THE ACTION OF HECKE OPERATORS ON HYPERGEOMETRIC FUNCTIONS

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Abstract

We study the action of the Hecke operators U_n on the set of hypergeometric functions, as well as on formal power series. We show that the spectrum of these operators on the set of hypergeometric functions is the set $\{n^a : n \in \mathbb{N}, a \in \mathbb{Z}\}$, and that the polylogarithms play an important role in the study of the eigenfunctions of the Hecke operators U_n on the set of hypergeometric functions. As a corollary of our results on simultaneous eigenfunctions, we also obtain an apparently unrelated result regarding the behavior of completely multiplicative hypergeometric coefficients.

1. Introduction

For each $n \in \mathbb{N}$, the space of formal power series $\mathfrak{F}$,

$$\mathfrak{F} := \left\{ f(x) = \sum_{k=0}^{\infty} c_k x^k : c_k \in \mathbb{C} \right\}, \tag{1.1}$$

admits the action of the linear operators U_n and V_n:

$$(U_n f)(x) := \sum_{k=0}^{\infty} c_{nk} x^k \tag{1.2}$$

and

$$(V_n f)(x) := f(x^n) = \sum_{k=0}^{\infty} c_k x^{nk}. \tag{1.3}$$

The spectral properties of these operators become more interesting when one considers their action on spaces with additional structure.

Historically, Hecke studied the vector spaces of modular forms of a fixed weight (see [2]), in which the set $\mathfrak{F}$ is replaced by the space $\mathfrak{M}_k$ of analytic functions in the upper half-plane $\mathbb{H} := \{\tau : \operatorname{Im} \tau > 0\}$ that satisfy the condition

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k f(\tau),$$

for every matrix in the modular group

$$\Gamma := \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}, \, ad - bc = 1\right\}.$$

These forms are also required to have an expansion at $\tau = i\infty$, or equivalently a Taylor series about $q = 0$:

$$f(\tau) = \sum_{n=0}^{\infty} c(n) q^n,$$

expressed in terms of the parameter $q = \exp(2\pi i \tau)$.

Hecke introduced a family of operators T_n, where $n \in \mathbb{N}$, which map the space $\mathfrak{M}_k$ into itself. The standard definition is

$$(T_n f)(\tau) := n^{k-1} \sum_{d \mid n} d^{-k} \sum_{b=0}^{d-1} f\left(\frac{n\tau + bd}{d^2}\right),$$

which, in the special case where n is a prime p, becomes

$$(T_p f)(\tau) = p^{k-1} f(p\tau) + \frac{1}{p} \sum_{b=0}^{p-1} f\left(\frac{\tau + b}{p}\right).$$

In terms of the Fourier expansion of $f \in \mathfrak{M}_k$, given by

$$f(\tau) = \sum_{m=0}^{\infty} c(m) q^m,$$

the action of T_n is

$$(T_n f)(\tau) = \sum_{m=0}^{\infty} \gamma_n(m) q^m,$$

where

$$\gamma_n(m) = \sum_{d \mid (n,m)} d^{k-1} c\left(\frac{mn}{d^2}\right).$$

In particular, when n is a prime p,

$$\gamma_p(m) = \begin{cases} c(pm) + p^{k-1} c(m/p) & \text{if } p \text{ divides } m, \\ c(pm) & \text{if } p \text{ does not divide } m. \end{cases}$$

History has shown that the study of Hecke operators is one of the most important tools in modern number theory, yielding results about the uniform distributions of

points, the eigenvalues of Laplacians on various domains, the asymptotic analysis of Fourier coefficients of modular forms, and other branches of number theory.

Interesting results were obtained in the last decade when the space of modular forms was replaced with the space of rational functions (see [4–6]). For example, the spectral properties of the operator U_n acting on rational functions were completely characterized, and corollaries about completely multiplicative functions that satisfy linear recurrence sequences were obtained (see [5]).

For the space $\mathfrak{R}$ of rational functions, the coefficients a_n in (1.1) are the Taylor coefficients of $A/B \in \mathfrak{R}$, where $B(x) = 1 + \alpha_1 x + \cdots + \alpha_d x^d$, and A is a polynomial of degree less than d. The coefficients a_n are known to satisfy the recurrence relation

$$a_{n+d} = -\alpha_1 a_{n+d-1} - \cdots - \alpha_d a_n$$

(see [7] for details). Thus the study of these coefficients employs the theory of linear recurrence sequences and their explicit solutions. One of the main results in [5] is the complete determination of the spectrum of U_n acting on $\mathfrak{R}$, namely

$$\mathrm{spec}(U_n) = \{\pm n^k : k \in \mathbb{N}\} \cup \{0\}.$$

Recent work has produced a description of the corresponding rational eigenfunctions (see [3, 6]).

In this paper, we consider the action of U_n on the *set* $\mathfrak{H}$ of hypergeometric functions:

$$\mathfrak{H} := \left\{ \sum_{k=0}^{\infty} c_k x^k : \frac{c_{k+1}}{c_k} \text{ is a rational function of } k \right\}.$$

We emphasize here that $\mathfrak{H}$ is a set rather than a vector space, because in general the sum of two hypergeometric functions is *not* another hypergeometric function. Nevertheless, this set includes many of the classical functions as well as all functions of the form

$$\sum_{k=0}^{\infty} R(k) x^k,$$

where R is a rational function.

Every hypergeometric function that we consider has a canonical Taylor series representation of the form

$$_pF_q(\mathbf{a}, \mathbf{b}; x) := \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \cdots (a_p)_k}{(b_1)_k (b_2)_k \cdots (b_q)_k} \frac{x^k}{k!},$$

where $\mathbf{a} := (a_1, a_2, \ldots, a_p) \in \mathbb{C}^p$ and $\mathbf{b} := (b_1, b_2, \ldots, b_q) \in \mathbb{C}^q$ are the *parameters* of $_pF_q$. These parameters satisfy $-b_i \notin \mathbb{N}$. Here we use the standard notation for the ascending factorial

$$(c)_k := c(c+1)(c+2) \cdots (c+k-1),$$

and $(c)_0 := 1$. For example, $(1)_k = k!$, and $(0)_k = 0$.

Hypergeometric functions include

$$f(x) = \sum_{k=0}^{\infty} \frac{x^k}{k^2 + 1},$$

as well as most of the elementary functions. For example, the hypergeometric representation of the exponential function is given by $e^x = {}_1F_1(a, a; x)$ for any nonzero $a \in \mathbb{C}$. Similarly, the error function

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt,$$

can also be represented as a hypergeometric function, namely

$$\mathrm{erf}(x) = \frac{2x}{\sqrt{\pi}} {}_1F_1\left(\frac{1}{2}, \frac{3}{2}; x\right).$$

For a more complete discussion of hypergeometric functions, see [1].

