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MAXIMAL QUOTIENT RINGS OF ENDOMORPHISM RINGS OF $E(R_R)$-TORSIONFREE GENERATORS

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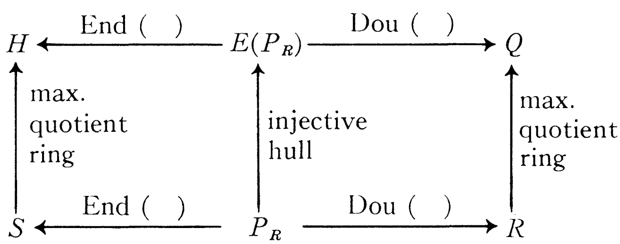
Introduction. Let R be a ring with identity and let $H = \text{End}\,(E(R_R))$ and $Q = \text{Dou}(E(R_R)) = \text{End}({}_HE(R_R))$. Then Lambek [**11**] showed that Q is always isomorphic to $Q_m(R)$, the maximal right quotient ring of R. And Johnson [**10**] and Wong-Johnson [**26**] proved that $Q_m(R)$ is regular and right self-injective if and only if R is right non-singular, and then H is isomorphic to $Q_m(R)$, too. Moreover, Sandomierski [**18**] showed that $Q_m(R)$ is semi-simple Artinian if and only if R is right finite dimensional and right non-singular. And it is well known that $Q_m(R)$ is a quasi-Frobenius ring if and only if $E(R_R)$ is a rational extension of R_R and the ACC holds on right annihilators of subsets of $E(R_R)$.

The purpose of this paper is to give some module-theoretic generalizations of these results. Let P_R be an $E(R_R)$-torsionless generator, and let $S = \text{End}(P_R)$, $H = \text{End}(E(P_R))$ and $Q = \text{Dou}(E(P_R))$. Then Q is always isomorphic to $Q_m(R)$ (Proposition 3.1). But, H is not necessarily isomorphic to $Q_m(S)$, the maximal right quotient ring of S. When is H isomorphic to $Q_m(S)$? In this paper we will investigate the necessary and sufficient conditions for H to be the right self-injective (right self-injective semi-perfect, quasi-Frobenius, regular, and semi-simple Artinian, respectively) maximal right quotient ring of S (Theorem 3.5, Theorem 4.2, Theorem 4.9, Theorem 5.1 and Theorem 5.3, respectively).

This situation is described by the diagram below:



This kind of investigation has been done in [**27**] in the special case where R is semi-prime. It was showed that if M_R is a torsionless, finite dimensional and non-singular right module over a semi-prime ring R, then

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End(M_R) has the semi-simple Artinian classical right quotient ring which is isomorphic to End$(E(M_R))$.

1. Preliminaries. Throughout this paper we assume that every ring has an identity element and every module is unital. Every homomorphism will be written on the side opposite the scalars. We denote by mod-R the category of all right R-modules. For any $M \in$ mod-R, we denote by $E(M_R)$, $\prod M_R$, End(M_R) and Dou(M_R), the injective hull of M_R, a direct product of copies of M_R, the endomorphism ring and the double centralizer of M_R, respectively. We can induce a partially ordered relation among the family of all injective right R-modules by setting $E_1 \geqq E_2$ if and only if $E_1 \hookrightarrow \prod E_2$. If $E_1 \geqq E_2$ and $E_2 \geqq E_1$, we say that E_1 and E_2 are *equivalent*. This is clearly an equivalence relation. Each equivalence class of injective right R-modules is called a *hereditary torsion theory on* mod-R. We will denote by tors-R the set of all hereditary torsion theories on mod-R. For each $\tau \in$ tors-R we call M_R τ*-torsionfree* if $E(M_R) \hookrightarrow \prod E$ for every $E \in \tau$, and will denote by $\mathscr{F}_\tau$ the class of all τ-torsionfree right R-modules. And we call M_R τ*-torsion* if $\operatorname{Hom}_R(M, E) = 0$ for every $E \in \tau$, and will denote by $\mathscr{T}\tau$ the class of all τ-torsion right R-modules.

For any $\tau \in$ tors-R and any $M \in$ mod-R, an R-submodule L of M is said to be τ*-dense* (resp. τ*-saturated*) if M/L is τ-torsion (resp. τ-torsionfree). We will denote by $\mathscr{L}_\tau$ the set of all τ-dense right ideals of R; i.e.,

$$\mathscr{L}_\tau = \{I_R \subseteq R_R | R/I \text{ is } \tau\text{-torsion}\}.$$

$\mathscr{L}_\tau$ is a so-called Gabriel topology with respect to τ. If R_R is τ-torsionfree, we call τ *faithful*. We can partially order tors-R by setting $\tau' \leqq \tau$ if and only if $\mathscr{T}_{\tau'} \subseteq \mathscr{T}_\tau$. There exists the largest element among the set of all faithful hereditary torsion theories on mod-R, which is sometimes called the Lambek torsion theory, and which will be denoted by $\chi(R)$ in this paper. It is well-known that $\chi(R)$ is cogenerated by $E(R_R)$; i.e., $E(R_R) \in \chi(R)$. Moreover, M_R is $\chi(R)$-torsionfree if and only if $M_R \hookrightarrow \prod E(R_R)$, and M_R is $\chi(R)$-torsion if and only if $\operatorname{Hom}_R(M, E(R_R)) = 0$. Any $\chi(R)$-torsionfree module is said to be $E(R_R)$-torsionfree, too, in this paper. And any $\chi(R)$-dense submodule is said to be a dense submodule for short. $T_\tau(M_R)$ denotes the τ-torsion submodule of M_R, which is the largest τ-torsion submodule of M_R. For any $\tau \in$ tors-R and any $M \in$ mod-R, $E_\tau(M_R)$ denotes the τ-injective hull of M_R. It is well known that

$$\begin{aligned} E_\tau(M_R) &= \{x \in E(M_R) | (M:x) \in \mathscr{L}_\tau\} \\ &= \{x \in E(M_R) | \alpha(x) = 0 \quad \text{for every} \quad \alpha: E(M_R) \rightarrow \mathrm{E} \\ &\qquad \text{with } \alpha(M) = 0, \text{ where } E \in \tau\}. \end{aligned}$$

2. Torsion theoretic investigation. Let P_R be a generator in mod-R and $S = \mathrm{End}(P_R)$. We will consider the two covariant functors, G: mod-$R \to$ mod-S, defined by

$$G(X_R) = \mathrm{Hom}_R(P, X)$$

and F: mod-$S \to$ mod-R, defined by

$$F(Y_S) = Y \otimes {}_SP.$$

Then it is well-known that $G(E(M_R))_S = E(G(M_R)_S)$ for every $M \in$ mod-R. For any $\tau \in$ tors-R, let us put

$$\mathscr{A} = \{Y \in \text{mod-}S | F(Y_S) \in \mathscr{T}_\tau\}.$$

Then we can easily show that $\mathscr{A}$ is closed under taking submodules, homomorphic images, direct sums and extensions. Hence $\mathscr{A} = \mathscr{T}_\sigma$ for some $\sigma \in$ tors-S. Then we will write it as $\sigma = \tilde{F}(\tau)$. The following Lemma 2.1 and Lemma 2.2 have been shown in [**9**].

LEMMA 2.1. *Let $\tau \in$ tors-R and $\sigma = \tilde{F}(\tau)$. Then we have that*

$$G(E_\tau(M_R))_S = E_\sigma(G(M_R)_S)$$

for each $M \in$ mod-R.

