \nabla \times \overrightarrow{A}, \tag{3.2}$$

where we define the *vector potential A* as

$$\overrightarrow{A} = \frac{\mu_0}{4\pi} \int \frac{\overrightarrow{J}}{R} dV. \tag{3.3}$$

This equation is valid in rectangular coordinates,[2] where the direction of $A$ is the same as that of $J$.

Since $B$ is defined as the curl of a vector by Equation 3.2, the divergence equation for $B$, $\nabla \cdot \overrightarrow{B} = 0$, is automatically satisfied because of the vector identity $\nabla \cdot \nabla \times \overrightarrow{V} = 0$. Using Stokes's theorem, we have

$$\oint \vec{A}\cdot\vec{dl} = \int(\nabla\times\vec{A})\cdot\vec{dS} = \int\vec{B}\cdot\vec{dS}.$$

Thus the magnetic flux through some surface $S$ is given by the contour integral of $A$ around the perimeter of $S$.

$$\Phi_B = \oint \vec{A}\cdot\vec{dl}. \tag{3.4}$$

Any solution for the vector potential is unique, provided that the sources are confined to a finite region of space.[3]

There is an uncertainty in the defining relation for $A$ in Equation 3.2. Any vector whose curl vanishes can be added to $A$ without affecting the value of $B$. For example, the gradient of a scalar function $\psi$ can be added because

$$\nabla\times\nabla\psi(r) = 0.$$

This freedom can be used to fix the divergence of $A$. Starting with Equation 3.3, the divergence of the vector potential can be written as

$$\nabla\cdot\vec{A} = \frac{\mu_0}{4\pi}\mathbb{I},$$

where

$$\mathbb{I} = \int\vec{J}(r')\nabla\cdot\left(\frac{1}{|\vec{r}-\vec{r}'|}\right)dV'.$$

Primes are used to indicate source coordinates. Note that the $\nabla$ operator is defined in terms of the observation point (field) coordinates, while the integration and current density depend on source coordinates. Replacing the $\nabla$ operator with one defined in terms of source coordinates, we find

$$\mathbb{I} = -\int\vec{J}(r')\nabla'\cdot\left(\frac{1}{|\vec{r}-\vec{r}'|}\right)dV'.$$

Now we can integrate using Equation B.3 and find that

$$\mathbb{I} = -\int\nabla'\cdot\left(\frac{\vec{J}\left(\vec{r}'\right)}{|\vec{r}-\vec{r}'|}\right)dV' + \int\frac{1}{|\vec{r}-\vec{r}'|}\nabla'\cdot\vec{J}\left(\vec{r}'\right)dV'.$$

The second term on the right-hand side vanishes because $\nabla'\cdot\vec{J'} = 0$ from Equation 1.4. We use the divergence theorem to transform the other integral and get

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$$\mathbb{I} = -\int \left( \frac{\overrightarrow{J}(\overrightarrow{r}')}{|\overrightarrow{r} - \overrightarrow{r}'|} \right) \cdot \overrightarrow{dS}.$$

We choose to evaluate this integral over a large radius sphere where all the current sources vanish. Then $\mathbb{I} = 0$ and we conclude that

$$\nabla \cdot \overrightarrow{A} = 0. \tag{3.5}$$

This result is equivalent to setting the arbitrary scalar function $\psi = 0$ and is known as the *Coulomb gauge*. Equation 3.5 represents a constraint on the components of the vector $A$.

Using Equation 1.24, we can relate the vector potential to the conduction current density $J$.

$$\nabla \times (\nabla \times \overrightarrow{A}) = \mu_0 \overrightarrow{J}, \tag{3.6}$$

Then, using the vector identity B.7, we obtain

$$\nabla(\nabla \cdot \overrightarrow{A}) - \nabla^2 \overrightarrow{A} = \mu_0 \overrightarrow{J}.$$

Using Equation 3.5, we can eliminate the gradient term and find that

$$\nabla^2 \overrightarrow{A} = -\mu_0 \overrightarrow{J}. \tag{3.7}$$

This is the vector *Poisson equation* for $A$. The solution is valid both inside and outside of the conductor. Equation 3.3 is a particular solution of this equation. When $J = 0$, Equation 3.7 is called the *Laplace equation*. Solutions of the Laplace equation are known as *harmonic functions*. In Cartesian coordinates, the quantity $\nabla^2 \overrightarrow{A}$ has the three scalar components $\nabla^2 A_\alpha$, where $\alpha$ corresponds to $x$, $y$, and $z$. For example,

$$\nabla^2 A_x = -\mu_0 J_x.$$

In non-Cartesian coordinate systems, the components of $\nabla^2 \overrightarrow{A}$ must be found from Equation B.7.

## 3.2 Vector potential in two dimensions

In cases where the current density is constant in one dimension, it is possible to develop a two-dimensional version of the theory that is significantly simpler than the general three-dimensional case. Of course this is only an approximation to the real world, but the approximation may be quite good, for example in the central

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region of a long magnet, far from its ends. The great power of the two-dimensional approximation will be most apparent in Chapter 5, where we develop the theory using complex analysis.

Let us consider an infinitely long current filament along the $z$ axis. Direct integration using Equation 3.3 to $\pm\infty$ diverges. It is possible to perform the integration over a long, but finite length and then to develop an expression for $A_z$ in a power series in $\rho/L$.[4] However, we will adopt another approach using the differential Equation 3.2. The field from the filament was given in Equation 1.15 and so we have

$$\nabla \times \overrightarrow{A} = \frac{\mu_0 I}{2\pi \rho}\hat{\phi}.$$

The $\rho$ component of the curl in cylindrical coordinates gives

$$\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho} = \frac{\mu_0 I}{2\pi \rho}.$$

The vector potential is constant in the $z$ direction, so the derivative with respect to $z$ vanishes and we have

$$-dA_z = \frac{\mu_0 I}{2\pi}\frac{d\rho}{\rho}.$$

Integrating this equation, we find

$$A_z = -\frac{\mu_0 I}{2\pi}\ln(\rho) + c.$$

If write the constant of integration $c$ in terms of the value of $A_z$ at some reference radius $\rho_0$, then

$$c = \frac{\mu_0 I}{2\pi}\ln(\rho_o)$$

and the vector potential for the infinite current filament is

$$A_z(\rho) = -\frac{\mu_0 I}{2\pi}\ln\left(\frac{\rho}{\rho_o}\right). \tag{3.8}$$

The presence of any constant terms in $A_z$ is not important since they will be removed when taking derivatives to find $B$.

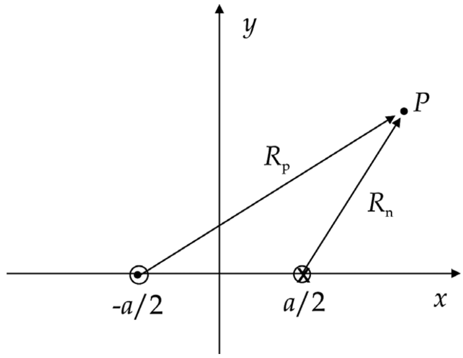


Figure 3.1 Bifilar conductors.

**Example 3.1:** bifilar conductors
One method of handling the issue of the reference radius is to argue that a single filament of current is not physical because any real current must be part of a closed circuit and therefore must have a return filament somewhere with the current flowing in the opposite direction. Consider the bifilar configuration shown in Figure 3.1. Using Equation 3.8 for both the positive ($p$) and negative ($n$) filaments, we find

$$A_z = -\frac{\mu_0 I}{2\pi}\left[\ln(R_p) - \ln(R_{po}) - \ln(R_n) + \ln(R_{no})\right],$$

where $R$ is the distance from the current element to the observation point $P$. If both filaments return at the same reference position, the dependence on $R_0$ drops out and we find

$$A_z(x,y) = -\frac{\mu_0 I}{2\pi}\ln\left(\frac{R_p}{R_n}\right). \tag{3.9}$$

The sign of $A_z$ depends on the relative magnitudes of $R_p$ and $R_n$.