In Section 5 we describe the action of the Hecke operator U_n on hypergeometric functions. To state the results, define

$$\mathfrak{F}_{(p,q)}^j := \{x^j\, {}_pF_q(\mathbf{a}, \mathbf{b}; x) : \mathbf{a} \in \mathbb{C}^p, \mathbf{b} \in \mathbb{C}^q\},$$

for fixed $j \in \mathbb{N}$ and $p, q \in \mathbb{N}$. This is the set of all hypergeometric functions that vanish to order j at the origin, and have hypergeometric coefficients

$$\frac{(a_1)_k(a_2)_k \cdots (a_p)_k}{(b_1)_k(b_2)_k \cdots (b_q)_k},$$

with p ascending factorials in the numerator and q ascending factorials in the denominator. Observe that

$$\mathfrak{H} := \{x^j\, {}_pF_q(\mathbf{a}, \mathbf{b}; x) : j, p, q \in \mathbb{N}, \mathbf{a} \in \mathbb{C}^p, \mathbf{b} \in \mathbb{C}^q\} = \bigcup_{j,p,q \in \mathbb{N}} \mathfrak{F}_{(p,q)}^j.$$

We establish first the identities

$$U_n(x^j\, {}_pF_q(\mathbf{a}, \mathbf{b}; x)) = x^{j/n} \sum_{k=0}^{\infty} n^{nk(p-q-1)} \frac{(c_1)_k(c_2)_k \cdots (c_{np})_k}{(d_1)_k(d_2)_k \cdots (d_{n(q+1)-1})_k} \frac{x^k}{k!}$$

in $\mathfrak{F}_{(p_1,q_1)}^j$, when n divides j, and

$$U_n(x^j\, {}_pF_q(\mathbf{a}, \mathbf{b}; x)) = x^{1+\lfloor j/n \rfloor} \sum_{k=0}^{\infty} n^{nk(p-q-1)} \frac{(c_1)_k \cdots (c_{pn})_k}{(d_1)_k \cdots (d_{(q+1)n-1})_k} \frac{x^k}{k!}$$

in $\mathfrak{F}_{(p_1,q_1)}^j$, if n does not divide j. Here $p_1 = np$ and $q_1 = n(q+1) - 1$, while the new parameters $\mathbf{c}$ and $\mathbf{d}$ are given in (5.1) and (5.2). In particular, we observe in Section 5 that U_n maps $\mathfrak{F}_{(p,q)}^j$ into itself if and only if $p = q + 1$. These are the *balanced* hypergeometric functions. Therefore, an eigenfunction of U_n must satisfy the condition $p = q + 1$.

The eigenfunctions of U_n on the space of formal power series are described in Section 4. We consider solutions of $U_n f = \lambda f$, where $f(x) = x^j \sum_{k=0}^\infty a_k x^k$, and show that if n divides j, then it follows that $j = 0$, $\lambda = 1$ and the eigenfunction f must reduce to the rational function $1/(1-x)$. On the other hand, in the case where n does not divide j, we show that j must be 1 and the eigenvalue λ must be of the form n^a, where $a \in \mathbb{Z}$.

One of the main results here is the complete characterization of the spectrum $\operatorname{spec}(U_n)$ of U_n on hypergeometric functions, yielding the result that

$$\operatorname{spec}(U_n) = \{n^k : k \in \mathbb{Z}\} \cup \{0\}.$$

As a corollary, we obtain a number-theoretic characterization of all completely multiplicative functions that are also hypergeometric ratios of ascending factorials.

An ultimate goal is to determine the spectrum of all linear combinations of hypergeometric series, but this seems far from the reach of current technology. Thus we focus first on the action of the Hecke operators on a single hypergeometric series.

2. A natural inner product on $\mathfrak{H}$

The set $\mathfrak{H}$ of all hypergeometric functions can be endowed with a natural inner product:

$$\langle f, g \rangle_R := \oint_{|z|=R} f(w) \overline{g(w)} \, \frac{dw}{w}.$$

We fix a real number R such that $0 < R < 1$, so that this inner product is now a function of R; as we shall see shortly, it is in fact a real analytic function of R. Moreover, we shall also see below that V_n is the natural conjugate linear operator to U_n with respect to this inner product. This fact is our motivation for introducing the linear operator V_n.

The next result describes this inner product in terms of the Taylor series expansions of f and g.

LEMMA 2.1. *If $f(z) = \sum_{n=0}^\infty c_n z^n$ and $g(z) = \sum_{n=0}^\infty d_n z^n$, then*

$$\langle f, g \rangle_R = 2\pi i \sum_{n=0}^\infty c_n \overline{d_n} R^{2n}.$$

PROOF. By definition,

$$\langle f, g \rangle_R = \oint_{|w|=R} \sum_{n=0}^\infty c_n w^n \times \sum_{m=0}^\infty \overline{d_m} \overline{w}^m \, \frac{dw}{w}$$

$$= \oint_{|w|=R} \sum_{n,m=0}^\infty c_n \overline{d_m} w^{n-m-1} R^{2m} \, dw,$$

because $\overline{w} = R^2/w$ on the circle of integration. Cauchy's integral formula shows that $n = m$, for otherwise the ensuing line integral is zero. This condition gives us the desired result. $\qquad\square$

This inner product makes sense even for formal power series, although we restrict our attention to the set $\mathfrak{H}$ of hypergeometric functions. On $\mathfrak{H}$, the inner product of any two hypergeometric functions is in fact a hypergeometric function of R, as is easily seen by noting that the product of two hypergeometric coefficients is another hypergeometric coefficient.

To further develop the algebra of the operators U_n and V_n, we now show that V_n is the natural conjugate linear operator for U_n, relative to the inner product introduced above.

LEMMA 2.2. *Let $f, g \in \mathfrak{H}$. Then*

$$\langle U_n f, g \rangle_R = \langle f, V_n g \rangle_{R^n}.$$

PROOF. On the one hand,

$$
\begin{aligned}
\langle U_n f, g \rangle_R &= \left\langle \sum_{k=0}^{\infty} c_{kn} z^k, \sum_{k=0}^{\infty} d_k z^k \right\rangle_R \\
&= \sum_{k=0}^{\infty} c_{nk} \overline{d_k} R^{2k}.
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
\langle f, V_n g \rangle_R &= \left\langle \sum_{k=0}^{\infty} c_k z^k, \sum_{k=0}^{\infty} d_k z^{kn} \right\rangle_R \\
&= \sum_{k=0}^{\infty} c_k \overline{h_k} R^{2k},
\end{aligned}
$$

where we define

$$
h_k = \begin{cases} 0 & \text{if } n \text{ does not divide } k, \\ d_{k/n} & \text{if } n \text{ divides } k. \end{cases}
$$

It follows that

$$
\langle f, V_n g \rangle_R = \sum_{k=0}^{\infty} c_{kn} \overline{d_k} (R^n)^{2k} = \langle U_n f, g \rangle_{R^n},
$$

as required.   □

Recall that Hadamard introduced an inner product in the space $\mathfrak{F}$ of formal power series:

$$(f * g)(x) := \sum_{k=0}^{\infty} c_k d_k x^k.$$

This product can be retrieved as a special case of $\langle \cdot, \cdot \rangle_R$; namely,

$$
\left\langle \sum_{k=0}^{\infty} c_k x^k, \overline{\sum_{k=0}^{\infty} d_k x^k} \right\rangle_{R^{1/2}} = \sum_{k=0}^{\infty} c_k d_k R^k.
$$

Thus the Hadamard product is completely equivalent to our inner product.

3. Elementary properties of the operators U_n and V_n

In this section we describe elementary properties of the operators defined in (1.2) and (1.3).