LEMMA 2.2. *Let $\tau \in$ tors-R and $\sigma = \tilde{F}(\tau)$. Then if σ is faithful, so also is τ. If, furthermore, P_R is $E(R_R)$-torsionfree, the converse is true and $\chi(S) = \tilde{F}(\chi(R))$. Conversely, if $\chi(S) = \tilde{F}(\chi(R))$, P_R is $E(R_R)$-torsionfree.*

For any $\tau \in$ tors-R and $M \in$ mod-R, we put

$$Q_\tau(M_R) = E_\tau(M/T_\tau(M)),$$

which is called the τ-localization module of M_R. And we call the endomorphism ring of $Q_\tau(R_R)$ the τ-localization ring of R and will denote it by R_τ. It is well-known that $R_\tau \cong Q_\tau(R_R)$ as right R-modules. In particular, the $\chi(R)$-localization ring of R is called the maximal right quotient ring of R and will be denoted by $Q_m(R)$ throughout this paper.

THEOREM 2.3. *Let P_R be an $E(R_R)$-torsionfree (resp. torsionless) generator in* mod-R *and let $S = \mathrm{End}(P_R)$, $H = \mathrm{End}(E_\tau(P_R))$ and $Q = \mathrm{Dou}(E_\tau(P_R))$ for any $\tau \in$ tors-R. Then if τ is faithful and if we put $\sigma = \tilde{F}(\tau)$, we have the following statements.*

(1) *Q is isomorphic to R_τ.*

(2) *H is isomorphic to S_σ.*

(3) *$E_\tau(P_R)$ is an $E(Q_Q)$-torsionfree (resp. torsionless) generator in* mod-Q *and*

$$H = \mathrm{Hom}_Q(E_\tau(P_R), E_\tau(P_R)).$$

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$$\begin{array}{ccccc} S_\sigma \cong H & \xleftarrow{\text{End}(\ )} & E_\tau(P_R) & \xrightarrow{\text{Dou}(\ )} & Q \cong R_\tau \\ \uparrow & & \uparrow & & \uparrow \\ S & \xleftarrow{\text{End}(\ )} & P_R & \xrightarrow{\text{Dou}(\ )} & R \end{array}$$

Proof. Since P_R is a generator, $R_R \ \langle\oplus\ (P \oplus \ldots . \oplus P)_R$; so

$$E_\tau(R_R) \ \langle\oplus\ (E_\tau(P_R) \oplus \ldots . \oplus E_\tau(P_R))_R,$$

where in general $X_R \ \langle\oplus\ (Y \oplus \ldots \oplus Y)_R$ implies that X_R is isomorphic to a direct summand of a finite direct sum of copies of Y_R. Let us consider the module

$$\text{Hom}_R(E_\tau(R_R), E_\tau(P_R)).$$

Since τ is faithful,

$$\begin{aligned} \text{End}(E_\tau(R_R)) &= \text{Hom}_R(E_\tau(R_R), E_\tau(R_R)) \\ &= \text{Hom}_R(Q_\tau(R_R), Q_\tau(R_R)) \\ &= R_\tau. \end{aligned}$$

And σ is faithful by Lemma 2.2, and the functor G is full and faithful. Hence we have that

$$\begin{aligned} H &= \text{Hom}_R(E_\tau(P_R), E_\tau(P_R)) \\ &\cong \text{Hom}_S(G(E_\tau(P_R)), G(E_\tau(P_R))) \\ &= \text{Hom}_S(E_\sigma(G(P_R)), E_\sigma(G(P_R))) \quad \text{by Lemma 2.1} \\ &= \text{Hom}_S(E_\sigma(S_S), E_\sigma(S_S)) \\ &= \text{Hom}_S(Q_\sigma(S_S), Q_\sigma(S_S)) \\ &= S_\sigma. \end{aligned}$$

Hence by a result of Hirata [**8**, Theorem 1.2], $\text{Hom}_R(E_\tau(R_R), E_\tau(P_R))$ is a finitely generated projective left H-module and

$$R_\tau \cong \text{End}({}_H\text{Hom}_R(E_\tau(R_R), E_\tau(P_R))).$$

And hence $\text{Hom}_R(E_\tau(R_R), E_\tau(P_R))$ is a generator as a right R_τ-module.

Next, consider the exact sequence

$$0 \to R_R \to E_\tau(R_R) \to E_\tau(R_R)/R_R \to 0.$$

Since $E_\tau(R_R)/R_R$ is τ-torsion and $E_\tau(P_R)$ is τ-injective, we get the exact sequence

$$\begin{aligned} 0 \to \text{Hom}_R(E_\tau(R_R)/R, E_\tau(P_R)) &\to \text{Hom}_R(E_\tau(R_R), E_\tau(P_R)) \\ &\to \text{Hom}_R(R, E_\tau(P_R)) \to 0. \end{aligned}$$

But, since $E_\tau(P_R)$ is τ-torsionfree because $P \in \mathscr{F}_{\chi(R)} \subseteq \mathscr{F}_\tau$, we have that

$$\mathrm{Hom}_R(E_\tau(R_R)/R, E_\tau(P_R)) = 0.$$

This implies that

$$\begin{aligned}\mathrm{Hom}_R(E_\tau(R_R), E_\tau(P_R)) &\cong \mathrm{Hom}_R(R_R, E_\tau(P_R)) \\ &\cong E_\tau(P_R)\end{aligned}$$

as right R-modules. Since $E_\tau(P_R)$ is τ-closed (= τ-torsionfree and τ-injective), $E_\tau(P_R)$ has an R_τ-module structure which extends its structure as a right R-module. And every R-homomorphism between two R_τ-modules, which are τ-closed as right R-modules, is necessarily an R_τ-homomorphism. Hence $E_\tau(P_R)$ is isomorphic to $\mathrm{Hom}_R(E_\tau(R_R), E_\tau(P_R))$ as a right R_τ-module. Therefore $E_\tau(P_R)$ is a generator as a right R_τ-module. And since τ is faithful, the canonical ring homomorphism $\hat{\tau}:R \to R_\tau$ is a monomorphism; so

$$\mathrm{Hom}_{R_\tau}(E_\tau(P_R), E_\tau(P_R)) \subseteq \mathrm{Hom}_R(E_\tau(P_R), E_\tau(P_R)).$$

Conversely, since $E_\tau(P_R)$ is τ-closed,

$$\mathrm{Hom}_R(E_\tau(P_R), E_\tau(P_R)) \subseteq \mathrm{Hom}_{R_\tau}(E_\tau(P_R), E_\tau(P_R)).$$

Hence we have that $H = \mathrm{Hom}_{R_\tau}(E_\tau(P_R), E_\tau(P_R))$. Therefore $E_\tau(P_R)$ is a finitely generated projective left H-module and

$$R_\tau \cong \mathrm{Hom}_H(E_\tau(P_R), E_\tau(P_R)).$$

Hence

$$Q = \mathrm{Dou}(E_\tau(P_R)) = \mathrm{Hom}_H(E_\tau(P_R), E_\tau(P_R)) \cong R_\tau.$$

Thus, $E_\tau(P_R)$ is a generator as a right Q-module and

$$H = \mathrm{Hom}_Q(E_\tau(P_R), E_\tau(P_R)).$$

Next, we want to show that $E_\tau(P_R)$ is an $E(Q_Q)$-torsionfree right Q-module. It is known that if M is a right Q-module, which is τ-closed as a right R-module, then $E(M_Q) = E(M_R)_Q$. Hence

$$E(Q_Q) = E(Q_R)_Q \cong E(E_\tau(R_R)_R)_Q = E(R_R)_Q,$$

since $Q_R \cong E_\tau(R_R)$. Now, since P_R is $E(R_R)$-torsionfree, $P_R \hookrightarrow \prod E(R_R)$; so $E_\tau(P_R) \hookrightarrow \prod E(R_R)$. Hence

$$E_\tau(P_R)_Q \hookrightarrow \prod E(R_R)_Q \cong \prod E(Q_Q).$$

Thus, $E_\tau(P_R)_Q$ is $E(Q_Q)$-torsionfree.

