ON A DOUBLE INTEGRAL VARIATIONAL PROBLEM

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1. Introduction. The results presented here were motivated by a desire to give a simple treatment of the non-linear elliptic partial differential equation governing the steady irrotational subsonic flow of an ideal compressible fluid. For two independent space variables x and y, the stream function ψ of such a flow satisfies an equation

$$\frac{\partial}{\partial x}\left[F'(\psi_x^2+\psi_y^2)\,\psi_x\right]+\frac{\partial}{\partial y}\left[F'(\psi_x^2+\psi_y^2)\,\psi_y\right]=0, \tag{1}$$

where $F = F(\psi_x{}^2 + \psi_y{}^2)$ is an analytic, increasing, convex function of $q = (\psi_x{}^2 + \psi_y{}^2)^{\frac{1}{2}}$ whose explicit form depends on the equation of state of the fluid in question. Our analysis of (1) will be based on the fact that it is the Euler-Lagrange equation for the double integral variational problem

$$\iint F(\psi_x^2+\psi_y^2)\,dx\,dy = \text{minimum}. \tag{2}$$

We shall introduce several devices for analyzing (2) which prove to be particularly successful for the case $F = (1 + \psi_x{}^2 + \psi_y{}^2)^{\frac{1}{2}}$ of the Plateau problem.

Shiffman **(5)** has given a proof of the existence of subsonic compressible flows based on (2). His work is part of an extensive literature on the calculus of variations and on non-linear elliptic partial differential equations of which the contributions by Haar **(2)** and Radó **(4)** come closest to the point of view of our paper. The deduction of a priori estimates on the derivatives of the solution ψ of (2) and the discussion of the analyticity of ψ are key developments in the theory, and it is in these two directions that our analysis applies.

In the second section of the paper, we study the minimum problem (2) by the method of interior variation **(1)**. This leads in a natural way to an integral H for which we can derive a second order partial differential equation from the existence of the first derivatives of ψ. In the case of the Plateau problem, H satisfies an elliptic Monge-Ampère equation which ties in with Radó's proof of the analyticity of a minimal surface.

As a preparation for the application of interior variations, we take up in the third section a construction based on symmetrization which yields for the Plateau problem a minimal sequence satisfying a uniform Lipschitz condition. For this construction, our assumption on the boundary data is more general than that usually required for the analogous conclusion using the three-point condition, and we are therefore able to discuss the Plateau problem in non-parametric form for a domain which need not be convex. Furthermore, the

Received June 19, 1953.

symmetrization method applies equally well for any finite number of independent variables.

2. Interior variation and the integral H. In a plane domain D with boundary curves C, let ψ be a solution of the minimum problem (2), with, for example, prescribed boundary values on the curves C. We shall assume merely that ψ satisfies a Lipschitz condition in D, so that the first derivatives ψ_x and ψ_y exist almost everywhere and are bounded. Instead of trying to derive the Euler-Lagrange equation (1) by the classical method of varying ψ, we study (2) in this section by performing infinitesimal transformations on the independent variables x and y and by considering the shift of ψ thus generated.

We let f be a continuous complex-valued function in D, with piece-wise continuous first derivatives, which vanishes on C. It is convenient to use the complex notation $2f_z = f_x - if_y$, $2f_{\bar{z}} = f_x + if_y$, $z = x + iy$, for derivatives. For small values of the complex parameter ϵ, the transformation

$$z^* = z + \epsilon f, \quad z^* = x^* + iy^*, \tag{3}$$

performs a one-to-one mapping of D onto itself. We define a new function ψ^* in D by the formula

$$\psi^*(x^*, y^*) = \psi(x, y), \tag{4}$$

and we compare the values of the integral in (2) associated with the two neighboring functions ψ and ψ^*.

Clearly

$$\iint_D F(4\psi^*_{z^*}\,\psi^*_{\bar{z}^*})\,dx^*\,dy^* = \iint_D F(4[\psi_z z_{z^*} + \psi_{\bar{z}}\,\bar{z}_{z^*}][\psi_z\,z_{\bar{z}^*} + \psi_{\bar{z}}\,\bar{z}_{\bar{z}^*}])\,\frac{\partial(x^*, y^*)}{\partial(x, y)}\,dx\,dy, \tag{5}$$

whence by elementary calculation

$$\iint_D F(4\psi^*_{z^*}\,\psi^*_{\bar{z}^*})\,dx^*\,dy^* - \iint_D F(4\psi_z\,\psi_{\bar{z}})\,dx\,dy = 2\,\Re\left\{\epsilon\iint_D [(F - q^2F')\,f_z - 4F'\,\psi_z^2 f_{\bar{z}}]\,dx\,dy\right\} + o(|\epsilon|^2), \tag{6}$$

where $q^2 = 4\psi_z\psi_{\bar{z}}$ and where F and F' stand for $F(q^2)$ and $F'(q^2)$. In the usual fashion, we conclude from (6), from the extremal property (2) of ψ, and from the arbitrary nature of ϵ that

$$\iint_D [(F - q^2F')\,f_z - 4F'\psi_z^2 f_{\bar{z}}]\,dx\,dy = 0 \tag{7}$$

for every continuous piece-wise continuously differentiable function f in D which vanishes on C.