THEOREM 3.1. *Let $m, n \in \mathbb{N}$. Then the following hold.*

(a) $U_n \circ U_m = U_m \circ U_n = U_{nm}$.
(b) $V_n \circ V_m = V_m \circ V_n = V_{nm}$.
(c) $U_n \circ V_n = \text{Id}$.
(d) $U_n \circ V_m = V_{m/\gcd(m,n)} \circ U_{n/\gcd(m,n)}$. *In particular, if m and n are relatively prime, then U_n and V_m commute.*

PROOF. Let $f \in \mathfrak{F}$ be a formal power series with coefficients c_k. The first two properties follow directly from the observation that

$$U_n U_m f(x) = U_n \sum_{k=0}^{\infty} c_{mk} x^k = \sum_{k=0}^{\infty} c_{nmk} x^k = U_{nm} f(x),$$

and similarly for $V_n V_m$. To establish the third property observe that

$$U_n V_n f(x) = U_n V_n \left(\sum_{k=0}^{\infty} c_k x^k \right) = U_n \left(\sum_{k=0}^{\infty} c_k x^{kn} \right)$$

$$= \sum_{k=0}^{\infty} c_k x^k = f(x).$$

Finally,

$$U_n \circ V_m \left(\sum_{k=0}^{\infty} c_k x^k \right) = U_n \left(\sum_{k=0}^{\infty} c_k x^{mk} \right).$$

To simplify this, let

$$d_k = \begin{cases} c_{k/m} & \text{if } m \text{ divides } k, \\ 0 & \text{if } m \text{ does not divide } k, \end{cases}$$

and write

$$U_n \circ V_m \left(\sum_{k=0}^{\infty} c_k x^k \right) = U_n \left(\sum_{k=0}^{\infty} d_k x^k \right) = \sum_{k=0}^{\infty} d_{nk} x^k.$$

Now observe that m divides kn if and only if $m/\gcd(m, n)$ divides k. Therefore, we define

$$N = \frac{n}{\gcd(m, n)} \quad \text{and} \quad M = \frac{m}{\gcd(m, n)},$$

and we obtain the following sum on k:

$$\sum_k d_{nk} x^k = \sum_{m | nk} d_{nk} x^k = \sum_{m | nk} c_{nk/m} x^k$$

$$= \sum_{M | Nk} c_{(Nk)/M} x^k = \sum_{M | k} c_{N(k/M)} x^k,$$

where each sum is defined for k from 0 to ∞, and subject to the constraints given. We now let $k = iM$, and sum over $i \geq 0$, and see that

$$\sum_{M|k} c_{N(k/M)} x^k = \sum_{i=0}^{\infty} c_{iN} x^{iM}.$$

Now define $h_i = c_{iN}$, and obtain

$$\sum_{i=0}^{\infty} h_i x^{iM} = V_M\left(\sum_{i=0}^{\infty} h_i x^i\right) = V_M\left(\sum_{i=0}^{\infty} c_{iN} x^i\right) = V_M \circ U_N\left(\sum_{i=0}^{\infty} c_i x^i\right),$$

and we have established part (d).   □

We now present an alternate proof for Theorem 3.1. To do this, we must prove an intermediate result.

LEMMA 3.2 (Associativity of U and V). *For all k, j, $m \in \mathbb{N}$,*

$$U_{kj} \circ V_m = U_k \circ (U_j \circ V_m).$$

PROOF. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$. For this proof, we will evaluate both operators and show they give the same result. First, for $U_{kj} \circ V_m$,

$$(U_{kj} \circ V_m) f(z) = U_{kj} f(z^m) = U_{kj}\left(\sum_{n=0}^{\infty} a_n z^{mn}\right).$$

Now we let

$$b_i = \begin{cases} a_{i/m} & \text{if } m \text{ divides } i, \\ 0 & \text{otherwise,} \end{cases}$$

so that we can write

$$U_{kj}\left(\sum_{n=0}^{\infty} a_n z^{mn}\right) = U_{kj}\left(\sum_{i=0}^{\infty} b_i z^i\right) = \sum_{i=0}^{\infty} b_{(kj)i} z^i.$$

Next, for $U_k \circ (U_j \circ V_m)$,

$$U_k \circ (U_j \circ V_m)\left(\sum_{n=0}^{\infty} a_n z^n\right) = U_k\left(U_j\left(\sum_{n=0}^{\infty} a_n z^{mn}\right)\right) = U_k\left(\sum_{i=0}^{\infty} c_i z^i\right),$$

where

$$c_i = \begin{cases} a_n & \text{if } ij = mn \text{ for } n \in \mathbb{N} \cup \{0\}, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$U_k\left(\sum_{i=0}^{\infty} c_i z^i\right) = \sum_{i=0}^{\infty} c_{ki} z^i.$$

We complete the proof by noting that $b_{(kj)i} = c_{ki}$.   □

ALTERNATIVE PROOF OF THEOREM 3.1(d). We can write

$$U_n \circ V_m = (U_{n/\gcd(m,n)} \circ U_{\gcd(m,n)}) \circ (V_{\gcd(m,n)} \circ V_{m/\gcd(m,n)}),$$

by parts (a) and (b) of the theorem. Now, using associativity, we can write

$$U_n \circ V_m = U_{n/\gcd(m,n)} \circ (U_{\gcd(m,n)} \circ V_{\gcd(m,n)}) \circ V_{m/\gcd(m,n)}.$$

By part (c) of the theorem, we see that $U_{\gcd(m,n)} \circ V_{\gcd(m,n)} = I$, and we are left with $U_n \circ V_m = U_{m/\gcd(m,n)} \circ V_{\gcd(m,n)}$. Thus, part (d) of the theorem is proven. $\qquad\square$

4. The action of U_n on formal power series

We now determine an expression for the action of the operator U_n on formal power series where we allow the first few coefficients to vanish. This result will be employed in our study of spectral properties of U_n acting on hypergeometric functions. For the rest of this section, all functions are assumed to be formal power series.

THEOREM 4.1. *Let* $j, n \in \mathbb{N}$. *Then*

$$U_n\left(x^j \sum_{k=0}^{\infty} a_k x^k\right) = \begin{cases} x^{1+\lfloor j/n \rfloor} \sum_{k=0}^{\infty} a_{n(k+1-\{j/n\})} x^k & \text{if } n \text{ does not divide } j, \\ x^{j/n} \sum_{k=0}^{\infty} a_{kn} x^k & \text{if } n \text{ divides } j. \end{cases}$$

PROOF. First observe that

$$U_n\left(x^j \sum_{k=0}^{\infty} a_k x^k\right) = U_n\left(\sum_{k=0}^{\infty} a_k x^{k+j}\right) = U_n\left(\sum_{k=j}^{\infty} a_{k-j} x^k\right),$$

and define

$$b_k = \begin{cases} 0 & \text{if } 0 \le k < j, \\ a_{k-j} & \text{if } k \ge j; \end{cases} \tag{4.1}$$

then we can write

$$U_n\left(x^j \sum_{k=0}^{\infty} a_k x^k\right) = U_n\left(\sum_{k=0}^{\infty} b_k x^k\right) = \sum_{k=0}^{\infty} b_{kn} x^k.$$

The discussion is divided into two cases, according to whether n divides j or not.