Finally, we assume that P_R is torsionless. Then $P_R \hookrightarrow \prod R_R$; so

$$E_\tau(P_R) \hookrightarrow E_\tau(\prod R_R) \hookrightarrow \prod E_\tau(R_R),$$

since any direct product of τ-injective right R-modules is τ-injective, too. Since $E_\tau(P_R)$ and $\prod E_\tau(R_R)$ are τ-closed right R-modules,

$$E_\tau(P_R)_Q \hookrightarrow \prod E_\tau(R_R)_Q = \prod Q_Q.$$

Thus, $E_\tau(P_R)$ is Q-torsionless. This completes the proof of Theorem 2.3.

COROLLARY 2.4. *If P_R is an $E(R_R)$-torsionfree (resp. torsionless) generator in* mod-R, *and if we put*

$$S = \mathrm{End}(P_R),\ H = \mathrm{End}(E_{\chi(R)}(P_R)) \text{ and } Q = \mathrm{Dou}(E_{\chi(R)}(P_R)),$$

then we have the following statements.

(1) *Q is isomorphic to $Q_m(R)$.*

(2) *H is isomorphic to $Q_m(S)$.*

(3) *$E_{\chi(R)}(P_R)$ is an $E(Q_Q)$-torsionfree (resp. torsionless) generator in* mod-Q *and*

$$H = \mathrm{Hom}_Q(E_{\chi(R)}(P_R), E_{\chi(R)}(P_R)).$$

Proof. By Lemma 2.2, $\chi(S) = \tilde{F}(\chi(R))$. Hence we have the assertions of this corollary by virtue of Theorem 2.3.

3. Self-injective maximal quotient rings. If $N_R \subseteq M_R$, then M is said to be a *rational extension of* N if for each module L_R such that $N \subseteq L \subseteq M$ and each $f:L \to M$, $f(N) = 0$ implies $f = 0$. There exists a unique maximal rational extension $\bar{M}$ of M which is obtained as follows:

$$\bar{M} = \{x \in E(M_R) | f(x) = 0 \text{ for all } f:E(M_R) \to E(M_R) \text{ with } f(M) = 0\}.$$

For any faithful $\tau \in$ tors-R, R_τ is a rational extension of R as a right R-module and $Q_m(R)_R$ is a maximal rational extension of R_R. A right ideal I of R is said to be *dense* if

$$\mathrm{Hom}_R(R/I, E(R_R)) = 0.$$

Hence $\mathscr{L}_{\chi(R)} = \{I_R \subseteq R_R | I$ is a dense right ideal$\}$. An R-module M is called *compactly faithful* if $R_R \hookrightarrow \bigoplus^n M_R$ (a finite direct sum of copies of M_R).

PROPOSITION 3.1. *If M_R is an $E(R_R)$-torsionfree, compactly faithful right R-module, then* $\mathrm{Dou}(E(M_R))$ *is isomorphic to $Q_m(R)$.*

Proof. Since $M_R \hookrightarrow \prod E(R_R)$, $E(M_R) \hookrightarrow \prod E(R_R)$. Since M_R is compactly faithful, $R_R \hookrightarrow \bigoplus^n M_R$, and hence

$$E(R_R) \hookrightarrow \bigoplus^n E(M_R).$$

Hence $E(M_R)$ and $E(R_R)$ are equivalent injective right R-modules. That is, $E(M_R) \in \chi(R)$. And hence, if we put $E_R = \bigoplus^n E(M_R)$, we have that $E_R \in \chi(R)$ and $E_R = E(R_R) \oplus C_R$ for some $C \in$ mod-R. Therefore $\mathrm{Dou}(E_R)$ is isomorphic to $Q_m(R)$ by [**21**, Theorem 8.4].

On the other hand, since $E_R = \bigoplus^n E(M_R)$, $\mathrm{Dou}(E(M_R))$ is isomorphic to $\mathrm{Dou}(E_R)$ by a well-known classical result. Thus, we have obtained the conclusion.

Throughout the remainder of this paper we assume that P_R is an $E(R_R)$-torsionfree generator in mod-R, and that $S = \mathrm{End}(P_R)$, $H = \mathrm{End}(E(P_R))$ and $Q = \mathrm{Dou}(E(P_R))$, unless otherwise stated. Masaike [**15**] has defined a concept which he called generalized non-singular and has characterized a ring which has the right self-injective maximal right quotient ring. Here we will generalize this concept to modules and characterize P_R for which H is the right self-injective maximal right quotient ring of S.

LEMMA 3.2. *Let P_R be an $E(R_R)$-torsionfree generator in* mod-R, *and let τ be a faithful hereditary torsion theory on* mod-R. *Then the following statements are equivalent.*

(a) *$E_\tau(P_R)$ is injective as a right R_τ-module.*

(b) *$E_\tau(P_R)$ is injective as a right R-module.*

(c) *For any R-submodule L of P and any R-homomorphism $\alpha: L \to P$, there exist a τ-dense submodule M of P and an R-homomorphism $\beta: M \to P$ such that $L \subseteq M$ and $\beta|L = \alpha$.*

Proof. It is well-known that (a) implies (b).

(b) $\Rightarrow$ (c). Let L be any submodule of P and let α be any R-map of L into P. Since $E_\tau(P_R)$ is R-injective, there exists an R-map $\gamma: P \to E_\tau(P_R)$, which extends α. Then, let us put $M = \gamma^{-1}(P)$ and $\beta = \gamma|M$. Clearly $L \subseteq M$ and $\beta|L = \alpha$. It remains to show that M is τ-dense in P. γ induces an R-monomorphism

$$\bar{\gamma}: P/M \to E_\tau(P_R)/P.$$

Since $E_\tau(P_R)/P$ is τ-torsion, so is also P/M. That is, M is τ-dense in P.

(c) $\Rightarrow$ (a). By Theorem 2.3, $E_\tau(P_R)$ is a generator in mod-R_τ. We want to show that $E_\tau(P_R)$ is a quasi-injective right R_τ-module. If it is shown, $E_\tau(P_R)$ becomes an injective right R_τ-module. Consider any R_τ-submodule L of $E_\tau(P_R)$ and any R_τ-map $\alpha: L \to E_\tau(P_R)$. Let us put $L' = \alpha^{-1}(P) \cap P$. By (c), there exist a τ-dense submodule M of P and an R-map $\beta: M \to P$ such that $L' \subseteq M$ and $\beta|L' = \alpha|L'$. Since M is τ-dense in $E_\tau(P_R)$ because $\mathscr{T}_\tau$ is closed under taking extensions, and since $E_\tau(P_R)$ is τ-injective, β can be extended to an R-map

$$\gamma: E_\tau(P_R) \to E_\tau(P_R).$$

Next, we will show that γ is an extension of α, as well. $\gamma - \alpha$ is a zero map

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on L' and hence $\gamma - \alpha$ induces an R-map

$$\overline{\gamma - \alpha}: L/L' \to E_\tau(P_R).$$

On the other hand, since α induces an R-monomorphism

$$\bar{\alpha}: L/\alpha^{-1}(P) \to E_\tau(P_R)/P,$$

$L/\alpha^{-1}(P)$ is τ-torsion. And since

$$\alpha^{-1}(P)/L' = \alpha^{-1}(P)/\alpha^{-1}(P) \cap P$$
$$\cong \alpha^{-1}(P) + P/P \hookrightarrow E_\tau(P_R)/P,$$

then $\alpha^{-1}(P)/L'$ is τ-torsion, too. Hence the exact sequence

$$0 \to \alpha^{-1}(P)/L' \to L/L' \to L/\alpha^{-1}(P) \to 0$$

shows that L/L' is τ-torsion. This, as well as the fact that $E_\tau(P_R)$ is τ-torsionfree, implies that $\overline{\gamma - \alpha}$ is zero. Therefore we get $\gamma = \alpha$ on L. Since γ is an R-endomorphism of the τ-closed module $E_\tau(P_R)$, γ is necessarily an R_τ-endomorphism. Hence γ is an extension of α as an R_τ-homomorphism. Therefore $E_\tau(P_R)$ is a quasi-injective right R_τ-module, as required.