We can find the two-dimensional vector potential for current distributions by superposition of the vector potential for a filament. The vector potential for a two-dimensional current sheet is given by

$$A_z(x,y) = -\frac{\mu_0}{2\pi}\int K_z(s')\,\ln\left(\frac{R}{R_o}\right)ds', \tag{3.10}$$

where $K_z = dI/ds$ is the sheet current density. The vector potential for a block conductor with finite cross-section is

$$A_z(x,y) = -\frac{\mu_0}{2\pi}\iint J_z(x',y')\ln\left(\frac{R}{R_o}\right)dx'dy'. \tag{3.11}$$

The magnetic field has two components given by

$$\begin{aligned} B_x &= \frac{\partial A_z}{\partial y} \\ B_y &= -\frac{\partial A_z}{\partial x} \end{aligned} \tag{3.12}$$

and, except inside a conductor, $A_z$ satisfies the scalar Laplace equation

$$\nabla^2 A_z = \frac{\partial^2 A_z}{\partial x^2} + \frac{\partial^2 A_z}{\partial y^2} = 0. \tag{3.13}$$

## 3.3 Boundary conditions on $A$

The boundary conditions on the vector potential $A$ can be determined from the boundary conditions on the magnetic field. Consider a boundary surface $S$ located at the intersection of two regions of space. We know from Equation 2.22 that the normal component of $B$ must be conserved across $S$. Thus the magnetic flux crossing $S$ is conserved and from Equation 3.4

$$\oint \vec{A} \cdot \vec{dl} = \Phi_B,$$

we see that the line integral of $A$ around the perimeter of $S$ must also be conserved. Therefore the tangential component of $A$ must be conserved on the boundary.

$$\vec{A}_t^{(1)} = \vec{A}_t^{(2)}. \tag{3.14}$$

The boundary condition on the tangential component of $H$ given in Equation 2.23 can be written in the form

$$\frac{1}{\mu^{(2)}} (\nabla \times \vec{A})_t^{(2)} - \frac{1}{\mu^{(1)}} (\nabla \times \vec{A})_t^{(1)} = K, \tag{3.15}$$

where $K$ is the surface current on $S$, if applicable. These two vector relations provide four constraints on $A$ at the boundary.[3]

The boundary conditions can be considerably simpler in two dimensions. Assume, for example, that the problem is uniform in the $z$ direction and that we have a boundary between two regions, as shown in Figure 3.2. All the current is along $z$, so $A$ only has the component $A_z$. In this case, the boundary surface is parallel to the $x$-$z$ plane and the tangential component of $A$ is along $z$. From Equation 3.14, we have

$$A_z^{(1)} = A_z^{(2)},$$

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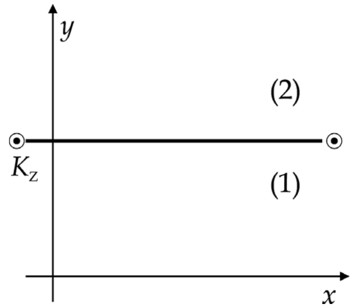


Figure 3.2 Boundary in two dimensions.

so the vector potential is continuous across the boundary. Since

$$\nabla \times \overrightarrow{A} = \hat{x}\partial_y A_z - \hat{y}\,\partial_x A_z,$$

the tangential component of the curl is along $x$. Thus Equation 3.15 gives

$$\frac{1}{\mu^{(2)}}\,\partial_y A_z^{(2)} - \frac{1}{\mu^{(1)}}\,\partial_y A_z^{(1)} = K_z.$$

In this case, we have two constraints that must be satisfied at the boundary.

## 3.4 Vector potential for a localized current distribution

Consider a localized distribution of current, as shown in Figure 3.3. We pick some origin $O$ inside the distribution and examine the potential at a field location $P$. Since the distance between the source point at $r'$ and the field point at $r$ is

$$R = \{r^2 + r'^2 - 2rr'\cos\theta\}^{1/2},$$

we can express the inverse distance as

$$\frac{1}{R} \simeq \frac{1}{r} + \frac{\overrightarrow{r}\cdot\overrightarrow{r}'}{r^3} + \cdots.$$

Then the first two terms in the multipole expansion for $A$ are

$$\overrightarrow{A}(\overrightarrow{r}) \simeq \frac{\mu_0}{4\pi}\left[\frac{1}{r}\int \overrightarrow{J}(\overrightarrow{r}')\,dV' + \frac{\overrightarrow{r}}{r^3}\cdot\int \overrightarrow{r}'\overrightarrow{J}(\overrightarrow{r}')\,dV' + \cdots\right]. \tag{3.16}$$

The first integral vanishes because the current distribution consists of closed loops. In the second integral, $J$ corresponds to one of the components of $A$,

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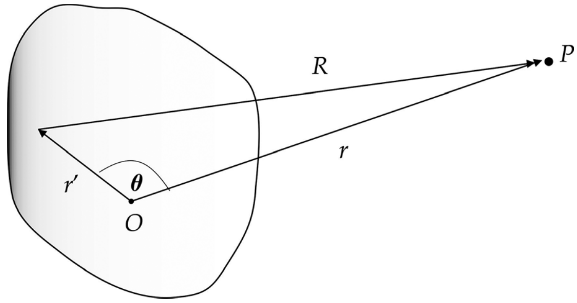


Figure 3.3 Localized current distribution.

which we specify with the index $i$, while $r'$ is part of the scalar product with $r$, which we specify with the index $j$. Thus the integral has the form

$$\mathbb{I} = \int r'_j J'_i \, dV'.$$

Using the vector identity B.3, we have

$$\nabla' \cdot (r'_i \overrightarrow{J'}) = r'_i \nabla' \cdot \overrightarrow{J'} + \overrightarrow{J'} \cdot \nabla' r'_i.$$

The first term vanishes because of Equation 1.4 and in the second term, we have $\nabla' r'_i = \hat{i}$. Thus

$$J'_i = \nabla' \cdot (r'_i \overrightarrow{J'})$$

and so we have

$$\mathbb{I} = \int \nabla' \cdot (r'_i \overrightarrow{J'}) r'_j \, dV'.$$

We can do the integral by parts with

$$u = r'_j$$
$$dv = \nabla' \cdot \left( r'_i \overrightarrow{J'} \right).$$

This gives

$$\mathbb{I} = \int r'_i r'_j \overrightarrow{J'} \, dS' - \int r'_i \overrightarrow{J'} \cdot \nabla' r'_j \, dV'.$$

The first term vanishes for a surface outside the charge distribution. In the second term, the gradient in the integrand vanishes except for the $j$ component. Thus

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$$\mathbb{I} = -\int r_i' J_j' \, dV'.$$

Comparing with Equation 3.16, this implies that

$$\int (\vec{r} \cdot \vec{r}') \, \vec{J} \, dV' = -\int (\vec{r} \cdot \vec{J}) \, \vec{r}' dV'. \tag{3.17}$$

Now consider the triple vector product from Equation B.1.

$$\vec{r} \times (\vec{r}' \times \vec{J}') = \vec{r}'(\vec{r} \cdot \vec{J}') - \vec{J}'(\vec{r} \cdot \vec{r}').$$

Substituting Equation 3.17, we have

$$2\int (\vec{r} \cdot \vec{r}')\vec{J}' dV' = -\int \vec{r} \times (\vec{r}' \times \vec{J}') dV'. \tag{3.18}$$

Substituting this back into Equation 3.16, we find that the vector potential is

$$\vec{A}(\vec{r}) \simeq -\frac{1}{2} \frac{\mu_0}{4\pi} \frac{\vec{r}}{r^3} \times \int \vec{r}' \times \vec{J}' dV'.$$

We define the magnetic moment of the current distribution as

$$\vec{m} = ½ \int \vec{r}' \times \vec{J}(\vec{r}') dV'. \tag{3.19}$$

Then the vector potential of the current distribution can be written as [5]

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \frac{\vec{m} \times \vec{r}}{r^3}. \tag{3.20}$$

Thus we find that the elementary form of magnetic matter is a magnetic dipole.