Case 1: n does not divide j. In this case, the restriction $kn \ge j$ in (4.1) is equivalent to $k \ge \lfloor j/n \rfloor + 1$. Thus,

$$\sum_{k=0}^{\infty} b_{kn} x^k = \sum_{k=0}^{\lfloor j/n \rfloor} b_{kn} x^k + \sum_{k=\lfloor j/n \rfloor + 1}^{\infty} b_{kn} x^k$$

$$= \sum_{k=\lfloor j/n \rfloor + 1}^{\infty} a_{kn-j} x^k.$$

Now let $i = k - \lfloor j/n \rfloor - 1$, to obtain

$$\sum_{k=0}^{\infty} b_{kn} x^k = x^{\lfloor j/n \rfloor + 1} \sum_{i=0}^{\infty} a_{ni + n \lfloor j/n \rfloor + n - j} x^i.$$

Now use the formula $j/n = \lfloor j/n \rfloor + \{j/n\}$ to obtain

$$U_n \left(x^j \sum_{k=0}^{\infty} a_k x^k \right) = x^{\lfloor j/n \rfloor + 1} \sum_{i=0}^{\infty} a_{n(i+1-\{j/n\})} x^i.$$

This is the result when n does not divide j.

Case 2: n divides j. In this case, the condition $kn \geq j$ is equivalent to $k \geq j/n = \lfloor j/n \rfloor$, and

$$\sum_{k=0}^{\infty} b_{kn} x^k = \sum_{k=\lfloor j/n \rfloor}^{\infty} b_{kn} x^k$$

$$= \sum_{k=\lfloor j/n \rfloor}^{\infty} a_{kn-j} x^k$$

$$= x^{\lfloor j/n \rfloor} \sum_{i=0}^{\infty} a_{ni} x^i,$$

so that

$$U_n \left(x^j \sum_{k=0}^{\infty} a_k x^k \right) = x^{\lfloor j/n \rfloor} \sum_{\nu \geq 0} a_{n\nu} x^{\nu}.$$

This concludes the proof. $\square$

The expressions for U_n are now used to derive some elementary properties of its eigenfunctions on the space of formal power series.

PROPOSITION 4.2. *Assume that U_n has an eigenfunction of the form*

$$f(x) = x^j \sum_{k=0}^{\infty} a_k x^k,$$

with eigenvalue λ. If n divides j, then $j = 0$ and $\lambda = 1$. If n does not divide j, then $j = 1$.

PROOF. Assume that n divides j and match the leading order terms of f and $U_n f$. Theorem 4.1 shows that $x^j = x^{j/n}$, which yields $j = 0$. Now compare the constant terms in the eigenvalue equations to see that $\lambda = 1$. In the case where n does not divide j, the same comparison yields

$$j = 1 + \lfloor j/n \rfloor.$$

This implies that $j = 1$. Indeed, let $j = \alpha n + \beta$ where $0 < \beta < n$. Then $j = 1 + \alpha$, and this yields

$$1 - \beta = \alpha(n - 1). \tag{4.2}$$

It follows that $\beta = 1$ and $\alpha = 0$, otherwise the two sides of (4.2) have different signs. We conclude that $j = 1 + \alpha = 1$. $\square$

Hence even for a formal power series f, we see that the assumption that f is an eigenfunction of the Hecke operator U_n imposes the restriction that f can only vanish to order zero or one.

For the sake of completeness, we describe the trivial eigenfunctions of the composition of operators $U_n \circ V_n$ and $V_n \circ U_n$. Theorem 3.1 shows that $U_n \circ V_n$ is the identity. The next result describes the composition $V_n \circ U_n$.

THEOREM 4.3. *The only eigenvalue of $V_n \circ U_n$ is 1. Moreover, given any formal power series $f(x) = \sum_{k=0}^{\infty} b_k x^k$, the function $g(x) = \sum_{k=0}^{\infty} a_k x^k$, where*

$$a_k = \begin{cases} b_k & \text{if } n \text{ divides } k, \\ 0 & \text{if } n \text{ does not divide } k, \end{cases}$$

is an eigenfunction of $V_n \circ U_n$, with eigenvalue 1.

PROOF. We consider the identity

$$(V_n \circ U_n)\left(\sum_{k=0}^{\infty} a_k x^k\right) = \sum_{k=0}^{\infty} a_{kn} x^k,$$

and we are done. $\square$

5. The hypergeometric functions

In this section we use Theorem 4.1 to describe the action of U_n on the set $\mathfrak{H}$ of all hypergeometric functions. We recall that a hypergeometric function is defined by the power series

$$_pF_q(\mathbf{a}, \mathbf{b}; x) := \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \cdots (a_p)_k}{(b_1)_k (b_2)_k \cdots (b_q)_k} \frac{x^k}{k!},$$

where $\mathbf{a} := (a_1, a_2, \ldots, a_p)$ and $\mathbf{b} := (b_1, b_2, \ldots, b_q)$ are the parameters of $_pF_q$. These parameters are nonzero complex numbers. We begin by stating explicitly the action of U_n on $\mathfrak{F}^j_{(p,q)}$ as the main result of this section.

THEOREM 5.1. *Let $j, n \in \mathbb{N}$. The action of U_n on the class $\mathfrak{F}^j_{(p,q)}$, that is, on functions of the form*

$$f_{p,q,j} = x^j \, _pF_q(\mathbf{a}, \mathbf{b}; x),$$

is characterized as follows.
    If n divides j, then

$$U_n\left(x^j \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_{q+1})_k} x^k\right) = x^{j/n} \sum_{k=0}^{\infty} \frac{(c_1)_k (c_2)_k \cdots (c_{np})_k}{(d_1)_k (d_2)_k \cdots (d_{n(q+1)-1})_k} \frac{x_1^k}{k!},$$

where the parameters **c** *and* **d** *are defined by*

$$c_{in+l} = \frac{a_{i+1} + l - 1}{n} \quad if \ 0 \le i \le p - 1, \ 1 \le l \le n,$$

$$d_{in+l} = \frac{b_{i+1} + l - 1}{n} \quad if \ 0 \le i \le q, \ 1 \le l \le n,$$

(5.1)

and $j_1 = j/n$. *The new variable is* $x_1 = n^{n(p-q-1)}x$.

If n *does not divide* j, *then*

$$U_n\left(x^j \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_{q+1})_k} x^k\right) = x^{1+\lfloor j/n \rfloor} \sum_{k=0}^{\infty} \frac{(c_1)_k \cdots (c_{pn})_k}{(d_1)_k \cdots (d_{(q+1)n-1})_k} \frac{x_1^k}{k!},$$

where the parameters **c** *and* **d** *are now defined by*

$$c_{in+l} = \frac{a_{i+1} + r + l}{n} \quad if \ 0 \le i \le p - 1, \ 1 \le l \le n,$$

$$d_{in+l} = \frac{b_{i+1} + r + l}{n} \quad if \ 0 \le i \le q, \ 1 \le l \le n,$$

(5.2)

and $j_2 = 1 + \lfloor j/n \rfloor$ *and* $r = n(1 - \{j/n\}) - 1$. *The new variable* x_1 *is defined as above.*

Before proving this theorem, we first need to state some intermediate results. The next lemma allows for a simplification of the ascending factorial function on an arithmetic progression of indices.

LEMMA 5.2. *Let* $k, n \in \mathbb{N}$ *and* $a \in \mathbb{R}$. *Then*

$$(a)_{kn} = n^{kn} \prod_{j=0}^{n-1} \left(\frac{a+j}{n}\right)_k.$$

PROOF. Start with the observation that

$$(a)_{kn} = \prod_{i=0}^{kn-1} (a+i) = n^{kn} \prod_{i=0}^{kn-1} \left(\frac{a}{n} + \frac{i}{n}\right),$$

and then collect terms according to their classes modulo n. □

In order to evaluate

$$U_n(x^j {}_pF_q(\mathbf{a}, \mathbf{b})) = U_n\left(x^j \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \cdots (a_p)_k}{(b_1)_k (b_2)_k \cdots (b_q)_k (b_{q+1})_k} x^k\right),$$

where we used the fact that $k! = (1)_k$ and defined $b_{q+1} = 1$, we observe that, by Theorem 4.1, the discussion should be divided into two cases according to whether or not n divides j.