LEMMA 3.3. *Let P_R be an $E(R_R)$-torsionfree generator in* mod-R *and* $S = \mathrm{End}(P_R)$. *An R-submodule L of P is dense (i.e., $\chi(R)$-dense) in P if and only if* $\mathrm{ann}_S(L:s) = 0$ *for every $s \in S$.*

Proof. First, assume that $\mathrm{ann}_S(L:s) = 0$ for all $s \in S$. Suppose that L is not dense in P. There exists a non-zero $\varphi \in \mathrm{Hom}_R(P/L, E(P_R))$, because $E(P_R)$ cogenerates $\chi(R)$. The exact sequence

$$0 \to L \xrightarrow{i} P \xrightarrow{j} P/L \to 0$$

induces the exact sequence

$$0 \to \mathrm{Hom}_R(P/L, E(P_R)) \to \mathrm{Hom}_R(P, E(P_R))$$
$$\to \mathrm{Hom}_R(L, E(P_R)) \to 0.$$

Hence $\varphi j \in \mathrm{Hom}_R(P, E(P_R))$ is non-zero and $\varphi j(L) = 0$. Since P_R is a generator and $S = \mathrm{End}(P_R)$,

$$\mathrm{Hom}_R(P, E(P_R))_S = E(S_S).$$

Hence there exists $s \in S$ such that $0 \neq \varphi js \in S$. But,

$$\varphi js(L:s) \subseteq \varphi j(L) = 0.$$

Hence by our assumption, we have $\varphi js = 0$, which is a contradiction. Therefore

$$\mathrm{Hom}_R(P/L, E(P_R)) = 0.$$

That is, L is a dense submodule in P.

Conversely, assume that L is dense in P. For any $s' \in \operatorname{ann}_S(L:s)$, we can induce the commutative diagram:

$$\begin{array}{ccc} P/(L:s) & \xrightarrow{\ \bar{s}\ } & P/L \\ \big\downarrow{\scriptstyle \bar{s}'} & & \big\downarrow{\scriptstyle \varphi} \\ P & \longrightarrow & E(P_R) \end{array}$$

because $\bar{s}$ is an R-monomorphism and $E(P_R)$ is injective. Since $\operatorname{Hom}_R(P/L, E(P_R)) = 0$ by assumption, $\varphi = 0$, and hence $s' = 0$. Thus, $\operatorname{ann}_S(L:s) = 0$ for every $s \in S$.

For any $M \in$ mod-R, let us put $K(M) = \{L_R \subseteq M_R |$ there exists an R-map $\varphi_L: L \to M$ such that φ_L cannot be extended to any submodule properly containing $L\}$.

LEMMA 3.4 *Let P_R be an $E(R_R)$-torsionfree generator in* mod-R *and* $S = \operatorname{End}(P_R)$. *Then the following statements are equivalent.*

(1) *For any R-submodule L of P and any R-homomorphism $\alpha: L \to P$, α can be extended to a dense submodule of P.*

(2) $\operatorname{ann}_S M = 0$ *for every* $M \in K(P)$.

Proof. (1) $\Rightarrow$ (2). Let $M \in K(P)$. Then M is dense in P by (1). Hence $\operatorname{ann}_S M = \operatorname{ann}_S(M:1_S) = 0$ by Lemma 3.3.

(2) $\Rightarrow$ (1). For any $\alpha: L \to P$, there exists a maximal $\beta: M \to P$ such that $L \subseteq M$ and $\beta|L = \alpha$ by Zorn's lemma. Then we want to show that M is dense in P. It suffices to show that $(M:s) \in K(P)$ for all $s \in S$, according to our assumption and Lemma 3.3. Define $\varphi: (M:s) \to P$ by $\varphi(y) = \beta(sy)$ for each $y \in (M:s)$. Then φ cannot be further extended. For, suppose there exist $X \supsetneqq (M:s)$ and $\psi: X \to P$ such that $\psi|(M:s) = \varphi$. Then, define $\gamma: M + sX \to P$ by

$$\gamma(y + sx) = \beta(y) + \psi(x).$$

If $y + sx = 0$, where $y \in M$ and $x \in X$,

$$\gamma(y + sx) = \beta(y) + \psi(x) = \beta(y) + \varphi(x) = \beta(y) + \beta(sx) = \beta(y + sx) = 0.$$

Hence γ is well-defined. And for any $y \in M$, $\gamma(y) = \beta(y)$. Since $X \supsetneqq (M:s)$, there exists $x \in X$ such that $sx \notin M$. Hence $M + sX \supsetneqq M$. This contradicts the maximality of M. Hence we have that $(M:s) \in K(P)$ for all $s \in S$.

We are now ready to prove the next theorem.

THEOREM 3.5. *Let P_R be an $E(R_R)$-torsionfree generator in* mod-R *and let* $S = \mathrm{End}(P_R)$, $H = \mathrm{End}(E(P_R))$ *and* $Q = \mathrm{Dou}(E(P_R))$. *Then the following statements are equivalent.*

(1) *H is a right self-injective ring and is isomorphic to $Q_m(S)$.*

(2) $H \cong Q_m(S)\,(H \ni \beta \to \beta|E_{\chi(R)}(P_R))$.

(3) $H_S \cong E(S_S)$.

(4) $l(J) = 0$ *for every* $J \in K(S)$, *where $l(J)$ denotes the left annihilator of J in S.*

(5) *$E(P_R)$ is a rational extension of P_R.*

(6) *$(P:x)$ is a dense right ideal of R for each $x \in E(P_R)$.*

(7) *For any R-submodule L of P and any R-homomorphism $\alpha: L \to P$, there exist a dense submodule M of P and an R-homomorphism $\beta: M \to P$ such that $L \subseteq M$ and $\beta|L = \alpha$.*

(8) $\mathrm{ann}_S\, M = 0$ *for every* $M \in K(P)$.

When these conditions are satisfied, $E(P_R)$ is a torsionless generator in mod-Q *and* $H = \mathrm{Hom}_Q(E(P_R), E(P_R))$, *and the following equivalent conditions hold.*

(9) *Q is right self-injective.*

(10) $Q_R \cong E(R_R)$.

(11) $l(I) = 0$ *for every* $I \in K(R)$, *where $l(I)$ denotes the left annihilator of I in R.*

If, furthermore, P_R is a finitely generated projective generator in mod-R, *all conditions* (1)–(11) *are equivalent.*

Proof. First, notice that Q is always isomorphic to $Q_m(R)$ by Proposition 3.1. And if M_R is a generator with $S = \mathrm{End}(M_R)$, it is well-known that (a) S is right self-injective if and only if M_R is injective, and (b) if M_R is injective, R is right self-injective.

(5) $\Rightarrow$ (6). Since $E(P_R)$ is rational over P_R,

$$\mathrm{Hom}_R(E(P_R)/P, E(P_R)) = 0.$$

This implies that

$$\mathrm{Hom}_R(E(P_R)/P, E(R_R)) = 0,$$

because $E(R_R)$ is cogenerated by $E(P_R)$. Hence $E(P_R)/P$ is $\chi(R)$-torsion. Therefore $(P:x)$ is a dense right ideal of R for each $x \in E(P_R)$.