In spherical coordinates, the vector potential for a magnetic dipole is directed in the azimuthal direction. We can find the magnetic field from the dipole by taking the curl of $A$.

$$\begin{aligned} B_r &= \frac{1}{r \sin\theta} \partial_\theta (A_\phi \sin\theta) \\ &= \frac{\mu_0}{4\pi} \frac{2m}{r^3} \cos\theta \end{aligned} \tag{3.21}$$

and

$$\begin{aligned} B_\theta &= -\frac{1}{r} \partial_r (r A_\phi) \\ &= \frac{\mu_0}{4\pi} \frac{m}{r^3} \sin\theta. \end{aligned} \tag{3.22}$$

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The remaining component $B_\phi = 0$. We see that the field of a magnetic dipole falls off like $1/r^3$.

We can relate this definition of the magnetic moment, Equation 3.19, with our discussion in Chapter 1 of the magnetic moment of a planar current loop. We let

$$\vec{J}\, dV' \to I\, \overrightarrow{dl'}\,.$$

Then Equation 3.19 gives

$$\vec{m} = \tfrac{1}{2}\, I \oint \vec{r}\,' \times \overrightarrow{dl'}\,.$$

The magnitude of the quantity $\tfrac{1}{2}\vec{r}\,' \times \overrightarrow{dl'}$ is the area of a triangular region inside the current loop. The closed integral then gives the total area $A$ enclosed by the loop. Thus

$$\vec{m} = I\, A\;\hat{n},$$

which agrees with the result in Equation 1.10.

Now consider a volume of magnetic material that contains many magnetic dipoles. The contribution to the vector potential from one small part of the overall volume at $r'$ can be written in terms of the magnetization vector $M$ as

$$\overrightarrow{dA} = \frac{\mu_0}{4\pi} \frac{\overrightarrow{M'} \times \vec{R}}{R^3}\, dV'.$$

Using the relation

$$\nabla\left(\frac{1}{R}\right) = -\frac{1}{R^2}\hat{r} = -\nabla'\left(\frac{1}{R}\right),$$

the vector potential for the whole volume is

$$\vec{A} = \frac{\mu_0}{4\pi}\int \overrightarrow{M'} \times \nabla'\left(\frac{1}{R}\right) dV'.$$

Using the vector identity B.6, we can write $A$ as the two terms

$$\vec{A} = -\frac{\mu_0}{4\pi}\int \nabla' \times \left(\frac{\overrightarrow{M'}}{R}\right) dV' + \frac{\mu_0}{4\pi}\int \frac{\nabla' \times \overrightarrow{M'}}{R}\, dV'.$$

Then using vector identity

$$\int \nabla \times \vec{W}\, dV = -\int \vec{W} \times \hat{n}\, dS$$

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in the first term and dropping the primes, we get

$$\vec{A} = \frac{\mu_0}{4\pi}\int \frac{\vec{M}\times\hat{n}}{R}\,dS + \frac{\mu_0}{4\pi}\int \frac{\nabla\times\vec{M}}{R}\,dV. \tag{3.23}$$

We have seen previously from Equation 2.2 that $\vec{K}_m = \vec{M}\times\hat{n}$ is the surface current density and from Equation 2.5 that $\vec{J} = \nabla\times\vec{M}$ is the volume current density.

## 3.5 Force on a localized current distribution

The force on a localized current in an applied magnetic field is

$$\vec{F} = \int \vec{J'}\times\vec{B}\,dV'.$$

If the field is non-uniform, each component of $B$ can be expanded in a Taylor's series.

$$B_i(\vec{r}) \simeq B_i(0) + \vec{r}\cdot\nabla B_i(0) + \cdots.$$

Then

$$\vec{F} \simeq -\vec{B}(0)\times\int \vec{J}(\vec{r}\,')dV' + \int \vec{J}(\vec{r}\,')\times\left[(\vec{r}\,'\cdot\nabla)\vec{B}(0)\right]dV'.$$

The first integral vanishes since $J$ consists of closed loops. Using Equation B.2, we have

$$\nabla(\vec{r}\,'\cdot\vec{B}) = \vec{r}\,'\times(\nabla\times\vec{B}) + \vec{B}\times(\nabla\times\vec{r}\,') + (\vec{B}\cdot\nabla)\,\vec{r}\,' + (\vec{r}\,'\cdot\nabla)\,\vec{B}.$$

The first term on the right-hand side vanishes outside the current distribution, while the second and third terms vanish because the $\nabla$ operator refers to field coordinates. Thus the force can be written as

$$\vec{F} = \int \vec{J'}\times\nabla(\vec{r}\,'\cdot\vec{B})dV'.$$

From Equation B.6, we have

$$\nabla\times[\vec{J'}(\vec{r}\,'\cdot\vec{B})] = (\vec{r}\,'\cdot\vec{B})\nabla\times\vec{J'} - \vec{J'}\times\nabla(\vec{r}\,'\cdot\vec{B}).$$

The first term on the right side vanishes because $\nabla$ operates on field coordinates, so

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$$\vec{F} = -\nabla \times \int \vec{J'} (\vec{r}' \cdot \vec{B}) dV'.$$

Using the relation in Equation 3.18 with $r$ replaced by $B$, we have

$$\int (\vec{B} \cdot \vec{r}') \vec{J'} \, dV' = -\tfrac{1}{2} \vec{B} \times \int \vec{r}' \times \vec{J'} dV'.$$

Thus

$$\vec{F} = -\nabla \times \int \left[ -\tfrac{1}{2} \vec{B} \times (\vec{r}' \times \vec{J'}) \right] dV'.$$

Using Equation 3.19, we can write this in terms of the magnetic moment as

$$\vec{F} = \nabla \times (\vec{B} \times \vec{m}').$$

Finally, using Equation B.9

$$\vec{F} = \vec{B} \, (\nabla \cdot \vec{m}') - \vec{m}' \, (\nabla \cdot \vec{B}) + (\vec{m}' \cdot \nabla) \, \vec{B} - (\vec{B} \cdot \nabla) \vec{m}'.$$

The first and fourth terms vanish because $m$ is independent of the field coordinates. Thus dropping the primes, we find that the force on a magnetic dipole in an inhomogeneous magnetic field is given by

$$\vec{F} = (\vec{m} \cdot \nabla) \, \vec{B}. \tag{3.24}$$

For a continuous magnetization distribution, the force is given by [6]

$$\vec{F} = \int (\vec{M} \cdot \nabla) \vec{B} \, dV. \tag{3.25}$$

## 3.6 Magnetic scalar potential

In regions of space where there are no conduction currents present, the magnetostatic field equations become

$$\begin{aligned} \nabla \times \vec{H} &= 0 \\ \nabla \cdot \vec{B} &= 0. \end{aligned}$$

From the curl equation, we know that the magnetic field can be expressed as the gradient of a scalar function

$$\vec{H} = -\nabla V_m, \tag{3.26}$$

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which we call the *magnetic scalar potential* $V_m$. Multiplying this equation by $\mu_0$ and substituting the expression for $B$ into the divergence equation, we find that

$$\nabla^2 V_m = 0. \tag{3.27}$$

Thus $V_m$ satisfies the Laplace equation in regions where the conduction current $J = 0$.