Let Ω be a closed subdomain of D, let ω be a continuously differentiable function in D which is identically 1 in Ω and which vanishes on C, let t be an interior point of Ω, and let $\rho > 0$ be so small that the circle $|z - t| \leqslant \rho$ is contained in the interior of Ω. We define

$$f = \frac{\bar{z} - \bar{t}}{\rho^2} \tag{8}$$

for $|z - t| \leqslant \rho$ and we define

$$f = \frac{\omega}{z - t} \tag{9}$$

in D for $|z - t| \geqslant \rho$. We can substitute this function f into (7) to obtain

$$\frac{4}{\rho^2} \iint_K F' \psi_z^2 \, dx\, dy = - \iint_{\Omega - K} \frac{F - q^2 F'}{(z - t)^2} \, dx\, dy + A(t), \tag{10}$$

where K denotes the circle $|z - t| \leqslant \rho$ and where $A(t)$ is the analytic function

$$A(t) = \iint_{D - \Omega} (F - q^2 F') \left[\frac{\omega_z}{z - t} - \frac{\omega}{(z - t)^2} \right] dx\, dy \tag{11}$$

in Ω.

Letting $\rho \to 0$, we find almost everywhere in Ω

$$4\pi F' \psi_t^2 = - \iint_{\Omega} \frac{F - q^2 F'}{(z - t)^2} \, dx\, dy + A(t), \tag{12}$$

where the integral on the right is to be interpreted in the sense of the Cauchy principal value. In the case of the Dirichlet problem, $F \equiv q^2$ and the formula (12) shows immediately that $\psi_t{}^2$ is an analytic function. Thus we obtain in the simplest situation, corresponding to an incompressible fluid, a quite elegant proof that the solution of the minimum problem (2) is a regular function.

In order to study the general non-linear problem, we introduce the integral

$$H = - \frac{2}{\pi} \iint_{\Omega} (F - q^2 F') \log |z - t| \, dx\, dy + B, \tag{13}$$

where B is a real harmonic function in Ω such that $\pi B_{tt} = -A(t)$. By (12) and by standard lemmas on the second derivatives of a logarithmic potential, we find almost everywhere in Ω

$$H_{t\bar{t}} = F - q^2 F', \tag{14}$$

$$H_{tt} = -4F'\psi_t^2, \quad H_{\bar{t}\bar{t}} = -4F'\psi_{\bar{t}}^2. \tag{15}$$

This gives in turn

$$H_{tt} H_{\bar{t}\bar{t}} - H_{t\bar{t}}^2 = 2q^2 F F' - F^2. \tag{16}$$

We can eliminate q from the equations (14) and (16) to obtain for the real function H a partial differential equation of the form

$$\Delta H = Q(H_{xx}H_{yy} - H_{xy}^2), \quad \Delta H = H_{xx} + H_{yy}, \tag{17}$$

where Q is a real analytic function of its real argument which is completely determined by F. For the most general function F corresponding to an arbitrary equation of state, (17) is a non-linear equation equivalent to (1) which involves only very special combinations of second derivatives of the auxiliary function H. The significance of this second order partial differential equation is that its derivation requires only a Lipschitz condition on the stream function ψ.

The form of (17) suggests finding those integrands F for which it reduces to the Monge-Ampère equation

$$\Delta H = H_{xx}H_{yy} - H_{xy}^2. \tag{18}$$

This reduction takes place, according to (14) and (16), when F satisfies the ordinary differential equation

$$2q^2F\,F' - F^2 = 2\lambda(q^2F' - F), \tag{19}$$

for a suitable value of the constant λ. We check immediately that (19) has the general solution

$$F = \lambda + \sqrt{\lambda^2 + \mu^2q^2}, \tag{20}$$

whence the Monge-Ampère equation (18) is seen to correspond to the case in which (2) is the Plateau problem.

We arrive in this way at a proof of the analyticity of a minimal surface. For we can apply the Legendre transformation

$$h + H = ux + vy, \quad u = H_x, \quad v = H_y \tag{21}$$

to (18) in order to obtain the linear elliptic equation

$$h_{uu} + h_{vv} = 1. \tag{22}$$

The solution h of (22) is evidently an analytic function of u and v, whence H is an analytic function of x and y, and by (14) and (15) we can conclude that the function ψ is also analytic in x and y. The Poisson equation (22) yields, furthermore, a procedure for constructing flow patterns explicitly according to the formulas of this section in the case where the equation of state of the fluid leads to an integrand of the type (20).