Case 1: n divides j. In this case, Theorem 4.1 yields

$$U_n(x^j {}_pF_q(\mathbf{a}, \mathbf{b})) = x^{j/n} \sum_{k=0}^{\infty} \frac{(a_1)_{kn} \cdots (a_p)_{kn}}{(b_1)_{kn} \cdots (b_{q+1})_{kn}} x^k,$$

and using Lemma 5.2 we can write this as

$$U_n(x^j {}_pF_q(\mathbf{a}, \mathbf{b})) = x^{j/n} \sum_{k=0}^{\infty} n^{(p-q-1)kn} \left(\prod_{j=1}^{p} \prod_{i=0}^{n-1} \left(\frac{a_j+i}{n} \right)_k \right)$$

$$\times \left(\prod_{j=1}^{q+1} \prod_{i=0}^{n-1} \left(\frac{b_j+i}{n} \right)_k \right)^{-1} x^k.$$

Now recall that $b_{q+1} = 1$, so the d-parameters when $i = q$ from the definition (5.2) are $1/n, 2/n, \ldots, (n-1)/n, 1$. The total number of d-parameters is now reduced by 1, in order to write the result in the canonical hypergeometric form:

$$U_n(x^j {}_pF_q(\mathbf{a}, \mathbf{b})) = x^{j/n} \sum_{k=0}^{\infty} n^{(p-q-1)kn} \frac{(c_1)_k (c_2)_k \cdots (c_{np})_k}{(d_1)_k (d_2)_k \cdots (d_{n(q+1)-1})_k} \frac{x^k}{k!}.$$

LEMMA 5.3. *The parameters $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ satisfy*

$$\sum_{i=1}^{np} c_i - \sum_{i=1}^{n(q+1)-1} d_i = \sum_{i=1}^{p} a_i - \sum_{i=1}^{q} b_i + \frac{(n-1)}{2}(p-q-1).$$

PROOF. The new parameters are

$$\frac{a_1}{n}, \frac{a_1+1}{n}, \ldots, \frac{a_1+n-1}{n}, \frac{a_p}{n}, \ldots, \frac{a_p+n-1}{n}$$

and

$$\frac{b_1}{n}, \frac{b_1+1}{n}, \ldots, \frac{b_1+n-1}{n}, \frac{b_{q+1}}{n} = \frac{1}{n}, \ldots, \frac{b_{q+1}+n-2}{n} = \frac{n-1}{n},$$

and the identity is now easy to check. □

COROLLARY 5.4. *If $p = q + 1$, then $\sum_{i=1}^{p} a_i - \sum_{i=1}^{q} b_i$ is preserved under the action of U_n.*

Case 2: n does not divide j. In this case, Theorem 4.1 gives

$$U_n\left(x^j \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_{q+1})_k} x^k \right) = x^{1+\lfloor j/n \rfloor} \sum_{k=0}^{\infty} \frac{(a_1)_N \cdots (a_p)_N}{(b_1)_N \cdots (b_{q+1})_N} x^k,$$

where $b_{q+1} = 1$ and we define $N = n(k+1-\{j/n\})$. Observe that $0 < \{j/n\} < 1$, and thus $nk < N < n(k+1)$. The next result simplifies the Pochhammer symbols.

LEMMA 5.5. *Let $a \in \mathbb{C}$ and $j, n \in \mathbb{N}$ where j is not divisible by n. Define $N = n(k + 1 - \{j/n\})$ and $r = n(1 - \{j/n\}) - 1$. Then*

$$(a)_N = n^{nk}(a)_{r+1} \prod_{i=r+1}^{r+n} \left(\frac{a+i}{n} \right)_k.$$

PROOF. We begin by recalling that

$$(a)_N = a(a+1)(a+2) \cdots (a+N-1)$$
$$= n^N \left[\frac{a}{n} \left(\frac{a}{n} + \frac{1}{n} \right) \cdots \left(\frac{a}{n} + \frac{N-1}{n} \right) \right].$$

We now regroup terms modulo n as follows:

$$n^{-N}(a)_N = \left(\frac{a}{n} \right) \cdot \left(\frac{a}{n} + 1 \right) \cdots \left(\frac{a}{n} + k - 1 \right)$$
$$\times \left(\frac{a}{n} + \frac{1}{n} \right) \cdot \left(\frac{a}{n} + \frac{1}{n} + 1 \right) \cdots \left(\frac{a}{n} + \frac{1}{n} + k - 1 \right)$$
$$\cdots$$
$$\times \left(\frac{a}{n} + \frac{n-1}{n} \right) \cdot \left(\frac{a}{n} + \frac{n-1}{n} + 1 \right) \cdots \left(\frac{a}{n} + \frac{n-1}{n} + k - 1 \right)$$
$$\times \left\{ \left(\frac{a}{n} + k \right) \cdot \left(\frac{a}{n} + \frac{1}{n} + k \right) \cdots \left(\frac{a}{n} + \frac{r}{n} + k \right) \right\}.$$

The last factor appears because n does not divide j. Therefore

$$(a)_N = n^N \prod_{i=0}^{n-1} \left(\frac{a+i}{n} \right)_k \times \prod_{i=0}^{r} \left(\frac{a+i}{n} + k \right),$$

where the second product does *not* involve the Pochhammer symbol. Now employ the relation

$$k + c = c \frac{(c+1)_k}{(c)_k},$$

to write

$$(a)_N = n^N \prod_{i=0}^{n-1} \left(\frac{a+i}{n} \right)_k \prod_{i=0}^{r} \left(\frac{a+i}{n} \right) \cdot \prod_{i=0}^{r} \left(\frac{a+i}{n} + 1 \right)_k \Big/ \prod_{i=0}^{r} \left(\frac{a+i}{n} \right)_k.$$

This expression reduces to the stated formula. □

The transformation above yields

$$U_n \left(x^j \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_{q+1})_k} x^k \right) = x^{1+\lfloor j/n \rfloor} \sum_{k=0}^{\infty} n^{nk(p-q-1)} \frac{(c_1)_k \cdots (c_{pn})_k}{(d_1)_k \cdots (d_{(q+1)n})_k} x^k.$$

The special case where $p = q + 1$ provides a simpler situation, in which the coefficient $n^{nk(p-q-1)}$ does not appear in the resulting series.

THEOREM 5.6. *Let j, $n \in \mathbb{N}$ and assume that $p = q + 1$.*
If n divides j, then

$$U_n(x^j {}_pF_q(\mathbf{a}, \mathbf{b}; x)) = x^{j/n} {}_{np}F_{np-1}(\mathbf{c}, \mathbf{d}; x),$$

where $\mathbf{c}$ and $\mathbf{d}$ are defined in (5.1).
If n does not divide j, then

$$U_n(x^j {}_pF_q(\mathbf{a}, \mathbf{b}; x)) = x^{1+\lfloor j/n \rfloor} {}_{np}F_{np-1}(\mathbf{c}, \mathbf{d}; x),$$

where $\mathbf{c}$ and $\mathbf{d}$ are defined in (5.2).