The boundary condition on the transverse component of the field $H_t$ in Equation 2.23 gives

$$\begin{aligned} H_t^{(2)} - H_t^{(1)} &= K \\ -\partial_t V_m^{(2)} + \partial_t V_m^{(1)} &= K. \end{aligned} \tag{3.28}$$

In the case where no surface current is present, the gradients in the two regions must be the same. This implies that the scalar potentials in the two regions can only differ at most by a constant factor $c$.

$$V_m^{(2)} = V_m^{(1)} + c$$

From the boundary condition on $B_n$ in Equation 2.22, we have

$$\begin{aligned} B_n^{(2)} &= B_n^{(1)} \\ \mu^{(2)}\, \partial_n V_m^{(2)} &= \mu^{(1)}\, \partial_n V_m^{(1)}. \end{aligned} \tag{3.29}$$

Thus the product of the permeability and the normal derivative of $V_m$ is continuous across the boundary.

Consider a field point $P$ nearby a current loop, as shown in Figure 3.4. Assign a normal vector $n$ to the loop according to the right hand rule. Now displace the loop by the vector $du$. The cross product of $dl$ with $du$ gives the area of the shaded area in the figure. In this case, the solid angle at $P$ subtended by the differential area $dS$ changes by an amount

$$d\Omega = \frac{d\overrightarrow{S}\cdot\hat{r}}{r^2} = \frac{(\overrightarrow{du} \times \overrightarrow{dl})\cdot\hat{r}}{r^2} = \frac{(\overrightarrow{dl} \times \hat{r})\cdot\overrightarrow{du}}{r^2}.$$

The last equation makes use of the fact that the terms in the triple vector product permute. To get the total change in solid angle, we sum over all the parts of the loop.

$$\begin{aligned} d\Omega &= \overrightarrow{du} \cdot \oint \frac{\overrightarrow{dl} \times \overrightarrow{r}}{r^3} \\ &= \overrightarrow{du} \cdot \nabla\Omega \end{aligned}$$

Comparing these two equations, we find that the gradient of the solid angle is given by

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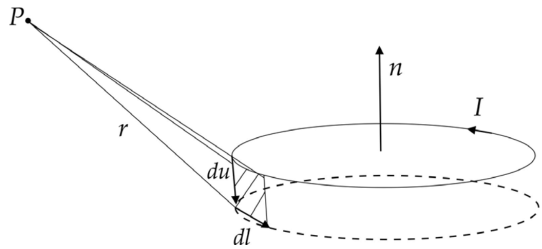


Figure 3.4 Solid angle of a displaced current loop.

$$\nabla\Omega = \oint \frac{\overrightarrow{dl} \times \overrightarrow{r}}{r^3}. \tag{3.30}$$

Now we want to relate this to the scalar potential

$$\nabla V_m = -\frac{\overrightarrow{B}}{\mu}.$$

Expressing $B$ using the Biot-Savart law and using Equation 3.30, we find

$$\begin{aligned}\nabla V_m &= -\frac{I}{4\pi}\oint \frac{\overrightarrow{dl} \times \overrightarrow{r}}{r^3} \\ &= -\frac{I}{4\pi}\nabla\Omega.\end{aligned}$$

The gradients on the two sides of the equation are proportional to each other, so $V_m$ must depend linearly on Ω.

$$V_m = -\frac{I}{4\pi}\Omega. \tag{3.31}$$

This indicates that the magnetic scalar potential is directly related to the solid angle that a current loop subtends at a field observation point. We can ignore any constant term since it drops out when calculating the field. Note that Equation 3.31 implies that the scalar potential is not a single-valued function. If $P$ moves from just in front of the loop to just behind it, the solid angle changes from $2\pi$ to $-2\pi$ since $\hat{n}\cdot\hat{r}$ changes sign.

## 3.7 Scalar potential for a magnetic body

In analogy with the potential due to dielectric polarization in electrostatics,[7] the scalar magnetic potential associated with the magnetization vector is given by

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$$V_m = \frac{1}{4\pi}\int \frac{\vec{M}\cdot\vec{R}}{R^3}dV, \tag{3.32}$$

which we can express as

$$V_m = \frac{1}{4\pi}\int \overrightarrow{M'}\cdot\nabla'\left(\frac{1}{R}\right)dV'.$$

Using the vector identity B.3, we can write this as

$$V_m = \frac{1}{4\pi}\left[\int \nabla'\cdot\frac{\overrightarrow{M'}}{R}dV' - \int\frac{\nabla'\cdot\overrightarrow{M'}}{R}dV'\right].$$

Using the divergence theorem for the first term and dropping the primes gives

$$V_m = \frac{1}{4\pi}\left[\int \frac{\vec{M}\cdot\hat{n}}{R}dS - \int\frac{\nabla\cdot\vec{M}}{R}dV\right]. \tag{3.33}$$

We can identify the first term on the right-hand side as coming from fictitious magnetic charges on the surface of the magnetized body with the surface current density[8]

$$K_m = \vec{M}\cdot\hat{n} \tag{3.34}$$

and the second term can be described as due to the volume charge density

$$\rho_m = -\nabla\cdot\vec{M}. \tag{3.35}$$

This shows that for the purpose of finding the magnetic field outside a magnetized body, we can replace the body with equivalent magnetic surface and volume charges.

## 3.8 General solutions to the Laplace equation

We have found that single components of the vector potential and the scalar potential both satisfy the scalar Laplace equation $\nabla^2 F = 0$ outside conductor regions. In general, this is a three-dimensional partial differential equation. Solutions of the scalar Laplace equation depend on the coordinate system that is used.

A common technique for solving the Laplace equation is to use the method of *separation of variables*. This method assumes that the solution is the product of three terms, each of which only depends on one of the coordinates. As a result, the partial differential equation in three variables is converted into three ordinary

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differential equations, each of which depends on a single variable. The method is most useful when a boundary surface in the problem lies along one of the coordinate directions. The Laplace equation is known to be separable in eleven coordinate systems.[9] We summarize results here for the three most common systems.

*Rectangular coordinates [10]*

The Laplace equation in rectangular coordinates is

$$\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} + \frac{\partial^2 F}{\partial z^2} = 0. \tag{3.36}$$

We assume the solutions can be written in the form

$$F(x, y, z) = X(x)Y(y)Z(z).$$

When this is substituted into Equation 3.36, we obtain

$$\frac{1}{X}\frac{d^2 X}{dx^2} + \frac{1}{Y}\frac{d^2 Y}{dy^2} + \frac{1}{Z}\frac{d^2 Z}{dz^2} = 0.$$

This equation can only be valid for all values of *x, y*, or *z* if each of the three terms is equal to a constant. Thus we have

$$\begin{aligned} \frac{1}{X}\frac{d^2 X}{dx^2} &= a^2 \\ \frac{1}{Y}\frac{d^2 Y}{dy^2} &= b^2 \\ \frac{1}{Z}\frac{d^2 Z}{dz^2} &= c^2, \end{aligned} \tag{3.37}$$

where the three constants have to satisfy the constraint

$$a^2 + b^2 + c^2 = 0. \tag{3.38}$$

In order to satisfy this equation, the set of constants $\{a, b, c\}$ must contain both real and imaginary members. The choice of which constants are real and which are imaginary depends on the specific conditions for the problem under consideration.