(6) $\Rightarrow$ (1) and (9). (6) implies that $E(P_R) = E_{\chi(R)}(P_R)$. Then we can induce $E(R_R) = E_{\chi(R)}(R_R)$. For, since P_R is a generator, $\bigoplus^n P_R = R_R \oplus X_R$ for some $X \in$ mod-R; so

$$\bigoplus{}^n E(P_R) = E(R_R) \oplus E(X_R).$$

Then we have that

$$\bigoplus{}^n (E(P_R)/P_R) \cong \bigoplus{}^n E(P_R)/\bigoplus{}^n P_R$$
$$\cong E(R_R)/R \oplus E(X_R)/X.$$

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Since $E(P_R)/P$ is $\chi(R)$-torsion and $\mathscr{T}_{\chi(R)}$ is closed under taking direct sums and submodules, also $E(R_R)/R$ is $\chi(R)$-torsion. Hence $E(R_R) = E_{\chi(R)}(R_R)$, as required. By Corollary 2.4, $E(P_R)$ is an $E(Q_Q)$-torsionfree generator in mod-Q and

$$H = \mathrm{Hom}_Q(E(P_R), E(P_R)),$$

and Q (resp. H) is isomorphic to $Q_m(R)$ (resp. $Q_m(S)$). But, on the other hand, since

$$Q_R \cong Q_m(R)_R \cong E_{\chi(R)}(R_R) = E(R_R),$$

Q is right self-injective by Lemma 3.2. Hence $E(P_R)$ is Q-torsionless. And since $E(P_R)$ is an injective right Q-module by Lemma 3.2, $H = \mathrm{End}(E(P_R)_Q)$ is right self-injective, too. Thus, we have (6) $\Rightarrow$ (1) and (9), and the assertion that $E(P_R)$ is a torsionless generator in mod-Q and

$$H = \mathrm{Hom}_Q(E(P_R), E(P_R)).$$

(1) $\Rightarrow$ (3). Since $H \cong Q_m(S)$,

$$H_S \cong Q_m(S)_S \cong E_{\chi(S)}(S_S),$$

and hence H is right self-injective if and only if H_S is injective by Lemma 3.2. Thus, we have $H_S \cong E(S_S)$.

(3) $\Rightarrow$ (5).

$$H_S \cong E(S_S) = E(G(P_R)_S) = G(E(P_R))_S.$$

This shows that

$$\mathrm{Hom}_R(E(P_R)/P, E(P_R)) = 0.$$

Hence we conclude that $E(P_R)$ is rational over P_R.

(2) $\Leftrightarrow$ (5). By virtue of Corollary 2.4, we can identify $\mathrm{Hom}_R(E_{\chi(R)}(P_R), E_{\chi(R)}(P_R))$ with $Q_m(S)$. Assume (2). Then every $\alpha \in Q_m(S)$ can be uniquely extended to $\beta \in H = \mathrm{Hom}_R(E(P_R), E(P_R))$. Hence $E(P_R)$ is a rational extension of $E_{\chi(R)}(P_R)$. Thus, we get $E(P_R) = E_{\chi(R)}(P_R)$. Hence (5) holds. Conversely, since (5) implies $E(P_R) = E_{\chi(R)}(P_R)$, (2) clearly holds.

(5) $\Rightarrow$ (7). By (5), we have $E(P_R) = E_{\chi(R)}(P_R)$. Hence we get (7) by Lemma 3.2.

(7) $\Rightarrow$ (5). By Lemma 3.2, $E_{\chi(R)}(P_R)$ is R-injective. Hence we get $E_{\chi(R)}(P_R) = E(P_R)$. Thus, (5) holds.

(7) $\Leftrightarrow$ (8). This is due to Lemma 3.4.

(9) $\Leftrightarrow$ (10). Since $Q \cong Q_m(R)$, we have

$$Q_R \cong E_{\chi(R)}(R_R).$$

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Hence Q is right self-injective if and only if

$$E_{\chi(R)}(\mathrm{R}_R) = E(R_R)$$

by Lemma 3.2.

Next, we assume that P_R is a finitely generated projective generator in mod-R. Then we can easily show that $E_{\chi(R)}(P_R)$ is a finitely generated projective generator as a right $Q_m(R)$-module and

$$Q_m(S) \cong \mathrm{Hom}_{Q_m(R)}(E_{\chi(R)}(P_R), E_{\chi(R)}(P_R))$$

by using Corollary 2.4. We want to show that (9) implies (6) in this case.

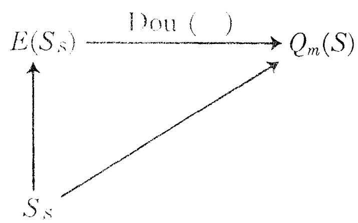
(9) $\Rightarrow$ (6). Since Q is right self-injective, so is also $Q_m(R)$. And since $E_{\chi(R)}(P_R)$ is a finitely generated projective right $Q_m(R)$-module by the above remark, $E_{\chi(R)}(P_R)$ is $Q_m(R)$-injective. Hence $E_{\chi(R)}(P_R)$ is R-injective by Lemma 3.2. Thus, we have

$$E_{\chi(R)}(P_R) = E(P_R).$$

Therefore (9) implies (6).

Thus, if P_R is a finitely generated projective generator, we have shown that all conditions but (4) and (11) are equivalent. Again we assume that P_R is an $E(R_R)$-torsionfree generator.

(1) $\Rightarrow$ (4). By (1), $Q_m(S)$ is right self-injective. Consider the diagram:



Then, applying (9) $\Rightarrow$ (8) to this situation, we get (4).

(4) $\Rightarrow$ (5). (4) implies that $Q_m(S)$ is right self-injective by using (8) $\Rightarrow$ (9). Since $E_{\chi(R)}(P_R)$ is a generator as a right $Q_m(R)$-module and

$$Q_m(S) = \mathrm{Hom}_{Q_m(R)}(E_{\chi(R)}(P_R), E_{\chi(R)}(P_R))$$

by Corollary 2.4, $E_{\chi(R)}(P_R)$ is $Q_m(R)$-injective. Hence $E_{\chi(R)}(P_R)$ is R-injective by Lemma 3.2. Therefore we have $E_{\chi(R)}(P_R) = E(P_R)$. Thus, (5) holds.

(11) $\Rightarrow$ (9). (11) implies that $Q_m(R)$ is right self-injective by using (8) $\Rightarrow$ (9). Hence Q is right self-injective by Proposition 3.1.

(9) $\Rightarrow$ (11). By (9), $Q_m(R)$ is right self-injective. Since $Q_m(R) = \mathrm{Dou}(E(R_R))$, R satisfies the condition (11) by using (9) $\Rightarrow$ (8).

This completes the proof of Theorem 3.5.

Putting $P = R$ in Theorem 3.5, we get the next well-known result.

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COROLLARY 3.6. *Let $H = \mathrm{End}(E(R_R))$ and $Q = \mathrm{Dou}(E(R_R))$. Then the following conditions are equivalent.*

(1) *H is right self-injective and isomorphic to $Q_m(R)$.*

(2) *$H \cong Q_m(R)$.*

(3) *$H_R \cong E(R_R)$.*

(4) *$E(R_R)$ is a rational extension of R_R.*

(5) *$(R:x)$ is a dense right ideal of R for each $x \in E(R_R)$.*

(6) *For any right ideal I of R and any R-homomorphism $\alpha: I \to R$, there exist a dense right ideal J and an R-homomorphism $\beta: J \to R$ such that $I \subseteq J$ and $\beta|I = \alpha$.*

(7) *$l(J) = 0$ for every $J \in K(R)$.*

(8) *Q is right self-injective.*

(9) *$Q_R \cong E(R_R)$.*

Remark. Masaike [**15**] has called a ring R right generalized non-singular if R satisfies the condition (7) of Corollary 3.6, and has proved the equivalence of (7) and (8) of Corollary 3.6.

4. Quasi-Frobenius maximal quotient rings. Let dim M_R denote the least integer n, if it exists, such that every direct sum of submodules of M_R has $\leqq n$ non-zero summands. If $\dim M_R = n < \infty$, M_R is said to be finite dimensional. In particular, if $\dim R_R = n < \infty$, R is called right finite dimensional. $E(M_R)$ is finite dimensional if and only if so also is M_R. The next lemma has its origin in [**27**, Lemma 1.4].