6. The eigenvalue equation

In this section we focus on the spectral properties of the operator U_n, as the spectral properties of the operators V_n are trivial. It is here that we encounter more subtle ideas. We describe first the eigenfunctions of the operator U_n of the form $x^j {}_pF_q(\mathbf{a}, \mathbf{b}; x)$. That is, we look for parameters $p, q \in \mathbb{N}$ and complex numbers

$$a_1, a_2, \ldots, a_p; b_1, b_2, \ldots, b_q$$

such that, with $\mathbf{a} = (a_1, \ldots, a_p)$ and $\mathbf{b} = (b_1, \ldots, b_q)$,

$$U_n(x^j {}_pF_q(\mathbf{a}, \mathbf{b}; x)) = \lambda x^j {}_pF_q(\mathbf{a}, \mathbf{b}; x). \tag{6.1}$$

The results from Theorem 5.1 showed that the action of U_n on $x_p^j F_q$ depends on whether or not n divides j, which by Proposition 4.2 reduces to the cases in which $j = 0$ and $j = 1$ when considering eigenfunctions of U_n.

Case 1: $j = 0$. Under this condition, we show that the eigenfunction reduces to a rational function.

LEMMA 6.1. *Assume that n divides j and that (6.1) has a nontrivial solution. Then, for all $k \in \mathbb{N}$,*

$$\prod_{j=1}^{p} \prod_{i=0}^{n-1} (a_j + nk + i) \times \prod_{j=1}^{q+1} (b_j + k)$$

$$= \prod_{j=1}^{p} (a_j + j) \times \prod_{j=1}^{q+1} \prod_{i=0}^{n-1} (b_j + nk + i).$$

PROOF. Comparing terms of the equation $U_n f = f$ yields

$$\frac{(a_1)_{nk}(a_2)_{nk} \cdots (a_p)_{nk}}{(b_1)_{nk}(b_2)_{nk} \cdots (b_{q+1})_{nk}} = \frac{(a_1)_k(a_2)_k \cdots (a_p)_k}{(b_1)_k(b_2)_k \cdots (b_{q+1})_k}. \tag{6.2}$$

Replace k by $k + 1$, divide the two equations and use the formulae

$$\frac{(a)_{k+1}}{(a)_k} = a + k \quad \text{and} \quad \frac{(a)_{n(k+1)}}{(a)_k} = (a + nk)(a + nk + 1) \cdots (a + nk + n - 1)$$

to produce the result. □

LEMMA 6.2. *Assume that n divides j and that (6.1) has a nontrivial solution. Then*
$p = q + 1$.

PROOF. Comparing the degrees of the left- and right-hand side of (6.2) gives the
equality $pn + q + 1 = p + n(q + 1)$. □

PROPOSITION 6.3. *Assume that n divides j and that (6.1) has a nontrivial solution.*
Then

$$
\left\{ a_i, \frac{b_i}{n}, \frac{b_i + 1}{n}, \ldots, \frac{b_i + n - 1}{n} \right\}_{i=1}^{p} = \left\{ b_i, \frac{a_i}{n}, \frac{a_i + 1}{n}, \ldots, \frac{a_i + n - 1}{n} \right\}_{i=1}^{p}.
$$

PROOF. The roots of the left- and right-hand side of (6.2) must match. □

We now show that the results of this proposition imply that the parameters must
match: $a_i = b_i$ for all indices.

PROPOSITION 6.4. *Assume that n divides j and that (6.1) has a nontrivial solution.*
Then, for any $k \in \mathbb{N}$,

$$
\sum_{i=1}^{p} a_i^k = \sum_{i=1}^{p} b_i^k.
$$

PROOF. Proof is by induction on k. The case where $k = 1$ comes from matching the
coefficients of the next to leading order in k. Indeed, this matching yields

$$
\sum_{i=1}^{p} a_i + \sum_{j=1}^{n-1} \sum_{i=1}^{p} (b_i + j) = \sum_{i=1}^{p} b_i + \sum_{j=1}^{n-1} \sum_{i=1}^{p} (a_i + j),
$$

and the case where $k = 1$ holds. In order to check it for $k = 2$, add the squares of the
elements in Proposition 6.3 to obtain, from the left-hand side, the expression

$$
\sum_{i=1}^{p} a_i^2 + \frac{1}{n^2} \sum_{i=1}^{p} \sum_{j=0}^{n-1} (b_i^2 + 2jb_i + j^2)
$$

$$
= \sum_{i=1}^{p} a_i^2 + \frac{1}{n} \sum_{i=1}^{p} b_i^2 + \frac{2}{n^2} \sum_{i=1}^{p} b_i \times \sum_{j=0}^{n-1} j + \frac{p}{n^2} \sum_{j=0}^{n-1} j^2.
$$

Matching with the corresponding expression from the right-hand side and using the
statement for $k = 1$ yields

$$
\sum_{i=1}^{p} a_i^2 = \sum_{i=1}^{p} b_i^2.
$$

The higher moments can be established along these lines. □

PROPOSITION 6.5. *Assume that two sets $\{a_j : 1 \le j \le p\}$ and $\{b_j : 1 \le j \le p\}$ of complex numbers satisfy*

$$\sum_{i=1}^{p} a_i^k = \sum_{i=1}^{p} b_i^k,$$

for every $k \in \mathbb{N}$. Then, after a possible rearrangement of the order of the terms of one of these sets, $a_i = b_i$ for all i.

PROOF. For $\mathbf{a}, \mathbf{b} \in \mathbb{C}^p$ and $N \in \mathbb{N}$, let

$$\mu_N(\mathbf{a}) = \sum_{i=1}^{p} a_i^N$$

and let

$$f_{\mathbf{a}}(t) = \sum_{j=0}^{\infty} \mu_j(\mathbf{a}) \frac{t^j}{j!} \quad \text{and} \quad f_{\mathbf{b}}(t) = \sum_{j=0}^{\infty} \mu_j(\mathbf{b}) \frac{t^j}{j!}$$

be the generating functions of $\mu_N(\mathbf{a})$ and $\mu_N(\mathbf{b})$, respectively. Now assume that $f_{\mathbf{a}} = f_{\mathbf{b}}$, that is,

$$\sum_{j=0}^{\infty} \mu_j(\mathbf{a}) \frac{t^j}{j!} = \sum_{j=0}^{\infty} \mu_j(\mathbf{b}) \frac{t^j}{j!}.$$

Expanding further gives

$$\sum_{j=0}^{\infty} \sum_{i=1}^{p} a_i^j \frac{t^j}{j!} = \sum_{j=0}^{\infty} \sum_{i=1}^{p} b_i^j \frac{t^j}{j!}.$$

Since μ_j is defined as a sum of finite terms, we can change the order of summation:

$$\sum_{i=1}^{p} \sum_{j=0}^{\infty} a_i^j \frac{t^j}{j!} = \sum_{i=1}^{p} \sum_{j=0}^{\infty} b_i^j \frac{t^j}{j!}.$$

This yields

$$e^{a_1 t} + e^{a_2 t} + \cdots + e^{a_p t} = e^{b_1 t} + e^{b_2 t} + \cdots + e^{b_p t}. \tag{6.3}$$

Suppose first that $a_i, b_i \in \mathbb{R}$ and order them as

$$a_1 \le a_2 \le \cdots \le a_p \quad \text{and} \quad b_1 \le b_2 \le \cdots \le b_p.$$

Eliminate from (6.3) all the terms for which the as and bs match, to assume that $a_1 < b_1$. Then

$$1 + e^{(a_2 - a_1)t} + \cdots + e^{(a_p - a_1)t} = e^{(b_1 - a_1)t} + e^{(b_2 - a_1)t} + \cdots + e^{(b_p - a_1)t}.$$

Finally, let $t \to -\infty$ to get a contradiction.   □

We summarize the previous discussion in the following theorem.