As an example, let us suppose that the constants $a$ and $b$ are imaginary and $c$ is real. Then we can define a new set of real constants $\{\alpha, \beta, \gamma\}$ such that

$$\begin{aligned} a^2 &= (i\alpha)^2 = -\alpha^2 \\ b^2 &= (i\beta)^2 = -\beta^2 \\ c^2 &= \gamma^2. \end{aligned}$$

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In the general case, there will be a set of constants that satisfy Equation 3.37.

$$\alpha_n = \{\alpha_1, \alpha_2, \ldots\}$$
$$\beta_m = \{\beta_1, \beta_2, \ldots\}$$

The constraint Equation 3.38 then becomes

$$-\alpha_n^2 - \beta_m^2 + \gamma_{nm}^2 = 0$$

and the separated differential equations are

$$\frac{1}{X_n}\frac{d^2X_n}{dx^2} = -\alpha_n^2$$
$$\frac{1}{Y_m}\frac{d^2Y_m}{dy^2} = -\beta_m^2$$
$$\frac{1}{Z_{nm}}\frac{d^2Z_{nm}}{dz^2} = \gamma_{nm}^2.$$

Then the general solution in Cartesian coordinates has the form

$$F(x,y,z) = \sum_{n,m=1}^{\infty}\left[C_n e^{i\alpha_n x} + D_n e^{-i\alpha_n x}\right]\left[E_m e^{i\beta_m y} + F_m e^{-i\beta_m y}\right]\left[G_{nm} e^{\gamma_{nm} z} + H_{nm} e^{-\gamma_{nm} z}\right]$$
$$+ I_0 + I_1 x + I_2 y + I_3 z.$$

The oscillatory terms could also be written in terms of sines and cosines and the nonoscillatory terms could be written using hyperbolic sines and cosines. The last four terms are also a solution of the Laplace equation that allow for continuity of the potential and the presence of external fields.

*Cylindrical coordinates [11]*

The Laplace equation in cylindrical coordinates is

$$\frac{1}{\rho}\partial_\rho(\rho\partial_\rho F) + \frac{1}{\rho^2}\partial_\phi^2 F + \partial_z^2 F = 0. \tag{3.39}$$

Using the method of separation of variables, we assume that

$$F(\rho, \phi, z) = R(\rho)\Phi(\phi)Z(z).$$

This leads to the three ordinary differential equations

$$\rho\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right) + (k^2\rho^2 - n^2)R = 0 \tag{3.40}$$

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$$\frac{d^2\Phi}{d\phi^2} + n^2\Phi = 0 \tag{3.41}$$

$$\frac{d^2 Z}{dz^2} - k^2 Z = 0, \tag{3.42}$$

where $k$ and $n$ are constants that may be real or imaginary.

In order for the azimuthal dependence to be single-valued, the parameter $n$ in Equation 3.41 must be a real integer. Solutions for the function $\Phi$ have the general form

$$\Phi_n(\phi) = C_n\cos(n\phi) + D_n\sin(n\phi) \tag{3.43}$$

when $n \neq 0$ and

$$\Phi_0(\phi) = C_0\,\phi + D_0 \tag{3.44}$$

when $n = 0$.

The equations for $R$ and $Z$ have different solutions, depending on the values for $n$ and $k$.

(1) real $k \neq 0$

The general form of the $z$ dependence in Equation 3.42 is

$$Z_k(z) = E_k e^{kz} + F_k e^{-kz}.$$

The solution for the radial dependence in Equation 3.40 has the form

$$R_n(k\rho) = G_n J_n(k\rho) + H_n N_n(k\rho),$$

where $J_n$ and $N_n$ are integer Bessel functions of the first and second kind.[12]

(2) imaginary $k \neq 0$

If $k = i\,\kappa$ where $\kappa$ is real, then the $z$ solution is oscillatory.

$$Z_k(z) = E_k e^{i\kappa z} + F_k e^{-i\kappa z}.$$

In this case, Equation 3.40 becomes

$$\rho\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right) - (\kappa^2\rho^2 + n^2)R = 0 \tag{3.45}$$

and the radial solution is

$$R_n(\kappa\rho) = G_n I_n(\kappa\rho) + H_n K_n(\kappa\rho),$$

where $I_n$ and $K_n$ are modified Bessel functions.[13]

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(3) $k = 0$

The $z$ solution has the form

$$Z_0(z) = E_0 z + F_0.$$

The general radial solution for $n \neq 0$ is

$$R_n(\rho) = G_n \rho^n + H_n \rho^{-n}. \tag{3.46}$$

If $n = 0$, the radial solution is

$$R_0(\rho) = G_0 \ln r + H_0. \tag{3.47}$$

The general form of the solution of the Laplace equation in cylindrical coordinates can then be written in the form

$$F(\rho, \phi, z) = \sum\nolimits_{k,n} C_{kn}\, R_n\, (k\rho)\Phi_n(\phi) Z_k(z).$$

Some additional information concerning Bessel functions can be found in Appendix C.

*Spherical coordinates [14]*

The Laplace equation in spherical coordinates is

$$\frac{1}{r^2}\partial_r(r^2 \partial_r F) + \frac{1}{r^2 \sin\theta}\partial_\theta(\sin\theta\, \partial_\theta F) + \frac{1}{r^2 \sin^2\theta}\partial_\phi^2 F = 0. \tag{3.48}$$

The radial and angular parts of this equation can be separated first in the form

$$F(r, \theta, \phi) = R(r) Y(\theta, \phi).$$

This leads to the radial equation

$$\frac{d}{dr}\left(r^2 \frac{dR}{dr}\right) - n(n+1)R = 0, \tag{3.49}$$

which has a general solution of the form

$$R_n(r) = G_n r^n + H_n r^{-n}.$$

The angular equation is

$$\frac{1}{\sin\theta}\partial_\theta(\sin\theta\, \partial_\theta Y) + \frac{1}{\sin^2\theta}\partial_\phi^2 Y + n(n+1)\; Y = 0. \tag{3.50}$$

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The solutions for this equation are known as *spherical harmonics*.[15] The constant $n$ must be an integer to avoid singularities in $Y(\theta,\phi)$ at $\theta = 0$ and $\theta = \pi$. The two angle coordinates can in turn be separated as

$$Y(\theta,\phi) = \Theta(\theta)\Phi(\phi).$$

This leads to the two ordinary differential equations

$$\frac{d}{dx}\left[(1-x^2)\frac{d\Theta}{dx}\right] + \left[n(n+1) - \frac{m^2}{1-x^2}\right]\Theta = 0 \tag{3.51}$$

and

$$\frac{d^2\Phi}{d\phi^2} + m^2\Phi = 0, \tag{3.52}$$

where $x = \cos\theta$. In order for the azimuthal dependence to be single-valued in Equation 3.52, $m$ must be an integer, and $\Phi$ has the solution

$$\Phi_m(\phi) = C_m\cos(m\phi) + D_m\sin(m\phi)$$

when $m \neq 0$ and

$$\Phi_0(\phi) = C_0\phi + D_0$$

when $m = 0$. The solution of Equation 3.51 has the form

$$\Theta_n^m(\theta) = E_{mn}P_n^m(\cos\theta) + F_{mn}Q_n^m(\cos\theta),$$

where $P_n^m$ and $Q_n^m$ are *associated Legendre functions* of the first and second kind.[16] In problems with azimuthal symmetry, we have $m = 0$ and the associated Legendre functions $P_n^m$ reduce to the ordinary Legendre polynomials.

$$P_n^0(\cos\theta) = P_n(\cos\theta).$$

The general form of the solution of the Laplace equation in spherical coordinates can then be written in the form

$$F(r,\theta,\phi) = \sum\nolimits_{n,m} C_{nm}R_n(r)\Theta_n^m(\theta)\Phi_m(\phi).$$

Some additional information concerning Legendre functions is given in Appendix D.