LEMMA 4.1. *Let H be a ring containing S. If H is a right quotient ring of S (i.e., H_S is a rational extension of S_S), then H is right finite dimensional if and only if so also is S.*

Proof. This proof is similar to that of [**27**, Lemma 1.4].

THEOREM 4.2. *Let P_R be an $E(R_R)$-torsionfree generator in* mod-R *and let $S = \mathrm{End}(P_R)$, $H = \mathrm{End}(E(P_R))$ and $Q = \mathrm{Dou}(E(P_R))$. If H is a right self-injective ring which is isomorphic to $Q_m(S)$, then the following statements are equivalent.*

(1) *H is semi-perfect.*

(2) *S is right finite dimensional.*

(3) *$E(P_R)$ is finite dimensional.*

(4) *P_R is finite dimensional.*

When this is so, Q is the right self-injective semi-perfect maximal right quotient ring of R.

Proof. (1) $\Leftrightarrow$ (3) and (3) $\Leftrightarrow$ (4) are well known. By our assumption and Lemma 4.1, S is right finite dimensional if and only if H is also. On the other hand, since H is right self-injective, H is semi-perfect if and only if H is right finite dimensional. Thus, we have (1) $\Leftrightarrow$ (2). And then, since ${}_H E(P_R)$ is a finitely generated projective left module over a semi-perfect ring H, ${}_H E(P_R)$ is a finitely generated semi-perfect module. Hence

$$Q = \mathrm{Hom}_H(E(P_R), E(P_R))$$

is a semi-perfect ring by [**14**, Theorem 6.1].

Putting $P = R$ in Theorem 4.2, we get the next result.

COROLLARY 4.3. *Let $H = \mathrm{End}(E(R_R))$ and $Q = \mathrm{Dou}(E(R_R))$. If H is right self-injective and isomorphic to $Q_m(R)$, then the following conditions are equivalent.*

(1) *H is semi-perfect.*

(2) *$E(R_R)$ is finite dimensional.*

(3) *R is right finite dimensional.*

(4) *Q is semi-perfect.*

For any $M \in$ mod-R, an R-submodule L of M is called rationally closed if L has no proper rational extension in M.

THEOREM 4.4. *Let P_R, S, H and Q be the same as in Theorem* 4.2. *If* $\mathrm{ann}_S M = 0$ *for each $M \in K(P)$ and P_R satisfies the* ACC *on rationally closed submodules, then H is right self-injective semi-primary and is isomorphic to $Q_m(S)$. When this is so, Q also is right self-injective and semi-primary.*

Proof. By Theorem 3.5, $E(P_R)$ is rational over P_R. Then it is known that the lattice of rationally closed submodules of $E(P_R)$ is isomorphic to that of P_R. Hence, since $E(P_R)$ is an injective module which satisfies the ACC on rationally closed submodules, $H = \mathrm{End}(E(P_R))$ is semi-primary by [**20**, Corollary 12]. This as well as Theorem 3.5 shows that H is a right self-injective semi-primary ring which is isomorphic to $Q_m(S)$. To show the last assertion of this theorem, it suffices to prove the next proposition.

PROPOSITION 4.5. *Let ${}_H M$ be a finitely generated projective left module over a semi-primary ring H. Then $Q = \mathrm{End}({}_H M)$ is a semi-primary ring.*

Proof. Since ${}_H M$ is a finitely generated projective module over a semi-perfect ring H, $Q = \mathrm{End}({}_H M)$ also is a semi-perfect ring by [**14**]. And since H is a semi-primary ring, ${}_H M$ can be regarded as a finite direct sum of indecomposable left ideals of H; say ${}_H M = \bigoplus_{i=1}^n He_i$, where each e_i is a primitive idempotent of H. Let $f_i : {}_H M \to {}_H He_i$, be the projection map of M onto He_i for each $i = 1, 2, \ldots, n$. Then $\{f_i\}$ is a set of orthogonal primitive idempotents of a semi-perfect ring Q and $1_Q = f_1 + \ldots + f_n$. And we have that

$$f_i Q f_i \cong \mathrm{Hom}_H(He_i, He_i) \cong e_i He_i$$

for each $i = 1, \ldots, n$. Since each e_i is a local idempotent of a semi-primary ring H, each e_iHe_i is semi-primary by [**16**, Theorem 2]. Hence, since each f_iQf_i also is semi-primary, then Q is a semi-primary ring by using [**16**] again.

COROLLARY 4.6. *If $l(I) = 0$ for every $I \in K(R)$, and if R satisfies the* ACC *on rationally closed right ideals, then $Q_m(R)$ is right self-injective and semi-primary.*

Remember that for any $\tau \in$ tors-R and any $M \in$ mod-R, a submodule L of M is called τ-saturated if M/L is τ-torsionfree. We denote by $\mathrm{Sat}_\tau(M)$ the lattice of all τ-saturated submodules of M.

LEMMA 4.7. *Let M_R be a right R-module the injective hull of which cogenerates a hereditary torsion theory τ on* mod-R. *Let $E(M_R) = E_\tau(M_R)$ and $H = \mathrm{End}(E(M_R))$. Then*

$$\mathrm{Sat}_\tau(M) = \{L_R \subseteq M_R | L = \mathrm{ann}_M X \text{ for some subset } X \text{ of } H\}.$$

Proof. $L \in \mathrm{Sat}_\tau(M)$ if and only if M/L is τ-torsionfree if and only if

$$M/L \hookrightarrow \prod E(M_R)$$

if and only if

$$L = \cap_{f_\alpha \in X} \mathrm{Ker}(f_\alpha)$$

for some $X \subseteq \mathrm{Hom}_R(M, E(M_R))$. Since $E(M_R)$ is injective, we have the exact sequence

$$0 \rightarrow \mathrm{Hom}_R(E(M_R)/M, E(M_R)) \rightarrow \mathrm{Hom}_R(E(M_R), E(M_R)) \rightarrow \mathrm{Hom}_R(M, \mathrm{E}(M_R)) \rightarrow 0.$$

Since $E(M_R)/M$ is τ-torsion and $E(M_R)$ is τ-torsionfree,

$$\mathrm{Hom}_R(E(M_R)/M, \mathrm{E}(M_R)) = 0,$$

and so we can identify $\mathrm{Hom}_R(M, \mathrm{E}(M_R))$ with $\mathrm{Hom}_R(E(M_R), E(M_R)) = H$. Hence $L \in \mathrm{Sat}_\tau(M)$ if and only if

$$L = \cap_{f_\alpha \in X} \mathrm{Ker}(f_\alpha) \quad \text{for some } X \subseteq H$$

if and only if

$$L = \mathrm{ann}_M X \quad \text{for some } X \subseteq H.$$

LEMMA 4.8. *Let $N_R \subseteq M_R$ such that M_R is a rational extension of N_R, and let $H = \mathrm{End}(M_R)$. For any two left ideals, L_1 and L_2 of H with $L_2 \subsetneqq L_1$, we have the following statements.*

(1) *If $r(L_1) \subsetneqq r(L_2)$, where $r(L_i)$ denotes the right annihilator of L_i in H, then we have that $\mathrm{ann}_M L_1 \subsetneqq \mathrm{ann}_M L_2$.*

(2) $\mathrm{ann}_M L_1 \subsetneqq \mathrm{ann}_M L_2$ *if and only if* $\mathrm{ann}_N L_1 \subsetneqq \mathrm{ann}_N L_2$.

Proof. (1) Assume that $r(L_1) \subsetneqq r(L_2)$. Clearly $\mathrm{ann}_M L_1 \subseteq \mathrm{ann}_M L_2$. Let $\alpha \in H$ be such that $L_2\alpha = 0$ and $x_1\alpha \neq 0$ for some $x_1 \in L_1$. Then $x_1\alpha(N) \neq 0$, since M_R is rational over N_R. Hence there exists $n \in N$ such that $x_1\alpha(n) \neq 0$. On the other hand, $x_2\alpha(n) = 0$ for all $x_2 \in L_2$. Hence $\alpha(n) \in \mathrm{ann}_M L_2$, but $\alpha(n) \notin \mathrm{ann}_M L_1$. So we have that $\mathrm{ann}_M L_1 \subsetneqq \mathrm{ann}_M L_2$.