3. Symmetrization and the Lipschitz condition. The analysis of the preceding section exploited formal manipulations in order to demonstrate, in certain important special cases, the analyticity of solutions of (2). In this section, we complete our discussion of the Plateau problem by constructing a minimal sequence which fulfills a uniform Lipschitz condition.

Along the curves C bounding the domain D, we assign boundary values which generate in space a system of smooth closed curves Γ. The problem is to span through the curves Γ a non-parametric minimal surface over the domain D.

For this problem of the type (2), we can clearly find a minimal sequence of surfaces whose areas approach the minimum value in question. There is no loss of generality if we assume that each of these surfaces lies in the convex hull of the system of curves Γ, since, when this is not true for a given surface, we can diminish the area of the surface by replacing portions of it by sections of planes tangent to the convex hull of Γ. Furthermore, it is permissible to diminish the area of any of the surfaces of the minimal sequence in a similar manner by replacing portions of them by simply-connected sections of catenoids, or of other specific known minimal surfaces, which do not intersect Γ. Thus we may suppose that all members of our minimal sequence lie in the largest closed region E projecting onto D which contains the curves Γ, but which does not intersect in a simply-connected surface element any plane or catenoid not meeting Γ. This latter condition means that E cannot be diminished infinitesimally by cutting off a volume element with a plane or catenoid which does not intersect Γ.

We now make the assumption on the boundary curves Γ that the closed set E is so situated that, for some $\theta > 0$, all lines making an angle smaller than θ with the normal to the plane domain D and intersecting Γ have only one point in common with E. This assumption restricts the curvature of Γ, but does not imply that D is a convex domain, since in some cases D can even be multiply-connected.

We consider any rectangular coordinate system in which the z-axis makes an angle less than θ with the normal to the plane of D. In this coordinate system, any element of our minimal sequence has a non-parametric representation $z = z(x, y)$ which may be multiple-valued, with branches

$$z = z_1(x, y),\ z = z_2(x, y), \ldots, z = z_{2m+1}(x, y), \qquad z_1 \leqslant z_2 \leqslant \ldots \leqslant z_{2m+1}.$$

We symmetrize such a surface by replacing it with the surface whose non-parametric representation is the single-valued expression

$$z = \sum_{k=1}^{2m+1} (-1)^{k+1} z_k(x, y). \tag{23}$$

It follows from the basic results of Steiner, or, more directly, from Minkowski's inequality **(3)**, that the symmetrization process (23) diminishes or leaves unchanged the area of the surface. Furthermore, if symmetrization in one such coordinate system is followed by symmetrization in another, the resulting surface still has a single-valued non-parametric representation in the first coordinate system. This can be checked by making an affine transformation such that the directions of the two symmetrizations become perpendicular, a case in which the result is evident. Finally, and most important of all, the symmetrization procedure (23) does not alter the boundary curves Γ of any of our surfaces, since the surfaces lie in the set E, which has the property that if any line parallel to one of our z-axes intersects Γ, then it intersects Γ and E in only one single point.

These considerations show that we can assume without loss of generality that each surface of our minimal sequence has a single-valued non-parametric representation in every coordinate system whose z-axis is inclined at an angle less than θ with the normal to the plane of D. But if we denote by $z = \psi(x, y)$ the representation of such a surface in a coordinate system such that D lies in the (x, y)-plane, then this result implies that ψ satisfies the Lipschitz condition

$$|\psi(x_2, y_2) - \psi(x_1, y_1)| \leqslant M[(x_2 - x_1)^2 + (y_2 - y_1)^2]^{\frac{1}{2}}, \tag{24}$$

with $M = \cot\theta$. Thus we obtain a minimal sequence of surfaces satisfying the Lipschitz condition (24), and from the lower semi-continuity of the area integral **(2)** we can deduce the existence of a solution of (2) satisfying the same Lipschitz condition, for $F = (1 + q^2)^{\frac{1}{2}}$.

A combination of the techniques of this section and of the previous section yields a solution of the Plateau problem in non-parametric form for a domain D which is not necessarily convex, but for a system of smooth boundary curves Γ satisfying the above geometrical condition relative to certain planes, catenoids, and projections. Our method for developing such a condition is a generalization to non-linear elliptic equations of the majorization principles of the theory of linear elliptic partial differential equations based on the maximum principle.

A point of interest in the symmetrization construction presented in this section is that it yields estimates of precisely the same nature in space of any number of dimensions. In particular, we can treat the variational problem of minimizing

$$\iiint (1 + \psi_x^2 + \psi_y^2 + \psi_z^2)^{\frac{1}{2}}\, dx\, dy\, dz, \tag{25}$$

which has applications to the study of three-dimensional flow of a Kármán-Tsien gas.

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