## 3.9 Boundary value problems

Unique solutions for boundary value problems for the Laplace equation require that either the potential $F$ or its normal derivative be specified on the boundary.[17] Problems where $F$ is specified on the boundary are known as *Dirichlet* boundary value problems, while problems where $\partial F/\partial n$ are specified are known as *Neumann* boundary value problems. Both of these types of problem give unique solutions for the Laplace equation. Solutions do not exist when both $F$ and $\partial F/\partial n$ are arbitrarily specified because the derivatives of $F$ have to be constrained to satisfy the Laplace equation.

The solution of boundary value problems begins by separating the problem space into regions with unique values for the current density and permeability. For each region, a potential function is written in the most general form possible. This introduces a set of unknown coefficients in the potential functions. Constraints on these coefficients are determined by demanding that the potential functions satisfy the boundary conditions at all the interfaces between different regions.

Fourier analysis is particularly useful in problems involving rectangular conductors in a space with rectangular boundaries.[18] If the boundaries are infinitely-permeable iron surfaces, they can be replaced with a set of image currents. Then the current density can be expressed as a Fourier series and the fields can be determined from the solution to a boundary value problem.

**Example 3.2:** rectangular conductor in an infinite slot
We first consider an example using the vector potential. Assume we have a rectangular-shaped conductor near the bottom of an infinitely deep slot with infinitely permeable walls, as shown in Figure 3.5. The current flows in the $z$ direction. The current density in the conductor is given by

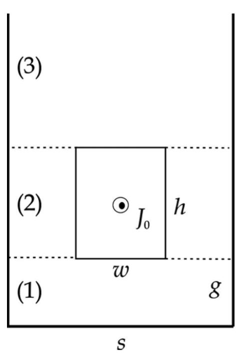


Figure 3.5 Rectangular conductor in a slot.

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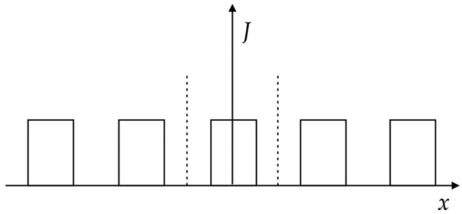


Figure 3.6 Periodic current distribution.

$$J(x) = \begin{cases} 0 & -s/2 \leq x \leq -w/2 \\ J_0 & -w/2 \leq x \leq w/2 \\ 0 & w/2 \leq x \leq s/2. \end{cases}$$

We replace the parallel side walls of the slot with an infinite set of image conductors, whose current density is shown in Figure 3.6. The Fourier series representing the current distribution is

$$J(x) = J_0 \frac{w}{s} + \frac{2J_0}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin\left(\frac{nkw}{2}\right) \cos(nkx), \tag{3.53}$$

where $k = 2\pi/s$.

Divide the problem space vertically into three regions, as shown in Figure 3.5. In region 1, there are no currents so the vector potential $A_1$ satisfies the Laplace equation. To satisfy the boundary conditions, we know that the $x$ dependence has to correspond with the $x$ dependence of $J(x)$. Thus we have

$$A_1 = \sum_{n=1}^{\infty} (C_n e^{nky} + D_n e^{-nky}) \cos(nkx),$$

where $C_n$ and $D_n$ are unknown coefficients. The solution $A_3$ for region 3 also has to satisfy the Laplace equation. Since region 3 extends to infinite values of $y$, the term proportional to $e^{nky}$ must vanish. Far from the conductor, the field must be uniform along the $x$ direction, so the potential must contain a term proportional to $y$. It must also contain a constant term to guarantee continuity of $A$. Thus the general form of the potential in region 3 is

$$A_3 = E_0 + E_1 y + \sum_{n=1}^{\infty} F_n e^{-nky} \cos(nkx).$$

Since region 2 contains the conductor, the vector potential $A_2$ has to satisfy the Poisson equation. The total potential $A_2$ has a general (or homogeneous) part plus a particular solution to the Poisson equation.

$$A_2 = A_{2h} + A_{2p}.$$

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To match the potential at the boundary with region 3, the general part of the potential has to include a constant term and a term linear in $y$. Thus we have

$$A_{2h} = G_0 + G_1 y + \sum_{n=1}^{\infty} (H_n e^{nky} + M_n e^{-nky}) \cos(nkx).$$

Since the current density $J$ has a constant term and a periodic term, we look for a particular potential of the form

$$A_{2p} = G_{2p}\, y^2 + \sum_{n=1}^{\infty} L_n \cos(nkx).$$

Substitute this expression into the Poisson equation, together with Equation 3.53 for the current density. Since the expansion makes use of orthogonal functions, we can equate the constant term and each term in the series independently. We find the coefficients for the particular solution are

$$G_{2p} = -\frac{\mu_0 J_0 w}{2s}$$

and

$$L_n = \frac{2\mu_0 J_0}{\pi n^3 k^2} \sin\left(\frac{nkw}{2}\right).$$

The three equations for $A$ contain a total of nine unknown coefficients. We find the values of these coefficients by imposing the boundary conditions.

Case 1: $y = 0$.

The bottom boundary is an infinite permeability surface, so $B = B_y$ must be perpendicular to this surface. Thus

$$B_x = \frac{\partial A_1}{\partial y} = 0,$$

which gives

$$C_n - D_n = 0. \tag{3.54}$$

Case 2: $y = g$

The nonperiodic and periodic parts of the equations for the continuity of $A$ and $\partial_y A$ give the four equations

$$0 = G_0 + G_1\, g - \frac{\mu_0 J_0 w}{2s}\, g^2 \tag{3.55}$$

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$$C_n e^{nkg} + D_n e^{-nkg} = H_n e^{nkg} + M_n e^{-nkg} + \frac{2\mu_0 J_0}{\pi n^3 k^2} \sin\left(\frac{nkw}{2}\right) \tag{3.56}$$

$$0 = G_1 - \frac{\mu_0 J_0 w}{s} g \tag{3.57}$$

$$C_n e^{nkg} - D_n e^{-nkg} = H_n e^{nkg} - M_n e^{-nkg}. \tag{3.58}$$

Case 3: $y = g + h$

The nonperiodic and periodic parts of the equations for the continuity of $A$ and $\partial_y A$ give the four equations

$$G_0 + G_1(g+h) - \frac{\mu_0 J_0 w}{2s}(g+h)^2 = E_0 + E_1(g+h) \tag{3.59}$$

$$H_n e^{nk(g+h)} + M_n e^{-nk(g+h)} + \frac{2\mu_0 J_0}{\pi n^3 k^2} \sin\left(\frac{nkw}{2}\right) = F_n e^{-nk(g+h)} \tag{3.60}$$

$$G_1 - \frac{\mu_0 J_0 w}{s}(g+h) = E_1 \tag{3.61}$$

$$H_n e^{nk(g+h)} - M_n e^{-nk(g+h)} = -F_n e^{-nk(g+h)}. \tag{3.62}$$

Case 4: $y \to \infty$

At large $y$, far from the conductor, the field must be uniform, so we have

$$B_x = \frac{\partial A_3}{\partial y} = E_1.$$

From far above, the conductor looks like an infinite current sheet with current density $J_0 h$, whose strength is reduced by the filling factor $w/s$ and is enhanced by a factor 2 due to the presence of the bottom permeable surface. Then using Equation 1.17, we find

$$\begin{aligned} E_1 = B_x &= -\tfrac{1}{2}\mu_0 J_0 h \frac{w}{s} 2 \\ &= -\frac{\mu_0 J_0 h w}{s}. \end{aligned} \tag{3.63}$$