(2) First, assume that $\mathrm{ann}_M L_1 \subsetneqq \mathrm{ann}_M L_2$. Then there exists $y \in M$ such that $L_2y = 0$ and $x_1y \neq 0$ for some $x_1 \in L_1$. Since M is rational over N, there exists an element $r \in R$ such that $yr \in N$ and $x_1yr \neq 0$. Hence

$$yr \in N \cap \mathrm{ann}_M L_2 = \mathrm{ann}_N L_2,$$

but

$$yr \notin N \cap \mathrm{ann}_M L_1 = \mathrm{ann}_N L_1.$$

Therefore we have that $\mathrm{ann}_N L_1 \subsetneqq \mathrm{ann}_N L_2$. Since the converse is trivial, this completes the proof of Lemma 4.8.

THEOREM 4.9. *Let P_R be an $E(R_R)$-torsionfree generator in* mod-R *and let $S = \mathrm{End}(P_R)$, $H = \mathrm{End}(E(P_R))$ and $Q = \mathrm{Dou}(E(P_R))$. If H is right self-injective and isomorphic to $Q_m(S)$, then the following statements are equivalent.*

(1) *H is a quasi-Frobenius ring.*

(2) $\mathrm{Sat}_{\chi(R)}(E(P_R))$ *is Noetherian.*

(2′) $\{Y_R \subseteq E(P_R) \mid Y = \mathrm{ann}_{E(P)}X$ *for some* $X \subseteq H\}$ *satisfies the* ACC.

(3) $\mathrm{Sat}_{\chi(R)}(P)$ *is Noetherian.*

(3′) $\{M_R \subseteq P_R \mid M = \mathrm{ann}_P X$ *for some* $X \subseteq H\}$ *satisfies the* ACC.

(4) $\mathrm{Sat}_{\chi(S)}(S)$ *is Noetherian.*

(4′) $\{I_S \subseteq S_S \mid I = r(X)$ *for some* $X \subseteq H\}$ *satisfies the* ACC.

When these conditions are satisfied, the next two equivalent conditions hold.

(5) *Q is a quasi-Frobenius ring.*

(6) $\mathrm{Sat}_{\chi(R)}(R)$ *is Noetherian.*

Proof. By Theorem 3.5, we have $E(P_R) = E_{\chi(R)}(P_R)$. Hence by Lemma 4.7, we get (2) $\Leftrightarrow$ (2′) and (3) $\Leftrightarrow$ (3′), and by Lemma 4.8 we get (2′) $\Leftrightarrow$ (3′).

(1) $\Rightarrow$ (2′). Let $Y_1 \subseteq Y_2 \subseteq Y_3 \ldots,$ be the ascending chain of annihilators of subsets of H in $E(P_R)$. Then, since H is left Artinian, the descending chain of left ideals,

$$\mathrm{ann}_H Y_1 \supseteq \mathrm{ann}_H Y_2 \supseteq \mathrm{ann}_H Y_3 \supseteq \ldots.$$

terminates; say $\mathrm{ann}_H Y_n = \mathrm{ann}_H Y_{n+1}$. Then

$$Y_n = \mathrm{ann}_{E(P)}\mathrm{ann}_H Y_n = \mathrm{ann}_{E(P)}\mathrm{ann}_H Y_{n+1} = Y_{n+1}.$$

Hence we get (1) $\Rightarrow$ (2′).

(3′) ⇒ (1). Let $r(X_1) \subsetneqq r(X_2) \subsetneqq r(X_3) \ldots$ be any strictly ascending chain of right annulets of H. Then by Lemma 4.8, we have the strictly ascending chain of right annihilators,

$$\mathrm{ann}_P X_1 \subsetneqq \mathrm{ann}_P X_2 \subsetneqq \mathrm{ann}_P X_3 \subsetneqq \ldots .$$

Hence H must satisfy the ACC on right annulets. Since H is right self-injective, too, H is a quasi-Frobenius ring. Thus, we get (3′) ⇒ (1).

Next, by our assumption and Theorem 3.5, $H_S = E(S_S)$ and H_S is rational over S_S, and $\mathrm{End}(H_S)$ is the right self-injective maximal right quotient ring of S. Hence the equivalence of (1), (3) and (3′) of this theorem guarantees that of (1), (4) and (4′).

(1) ⇒ (5). Since $E(P_R)$ is a faithful, finitely generated projective left H-module and $Q = \mathrm{End}({}_H E(P_R))$, Q is also a quasi-Frobenius ring.

(5) ⇔ (6). By our assumption, $Q_m(R)$ ($\cong Q$) is right self-injective, and so $\mathrm{End}(E(R_R))$ is right self-injective and is isomorphic to $Q_m(R)$ by Corollary 3.6. Hence the equivalence of (1) and (3) of this theorem guarantees that of (5) and (6). This completes the proof of Theorem 4.9.

Remark. In Theorem 4.9, the implication (5) ⇒ (1) does not necessarily hold. A result of [**19**, Theorem 2] shows this.

COROLLARY 4.10. *Let* $H = \mathrm{End}(E(R_R))$ *and* $Q = \mathrm{Dou}(E(R_R))$. *If* H *is right self-injective and isomorphic to* $Q_m(R)$, *then the following conditions are equivalent.*

(1) *H is a quasi-Frobenius ring.*

(2) $\mathrm{Sat}_{\chi(R)}(E(R_R))$ *is Noetherian.*

(2′) $\{Y_R \subseteq E(R_R) \mid Y = \mathrm{ann}_{E(R)} X$ *for some* $X \subseteq H\}$ *satisfies the* ACC.

(3) $\mathrm{Sat}_{\chi(R)}(R)$ *is Noetherian.*

(3′) $\{I_R \subseteq R_R \mid I = \mathrm{ann}_R X$ *for some* $X \subseteq E(R_R)\}$ *satisfies the* ACC.

(4) *Q is a quasi-Frobenius ring.*

5. Semi-simple Artinian maximal quotient rings. By $Z(M_R)$ we denote the singular submodule of M_R. For any $M \in$ mod-R, if $Z(M_R) = 0$, M is called a *non-singular module.* In particular, if $Z(R_R) = 0$, then R is said to be a *right non-singular ring.* It is well known that (a) if N is an essential submodule of M, then $Z(N_R) = 0$ if and only if $Z(M_R) = 0$, and (b) if $N_R \subseteq M_R$ and if $Z(N_R) = 0$, then M is rational over N if and only if M is essential over N.

THEOREM 5.1. *Let* P_R *be an* $E(R_R)$*-torsionfree generator in* mod-R *and let* $S = \mathrm{End}(P_R)$, $H = \mathrm{End}(E(P_R))$ *and* $Q = \mathrm{Dou}(E(P_R))$. *Then the following statements are equivalent.*

(1) *H is a regular, right self-injective ring and is isomorphic to* $Q_m(S)$.

(2) *H is semi-primitive.*

(3) *S is a right non-simgular ring.*

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(4) *P_R is non-singular.*
(5) *Q is regular and right self-injective.*
(6) *Q is regular.*
(7) *R is a right non-singular ring.*
In particular, if P_R is a non-singular generator in mod-R, *then all conditions* (1) — (7) *hold.*

Remark. (3) $\Leftrightarrow$ (4) and (3) $\Rightarrow$ (7) have been obtained in [**2**, Theorem 4.9] in a slightly different form.