THEOREM 6.6. *Suppose there exists an eigenfunction of U_n of the form*

$$f(x) = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \cdots (a_p)_k}{(b_1)_k (b_2)_k \cdots (b_{q+1})_k} x^k,$$

corresponding to the eigenvalue λ (this is the case where $j = 0$ in (6.1)). Then $\lambda = 1$, $p = q + 1$ and $a_i = b_i$ for all i. Therefore $f(x) = 1/(1 - x)$.

Proposition 4.2 shows that the only possible values for j are 0 and 1. The rest of the section treats the case where $j = 1$.

Case 2: $j = 1$. Under this condition we show that the spectrum of the operator U_n is the set $\{n^i : i \in \mathbb{Z}\}$. The corresponding eigenfunctions are the polylogarithm functions

$$\text{PolyLog}_i(x) := \sum_{k=1}^{\infty} k^i x^k,$$

corresponding to the eigenvalue n^i when i is negative, and the eigenfunctions

$$\left(x \frac{d}{dx} \right)^i \left(\frac{1}{1 - x} \right),$$

corresponding to the eigenvalue n^i when i is nonnegative.

EXAMPLE 6.7. The dilogarithm function

$$\text{Li}_2(x) := \sum_{k=1}^{\infty} \frac{x^k}{k^2}$$

satisfies

$$U_n(\text{Li}_2(x)) = \frac{1}{n^2} \text{Li}_2(x).$$

Therefore $1/n^2 \in \text{Spec}(U_n)$. The dilogarithm function admits the hypergeometric representation

$$\text{Li}_2(x) = x \, {}_3F_2(\mathbf{1}_3, \mathbf{2}_2; x),$$

where $\mathbf{1}_3$ stands for $(1, 1, 1)$ and $\mathbf{2}_2$ stands for $(2, 2)$.

We now explore properties of eigenfunctions of the operator U_n.

PROPOSITION 6.8. *Assume that (6.1) has a nontrivial solution for which $j = 1$. Then, for all $k \in \mathbb{N}$,*

$$\prod_{j=1}^{p} (a_j + k - 1) \cdot \prod_{j=1}^{q+1} \prod_{i=-1}^{n-2} (b_j + nk + i) = \prod_{j=1}^{q+1} (b_j + k - 1) \cdot \prod_{j=1}^{p} \prod_{i=-1}^{n-2} (a_j + nk + i).$$

PROOF. Assume that the eigenfunction has the form

$$f(x) = x \sum_{k=0}^{\infty} c_k x^k.$$

Comparing coefficients in the equation $U_n f = \lambda f$ gives

$$c_{nk-1} = \lambda c_{k-1}.$$

Replacing the standard hypergeometric type yields

$$\frac{(a_1)_{nk-1} \cdots (a_p)_{nk-1}}{(b_1)_{nk-1} \cdots (b_{q+1})_{nk-1}} = \lambda \frac{(a_1)_{k-1} \cdots (a_p)_{k-1}}{(b_1)_{k-1} \cdots (b_{q+1})_{k-1}}. \tag{6.4}$$

Replace k by $k + 1$ and divide the two corresponding equations to obtain the result. $\square$

LEMMA 6.9. *Assume that $j = 1$ and that (6.1) has a nontrivial solution of the form*

$$f(x) = x \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_{q+1})_k} x^k.$$

Then $p = q + 1$.

PROOF. Compare the degrees on both sides of the polynomial in Proposition 6.8. $\square$

PROPOSITION 6.10. *Assume that*

$$f(x) = x \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_p)_k} x^k$$

is an eigenfunction for U_n. Then

$$\bigcup_{i=1}^{p} \left\{ a_i - 1, \frac{b_i - 1}{n}, \frac{b_i}{n}, \ldots, \frac{b_i + n - 2}{n} \right\}$$

$$= \bigcup_{i=1}^{p} \left\{ b_i - 1, \frac{a_i - 1}{n}, \frac{a_i}{n}, \ldots, \frac{a_i + n - 2}{n} \right\}.$$

PROOF. These are the roots of both sides of the polynomial in Proposition 6.8. $\square$

We now show that this equality of sets imposes severe restrictions on the eigenvalues and eigenfunctions of the operator U_n. We discuss first the eigenvalues.

PROPOSITION 6.11. *Assume that*

$$f(x) = x \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_p)_k} x^k$$

is an eigenfunction for U_n. Let

$$\gamma_a := |\{i \in \{1, 2, \ldots, p\} : a_i = 1\}|,$$

and

$$\gamma_b := |\{i \in \{1, 2, \ldots, p\} : b_i = 1\}|.$$

Then

$$n^{\gamma_a}(a_1)_{n-1}(a_2)_{n-1} \cdots (a_p)_{n-1} = n^{\gamma_b}(b_1)_{n-1}(b_2)_{n-1} \cdots (b_p)_{n-1}. \quad (6.5)$$

PROOF. Consider the product of all the nonzero terms on the left-hand side of Proposition 6.10. The removal of the zero terms, corresponding to those a_i equal to 1, carries with it the removal of a power of n. □

EXAMPLE 6.12. In Example 6.7 we see that $\gamma_a = 3$ and $\gamma_b = 1$. Thus (6.5) states correctly that

$$n^3 \times (1)_{n-1}(1)_{n-1}(1)_{n-1} = n \times (2)_{n-1}(2)_{n-1}(1)_{n-1}.$$

THEOREM 6.13. *Assume that $j = 1$ and*

$$f(x) = x \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_p)_k}$$

is an eigenfunction for U_n with eigenvalue λ. Then

$$\lambda = n^{\gamma_b - \gamma_a},$$

where γ_a and γ_b are defined in (6.5).

PROOF. Put $k = 1$ in the relation (6.4) to obtain

$$\lambda = \frac{(a_1)_{n-1} \cdots (a_p)_{n-1}}{(b_1)_{n-1} \cdots (b_{q+1})_{n-1}}.$$

Now use (6.5) to conclude. □

The result above shows that the spectrum of U_n satisfies

$$\mathrm{Spec}(U_n) \subseteq \{n^a : a \in \mathbb{Z}\}.$$

The example below shows that actually equality holds.

EXAMPLE 6.14. Let $i \in \mathbb{Z}$. Then the hypergeometric series

$$f_i(x) := \sum_{k=1}^{\infty} k^i x^k,$$

is an eigenfunction of U_n, with eigenvalue n^i. The dilogarithm corresponds to the case where $i = -2$.