Equations 3.54–3.63 give ten constraints on the nine unknown coefficients. However, Equation 3.61 is redundant since it is equivalent to Equations 3.57 and 3.63. Thus we have nine equations in nine unknowns. After solving this system of equations, the resulting vector potentials are:

$$A_1 = \alpha \sum_{n=1}^{\infty} (e^{nky} + e^{-nky}) \cos(nkx), \tag{3.64}$$

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where

$$\alpha = -\frac{\mu_0 J_0}{\pi n^3 k^2} \sin\left(\frac{nkw}{2}\right) e^{-nkg} \left(e^{-nkh} - 1\right); \tag{3.65}$$

$$A_2 = -\frac{\mu_0 J_0 w}{2s}(y-g)^2 + \sum_{n=1}^{\infty}(\beta_1\, e^{nky} + \beta_2\, e^{-nky} + \beta_3)\cos(nkx), \tag{3.66}$$

where

$$\beta_1 = -\frac{\mu_0 J_0}{\pi n^3 k^2} \sin\left(\frac{nkw}{2}\right) e^{-nk(g+h)} \tag{3.67}$$

$$\beta_2 = \frac{\mu_0 J_0}{\pi n^3 k^2} \sin\left(\frac{nkw}{2}\right)\left(-\,e^{-nk(g+h)} - e^{nkg} + e^{-nkg}\right) \tag{3.68}$$

$$\beta_3 = 2\mu_0 J_0 \sin\left(\frac{nkw}{2}\right); \tag{3.69}$$

$$A_3 = \frac{\mu_0 J_0 hw(2g+h)}{2s} - \frac{\mu_0 J_0 hw}{s} y + \sum_{n=1}^{\infty} \gamma e^{-nky} \cos(nkx), \tag{3.70}$$

where

$$\gamma = -\frac{\mu_0 J_0}{\pi n^3 k^2} \sin\left(\frac{nkw}{2}\right) \left(-\,e^{nk(g+h)} + e^{-nk(g+h)} + e^{nkg} - e^{-nkg}\right). \tag{3.71}$$

The series for the vector potential converge rapidly because of the $n^3$ factor in the denominators of the coefficients.

We find $B$ by taking the curl of the vector potential. The resulting field in the slot is shown in Figure 3.7. The dotted lines show the location of the conductor.

**Example 3.3:** permeable sphere in external magnetic field

We next consider an example of using the scalar potential. Assume we have a sphere of some magnetic material located in an external magnetic field, as shown in Figure 3.8. We choose the $z$ coordinate of a spherical coordinate system to lie along the direction of the external field $B_0 = \mu_0 H_0$. The problem has azimuthal symmetry, so the results cannot depend on $\phi$. The magnetic scalar potential for the external field can be written as

$$\begin{aligned} V_0 &= -H_0 z = -H_0 r \cos\theta \\ &= -H_0 r\, P_1(\cos\theta), \end{aligned}$$

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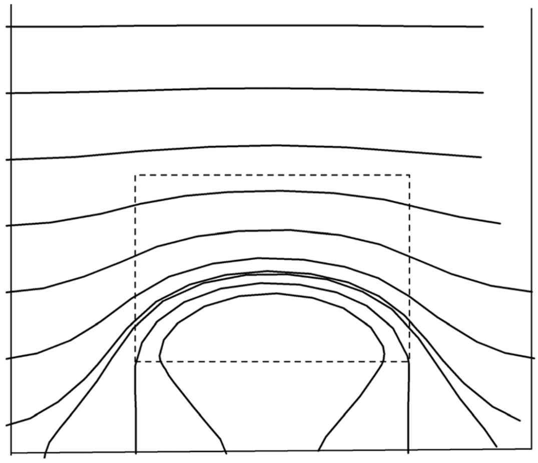

Figure 3.7 Magnetic flux density inside the slot.

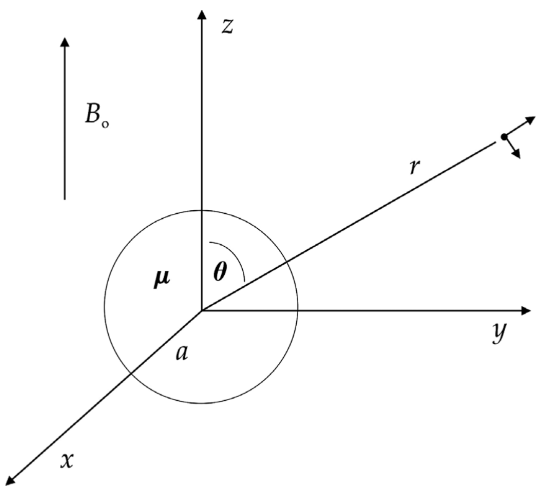


Figure 3.8 Permeable sphere in an external magnetic field.

where $P_1$ is a Legendre polynomial. The potential for the problem including the sphere must remain finite as $r \to \infty$. Therefore the potential outside the sphere is given by

$$V_m^{ext} = V_0 + \sum_{n=0}^{\infty} c_n \; r^{-n-1} \; P_n(\cos\theta).$$

The potential must also remain finite at $r = 0$, so the potential inside the sphere has the form

$$V_m^{int} = V_0 + \sum_{n=0}^{\infty} d_n \, r^n P_n(\cos\theta).$$

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At the boundary surface $r = a$, $H_t = H_\theta$ must be continuous, so

$$-\sum_{n=0}^{\infty} c_n\, a^{-n-2} \frac{dP_n}{d\theta} = -\sum_{n=0}^{\infty} d_n\, a^{n-1} \frac{dP_n}{d\theta}.$$

Since this must true for any $\theta$, the coefficients must satisfy the relation

$$c_n = d_n a^{2n+1}. \tag{3.72}$$

At the boundary surface $r = a$, $B_n = B_r$ must also be continuous. After simplifying we have

$$\mu_0 \left[ H_0 P_1 + \sum_{n=0}^{\infty} c_n (n+1) a^{-n-2} P_n \right] = \mu \left[ H_0 P_1 - \sum_{n=0}^{\infty} d_n n a^{n-1} P_n \right].$$

We require that this relation hold for any value of $n$. For $n = 0$, we obtain

$$\mu_0\, c_0\, a^{-2} = 0.$$

This shows that $c_0 = 0$ and from Equation 3.72 we find that $d_0 = 0$. For $n = 1$, we find that

$$\mu_0 (H_0 + 2c_1 a^{-3}) = \mu (H_0 - d_1). \tag{3.73}$$

Substituting Equation 3.72 for $c_1$, we obtain

$$d_1 = \frac{\mu - \mu_0}{\mu + 2\,\mu_0} H_0.$$

Substituting this back into Equation 3.72, we find

$$c_1 = \frac{\mu - \mu_0}{\mu + 2\,\mu_0} a^3 H_0.$$

For $n > 1$, we have

$$\mu_0\, c_n (n+1) a^{-n-2} = -\mu n d_n\, a^{n-1}. \tag{3.74}$$

Using Equation 3.72 for $c_n$, we obtain $d_n = 0$. Using this in Equation 3.72, we find $c_n = 0$. Thus the only nonvanishing coefficients are $c_1$ and $d_1$. The solution for the potential outside the sphere is

$$V_m^{ext} = -H_0\, r \cos\theta + \frac{\mu - \mu_0}{\mu + 2\,\mu_0} \frac{a^3}{r^2} H_0 \cos\theta$$

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and the field components are

$$B_r = B_0 \cos\theta \left[1 + 2\left(\frac{\mu_r - 1}{\mu_r + 2}\right)\frac{a^3}{r^2}\right]$$

$$B_\theta = -B_0 \sin\theta \left[1 - \left(\frac{\mu_r - 1}{\mu_r + 2}\right)\frac{a^3}{r^2}\right].$$

The potential inside the sphere is

$$V_m^{int} = -H_0\, r \cos\theta + \frac{\mu - \mu_0}{\mu + 2\,\mu_0} H_0 \cos\theta$$

and the field components are

$$B_r = \mu H_0 \cos\theta \left[1 - \left(\frac{\mu_r - 1}{\mu_r + 2}\right)\right]$$

$$B_\theta = -\mu H_0 \sin\theta \left[1 - \left(\frac{\mu_r - 1}{\mu_r + 2}\right)\right].$$

The magnetic flux density in the vicinity of a sphere with $\mu_r = 20$ is shown in Figure 3.9. The lines of $B$ are pulled into the sphere and approach the boundary approximately along a normal. The field inside the sphere is parallel to the external field.