Proof of Theorem 5.1. (4) $\Rightarrow$ (1). Since $Z(P_R) = 0$, $E(P_R)$ is a rational extension of P_R and $Z(E(P_R)) = 0$. Hence H is regular, right self-injective and is isomorphic to $Q_m(S)$ by a result of [**26**] and Theorem 3.5.

(1) $\Rightarrow$ (5). By Theorem 3.5, $E(P_R)$ is a finitely generated projective left H-module and

$$Q = \mathrm{Hom}_H(E(P_R), E(P_R)).$$

Then Q is regular and right self-injective by a result of [**1**, Corollary 2.6], since so also is H.

(5) $\Rightarrow$ (6). This is trivial.

(6) $\Rightarrow$ (7). Since $Q \cong Q_m(R)$, R_R is essential in Q_R. Hence we can easily show that $Z(R_R) \subseteq Z(Q_R) \subseteq Z(Q_Q)$. Since Q is regular, $Z(Q_Q) = 0$, and hence $Z(R_R) = 0$.

(7) $\Rightarrow$ (4). Since P_R is $E(R_R)$-torsionfree,

$$P_R \hookrightarrow \prod E(R_R).$$

Then $Z(R_R) = 0$ implies that $Z(\prod E(R_R)) = 0$, and hence we get $Z(P_R) = 0$.

(1) $\Rightarrow$ (2). This is trivial.

(2) $\Rightarrow$ (1). Consider any $h: E(P_R) \to E(P_R)$ such that $h(P) = 0$. Hence

$$h \in \{f \in H | \mathrm{Ker}(f) \text{ is essential in } E(P_R)\} = \mathrm{Rad}\, H.$$

Therefore $h = 0$ by assumption (2). This shows that $E(P_R)$ is a rational extension of P_R. Hence $H \cong Q_m(S)$ by Theorem 3.5. On the other hand, $H = H/\mathrm{Rad}\, H$ is regular and right self-injective by results of Utumi and Osofsky (e.g., see [**17**, Lemma 7 and Theorem 12]).

(3) $\Leftrightarrow$ (4). This is due to [**2**, Theorem 4.9]. This completes the proof of Theorem 5.1.

Putting $P = R$ in Theorem 5.1, we get the next well-known result.

COROLLARY 5.2. *Let $H = \mathrm{End}(E(R_R))$ and $Q = \mathrm{Dou}(E(R_R))$. Then the following statements are equivalent.*

(1) *H is a regular, right self-injective ring and is isomorphic to $Q_m(R)$.*
(2) *H is a semi-primitive ring.*

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(3) *R is a right non-singular ring.*

(4) *Q is a regular and right self-injective ring.*

(5) *Q is a regular ring.*

THEOREM 5.3. *Let P_R be an $E(R_R)$-torsionfree generator in* mod-R *and let $S = \text{End}(P_R)$, $H = \text{End}(E(P_R))$ and $Q = \text{Dou}(E(P_R))$. Then the following statements are equivalent.*

(1) *H is semi-simple Artinian and is isomorphic to $Q_m(S)$.*

(2) *H is semi-simple Artinian.*

(3) *S is right finite dimensional and right non-singular.*

(4) *P_R is finite dimensional and non-singular.*

When these conditions are satisfied, the next two equivalent conditions hold.

(5) *Q is semi-simple Artinian.*

(6) *R is right finite dimensional and right non-singular.*

And then H and Q are Morita equivalent via ${}_HE(P_R)_Q$.

Remark. The equivalence of (1) and (4), and the last statement of this theorem are in [**2**, Corollary 4.10].

Proof. (1) $\Leftrightarrow$ (4). In this case, H is regular and right self-injective by Theorem 5.1. Then H is semi-simple Artinian if and only if H is semi-perfect if and only it $E(P_R)$ is finite dimensional if and only if P_R is finite dimensional.

(1) $\Leftrightarrow$ (2). This is clear by Theorem 5.1.

(2) $\Leftrightarrow$ (3). In this case, all conditions of Theorem 5.1 hold. Then H is right finite dimensional if and only if S is also by Lemma 4.1. And, since H is regular and right self-injective, H is semi-simple Artinian if and only if H is right finite dimensional. Thus, we have (2) $\Leftrightarrow$ (3).

(1) $\Rightarrow$ (5). Since ${}_H\text{E}(P_R)$ is faithful, finitely generated projective and $Q = \text{End}({}_HE(P_R))$, Q is semi-simple Artinian.

(5) $\Leftrightarrow$ (6). Let us put $Q' = \text{End}(E(R_R))$. Under the assumption (5) or (6), we have that $Q' \cong Q_m(R)$ and that Q' is regular and right self-injective. Then the equivalence of (2) and (4) of this theorem guarantees that of (5) and (6).

COROLLARY 5.4. *Let $H = \text{End}(E(R_R))$ and $Q = \text{Dou}(E(R_R))$. Then the following conditions are equivalent.*

(1) *H is semi-simple Artinian and isomorphic to $Q_m(R)$.*

(2) *H is semi-simple Artinian.*

(3) *R is right finite dimensional and right non-singular.*

(4) *Q is semi-simple Artinian.*

And then H and Q are Morita equivalent via ${}_HE(R_R)_Q$.

COROLLARY 5.5. *If P_R is a finite dimensional, non-singular generator in* mod-R, *then $S = \text{End}(P_R)$ has the semi-simple Artinian maximal right quotient ring which is isomorphic to $H = \text{End}(E(P_R))$.*

Proof. Since P_R is $E(R_R)$-torsionfree by our assumption, our assertion is clear by virtue of Theorem 5.3.

Remark. In Theorem 5.3, the implication (5) $\Rightarrow$ (1) does not necessarily hold. The next corollary shows this.

COROLLARY 5.6. *Let R be a right finite dimensional, right non-singular ring, and let F_R be an infinitely generated free right R-module. Then $H = \mathrm{End}(E(F_R))$ is a regular, right self-injective, but not left self-injective ring which is isomorphic to $Q_m(S)$, where $S = \mathrm{End}(F_R)$. And then $Q = \mathrm{Dou}(E(F_R))$ is the semi-simple Artinian maximal right quotient ring of R.*

Proof. Let $F_R = \bigoplus_{\alpha\in\Lambda} x_\alpha R$ be a non-finitely generated free right R-module with free basis $\{x_\alpha\}_{\alpha\in\Lambda}$. Since R is right finite dimensional and right non-singular, every direct sum of non-singular, injective right R-modules is injective, too. Hence $\bigoplus_{\alpha\in\Lambda} E(x_\alpha R)$ is an injective right R-module. Then we can easily verify that

$$E(F_R) = \bigoplus_{\alpha\in\Lambda} E(x_\alpha R).$$

By Theorem 5.1, H is a regular, right self-injective ring which is isomorphic to $Q_m(S)$. And since $E(F_R)$ is a generator in mod-Q and $H = \mathrm{Hom}_Q(E(F_R), E(F_R))$ by Theorem 3.5, $E(F_R)$ is a finitely generated projective left H-module. On the other hand, $E(F_R)$ is not an injective left H-module by [**19**, Theorem 1], since it is a direct sum of infinite non-zero submodules. Therefore H cannot be left self-injective. But, Q is the semi-simple Artinian maximal right quotient ring of R by Theorem 5.3 and Proposition 3.1.

As an immediate consequence of Corollary 5.6, we have the next result.

COROLLARY 5.7. *The endomorphism ring of an infinitely generated free right module over a semi-simple Artinian ring is a regular and right self-injective ring which is not left self-injective.*

Note added in proof. After having type-written this manuscript, the author found that we can easily deduce Proposition 4.5 also from a result of J.-E. Bjork [*Conditions which imply that subrings of semi-primary rings are semi-primary*, J. Algebra *19* (1971), 384–395, Theorem 4.1], which states that if M is a left module of a finite presentation over a semi-primary ring R, then $\mathrm{End}({}_RM)$ is semi-primary.

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