THEOREM 6.15. *The spectrum of U_n is the set $\{n^i : i \in \mathbb{Z}\}$.*

We now discuss the eigenfunctions of U_n, starting with the sets

$$\bigcup_{i=1}^{p} \left\{ a_i - 1, \frac{b_i - 1}{n}, \frac{b_i}{n}, \dots, \frac{b_i + n - 2}{n} \right\}$$

$$= \bigcup_{i=1}^{p} \left\{ b_i - 1, \frac{a_i - 1}{n}, \frac{a_i}{n}, \dots, \frac{a_i + n - 2}{n} \right\},$$

that appeared in Proposition 6.10. Recall that $\{a_i, b_i\}$ are the parameters of the eigenfunction

$$f(x) = x \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_{q+1})_k} x^k.$$

We may assume that $a_i \neq b_j$, otherwise we just cancel the equal terms $(a_i)_k$ and $(b_j)_k$. Observe that if we let $c_i = a_i - 1$ and $d_i = b_i - 1$, the basic set identity in Case 2 of Proposition 6.10 becomes the basic set identity of Case 1, with c_i instead of a_i and d_i instead of b_i. The reason why one cannot deduce $a_i = b_i$ is that some of the a_i in Case 2 might be 1, and the corresponding c_i would vanish. This would violate the basic assumption of Proposition 6.5.

   We bypass this difficulty by defining

$$a_i' = \begin{cases} a_i & \text{if } a_i \neq 1, \\ 2 & \text{if } a_i = 1, \end{cases}$$

and

$$b_i' = \begin{cases} b_i & \text{if } b_i \neq 1, \\ 2 & \text{if } b_i = 1. \end{cases}$$

The sets described above, obtained by replacing a_i and b_i by a_i' and b_i', remain equal. In order to see this, observe that if $a_1 = 1$, then the elements containing a_1 are

$$\left\{ \frac{a_1 - 1}{n}, \frac{a_1}{n}, \ldots, \frac{a_1 + n - 2}{n} \right\};$$

these are replaced by

$$\left\{ \frac{a_1}{n}, \frac{a_1 + 1}{n}, \ldots, \frac{a_1 + n - 1}{n} \right\},$$

which amounts to simply replacing a_1 by $a_1 + 1$. Theorem 6.6 shows that $a_i' = b_i'$. Whether a_i and b_i are both equal to 1 *or* both different from 1, we still conclude that $a_i = b_i$. This contradicts our original assumption. In the case where $a_i' = b_i'$ and $a_i = 1$, we conclude that $b_i = 2$. Similarly, if $a_i' = b_i'$ and $b_i = 1$, then we obtain $a_i = 2$. Therefore, any instance where $a_i = 1$ produces the valid pair $\{1, 2\}$, and any instance where $b_i = 1$ leads to $\{2, 1\}$. Using the fact that there are no common parameters in the numerator and denominator, we conclude that an eigenfunction must have the following structure.

THEOREM 6.16. *Assume that*

$$f(x) = x \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_{q+1})_k} x^k$$

is an eigenfunction of U_n, with the usual normalization $b_{q+1} = 1$. Then, either $a_i = 2$ and $b_i = 1$ whenever $1 \leq i \leq p$, or $a_i = 1$ and $b_i = 2$ whenever $1 \leq i \leq p$.

In other words, if f is an eigenfunction of any particular Hecke operator U_n, then

$$f(x) = {}_aF_{a-1}(\mathbf{1}_a, \mathbf{2}_{a-1}; x) = \sum_{k=0}^{\infty} \frac{1}{k^a} x^k,$$

or

$$f(x) = {}_aF_{a-1}(\mathbf{2}_a, \mathbf{1}_{a-1}; x) = \sum_{k=0}^{\infty} k^a x^k,$$

where $a \in \mathbb{N}$ and $\mathbf{1}_a$ stands for the a-vector $(1, 1, \ldots, 1)$ and 2_b is defined analogously.

7. The simultaneous eigenfunctions of U_n for all n

In this section we completely characterize those hypergeometric functions that are simultaneous eigenfunctions of U_n for all n, and give an application to the theory of completely multiplicative functions.

What do the hypergeometric functions $f(x) = \sum_{k=1}^{\infty} c_k x^k$ that are simultaneous eigenfunctions of all of the linear operators U_n look like? It turns out that there is a simple answer: they are precisely the polylogarithms and the rational functions $(x\,d/dx)^a(1/(1-x))$, as given by the following theorem.

THEOREM 7.1. *Let*

$$f(x) = \sum_{k=1}^{\infty} c_k x^k$$

be a hypergeometric function with no constant term. Then f is a simultaneous eigenfunction for the set of all Hecke operators $\{U_n\}_{n=1}^{\infty}$ if and only if

$$f(x) = C \sum_{k=1}^{\infty} k^a x^k,$$

where $a \in \mathbb{Z}$ and $C \in \mathbb{C}$. In other words, either f is a polylogarithm, or $f(x) = (x\,d/dx)^a(1/(1-x))$.

Before proving this theorem, we prove an interesting corollary regarding a number-theoretic fact concerning hypergeometric coefficients that are completely multiplicative functions of the summation index.

COROLLARY 7.2. *The hypergeometric coefficient*

$$c(n) = \frac{(a_1)_{n-1} \cdots (a_p)_{n-1}}{(b_1)_{n-1} \cdots (b_q)_{n-1}},$$

is a completely multiplicative function of n if and only if it is of the form Cn^a for some $a \in \mathbb{Z}$ and $C \in \mathbb{C}$.

PROOF. By the definition of completely multiplicative functions, we know that

$$c(nk) = c(n)c(k)$$

for all $k, n \in \mathbb{N}$. This implies that

$$f(x) = \sum_{k=1}^{\infty} c(k) x^k$$

is an eigenfunction of U_n, with eigenvalue $c(n)$. This holds for all n, thus it is a simultaneous eigenfunction. The result now follows from Theorem 7.1. □

We recall that by definition f is a simultaneous eigenfunction of all of the Hecke operators if and only if, for all $n \in \mathbb{N}$,

$$U_n f = \lambda_n f.$$

Treating f as a power series without even considering its hypergeometric properties, we see that

$$\sum_{k=1}^{\infty} c_{nk} x^k = \lambda_n \sum_{k=1}^{\infty} c_k x^k,$$

so that

$$c_{nk} = \lambda_n c_k.$$

This is true for all $k \geq 1$, so we can set $k = 1$, which shows that

$$c_n = \lambda_n c_1.$$

This, in turn, is true for all n, and proves the following result.

LEMMA 7.3. *Let*

$$f(x) = \sum_{k=1}^{\infty} c_k x^k$$

be a power series with no constant term that is a simultaneous eigenfunction for the set of operators $\{U_n\}_{n=1}^{\infty}$ with respective eigenvalues $\{\lambda_n\}_{n=1}^{\infty}$. If $i \geq 2$, then

$$c_i = \lambda_i c_1,$$

and thus

$$f(x) = c_1 \sum_{k=1}^{\infty} \lambda_k x^k. \tag{7.1}$$

PROOF OF THEOREM 7.1. Let us suppose now that f, as defined in (7.1), is a hypergeometric function. By Theorem 6.13,

$$\lambda_n = n^{\gamma_b - \gamma_a},$$

where γ_b and γ_a are defined in (6.5). This proves Theorem 7.1. □

In our formal hypergeometric notation, we see that any simultaneous eigenfunction must be the polylogarithm

$$\sum_{k=0}^{\infty} \frac{1}{k^a} x^k$$

or the rational function

$$\sum_{k=0}^{\infty} k^a x^k,$$

where a is a nonnegative integer. In conclusion, we see that if a hypergeometric function is an eigenfunction for a single operator U_j, then it is automatically a simultaneous eigenfunction for all of the Hecke operators U_n, as n varies over all positive integers. This situation stands in sharp contrast to the space of rational functions studied recently in [4–6].

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