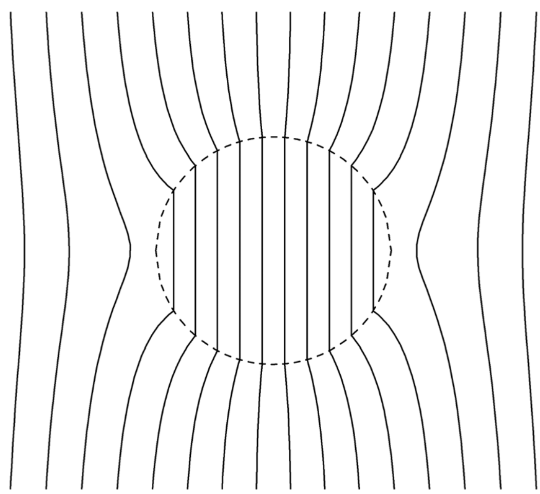

Figure 3.9 Cross-section of the permeable sphere in an external magnetic field.

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## 3.10 Green's theorem

Let $V$ be a region of space enclosed by the surface $S$. Let $\psi$ and $\phi$ be scalar functions of position that have continuous first and second derivatives. Applying the divergence theorem to the vector $\psi\nabla\phi$, we get

$$\int \nabla\cdot(\psi\nabla\phi)dV = \int(\psi\nabla\phi)\cdot\hat{n}\ dS. \tag{3.75}$$

Since

$$\nabla\cdot(\psi\nabla\phi) = \nabla\psi\cdot\nabla\phi + \psi\nabla^2\phi$$

and

$$\nabla\phi\cdot\hat{n} = \frac{\partial\phi}{\partial n},$$

Equation 3.75 becomes

$$\int \nabla\psi\cdot\nabla\phi\ dV + \int \psi\nabla^2\phi\ dV = \int \psi\frac{\partial\phi}{\partial n}dS. \tag{3.76}$$

If we repeat this calculation, interchanging the role of $\phi$ and $\psi$, we obtain

$$\int \nabla\phi\cdot\nabla\psi\ dV + \int \phi\nabla^2\psi\ dV = \int \phi\frac{\partial\psi}{\partial n}dS. \tag{3.77}$$

Subtracting Equation 3.77 from Equation 3.76, we obtain Green's second identity or *Green's theorem* [19]

$$\int(\psi\nabla^2\phi - \phi\nabla^2\psi)dV = \int\left(\psi\frac{\partial\phi}{\partial n} - \phi\frac{\partial\psi}{\partial n}\right)dS. \tag{3.78}$$

For two-dimensional problems, there is a corresponding Green's theorem on the plane given by [20]

$$\iint\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dx\,dy = \oint (P\ dx + Q\ dy), \tag{3.79}$$

where $P$ and $Q$ are continuous functions of $x$ and $y$ and have continuous partial derivatives.

To apply Green's theorem to magnetostatics, let us choose the function $\psi$ to be proportional to the inverse distance between an element of current at $r'$ and a field observation point at $r$.

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$$\psi = \frac{1}{4\pi R} = \frac{1}{4\pi|\vec{r} - \vec{r}'|}. \tag{3.80}$$

In the case where currents are present inside $V$, we choose $\phi$ to be one of the components of the vector potential, e.g., $A_x$. Then we have

$$\nabla^2 A_x = -\mu_0 J_x$$

and from Equation 1.23

$$\nabla^2\left(\frac{1}{4\pi R}\right) = -\delta(\vec{r} - \vec{r}').$$

Substituting this into Green's theorem, we get

$$\int\left[\frac{1}{4\pi R}(-\mu_0 J_x) - A_x\Big(-\delta(\vec{r} - \vec{r}')\Big)\right]dV' = \int\left[\frac{1}{4\pi R}\frac{\partial A_x}{\partial n} - A_x\frac{\partial}{\partial n}\left(\frac{1}{4\pi R}\right)\right]dS'.$$

Because of the delta function, we can solve this equation for $A_x(r)$.

$$A_x(\vec{r}) = \frac{\mu_0}{4\pi}\int\frac{J_x}{R}dV' + \frac{1}{4\pi}\int\left[\frac{1}{R}\frac{\partial A_x}{\partial n} - A_x\frac{\partial}{\partial n}\left(\frac{1}{R}\right)\right]dS'. \tag{3.81}$$

The volume integral represents a particular solution of the Poisson equation and only includes the effects of currents inside $V$. Any additional currents outside $V$ influence the value of the surface integral. If we let $r$ be a set of points on the surface $S$, then this represents an integral equation for the unknown vector potential.

We can use Green's theorem to develop integral equation solutions for Dirichlet and Neumann boundary value problems. We generalize Equation 3.80 used in the previous derivation by defining the *Green's function* [21]

$$G(\vec{r}, \vec{r}') = \frac{1}{4\pi|\vec{r} - \vec{r}'|} + L(\vec{r}, \vec{r}'),$$

where $L$ is an arbitrary solution of the Laplace equation inside $V$. Substituting this into Green's theorem, we obtain an equation similar to Equation 3.81 with $1/R$ replaced with $G$. In the case where there are no currents inside $V$, we have

$$\phi(\vec{r}) = \frac{1}{4\pi}\int\left[G(\vec{r}, \vec{r}')\frac{\partial\phi}{\partial n} - \phi\frac{\partial}{\partial n}G(\vec{r}, \vec{r}')\right]dS', \tag{3.82}$$

where $\phi$ represents either the magnetic scalar potential or one of the components of $A$.

For Dirichlet boundary value problems, if we choose $L$ such that the Green's function

$$G_D = 0 \tag{3.83}$$

when $r'$ is on the surface, we obtain the integral equation

$$\phi(\vec{r}) = -\frac{1}{4\pi}\int \phi(\vec{r}\,')\frac{\partial}{\partial n}G_D(\vec{r}, \vec{r}\,')dS'. \tag{3.84}$$

For Neumann boundary value problems for the exterior of the volume $V$, if we instead choose $L$ such that

$$\frac{\partial G_N}{\partial n} = 0 \tag{3.85}$$

when $r'$ is on the surface, we obtain the integral equation

$$\phi(\vec{r}) = \frac{1}{4\pi}\int G_N(\vec{r}, \vec{r}\,')\frac{\partial \phi(\vec{r}\,')}{\partial n}dS'. \tag{3.86}$$

Thus once we have an appropriate Green's function for a given geometry, the potential at some field point $r$ can be found by integration over the boundary surface. The Green's function is the solution of the Poisson equation for a delta function source.

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