Sixth Meeting, April 7th, 1897.

Dr SPRAGUE, President, in the Chair.

Certain Expansions of x^n in Hypergeometric Series.

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In this paper the following expansion will be obtained:

$$\begin{aligned}(-1)^{n+1}x^n = {} & \frac{(n)_1}{1!}\left[\frac{(x)_r}{0!r!}+\frac{(n-r)_1(x)_{r+1}}{1!r+1!}+\frac{(n-r)_2(x)_{r+2}}{2!r+2!}+\ \ldots\right] \\ & -\frac{(n)_2}{2!}\left[\frac{(2x)_r}{0!r!}+\frac{(n-r)_1(2x)_{r+1}}{1!r+1!}+\frac{(n-r)_2(2x)_{r+2}}{2!r+2!}+\ \ldots\right] \quad (1) \\ & +\frac{(n)_3}{3!}\left[\frac{(3x)_r}{0!r!}+\frac{(n-r)_1(3x)_{r+1}}{1!r+1!}+\frac{(n-r)_2(3x)_{r+2}}{2!r+2!}+\ \ldots\right] \\ & \qquad \vdots \qquad\qquad\qquad \vdots\end{aligned}$$

in which n and r are positive integers. The Series in the square brackets are Hypergeometric Series with a finite number of terms.

Let $\prod\limits_{r=1}^{r=n}(b+a_r)$ denote the product of the n factors

$$b+a_1 \quad b+a_2 \quad b+a_3 \ .\ .\ \text{etc.} \quad b+a_n$$

https://doi.org/10.1017/S0013091500032089 Published online by Cambridge University Press

Then we know

$$\prod_{r=1}^{r=n}(a_r) - n\prod_{r=1}^{r=n}(b+a_r) + \frac{n.n-1}{2!}\prod_{r=1}^{r=n}(2b+a_r) + \ \ldots\ldots$$
$$+(-1)^n\prod_{r=1}^{r=n}(nb+a_r) \equiv n!(-b)^n \qquad - \qquad (2)$$

[*Edin. Math. Socy. Proc.*, Vol. XIII., 1895, p. 115 (4).]

If $a_1 = a$

$a_2 = a - 1$

$\cdots$

$\cdots$

$a_n = a - n + 1$

$$\prod_{r=1}^{r=n}(sb+a_r) = (sb+a)(sb+a-1)\ \ldots\ldots\ (sb+a-n+1) = (sb+a)_n$$

And we have the identity

$$(a)_n - n(a+b)_n + \frac{n.n-1}{2!}(a+2b)_n - \ \ldots\ldots\ \equiv n!(-b)^n \qquad - \qquad (3)$$

In most of the subsequent work the series on the left side of (3) will be considered for all values of n

The function $(a)_n$ being $\dfrac{\Pi(a)}{\Pi(a-n)}$ in terms of Gauss's Π Function.

The series (3) when n is a positive integer may be written

$$(a)_n - n(a-b)_n + \frac{n.n-1}{2!}(a-2b)_n - \ \ldots\ldots\ \equiv n!b^n \qquad - \qquad (4)$$

When n is unrestricted let us write

$$(a)_n - n(a-b)_n + \ \ldots\ldots\ = f(n,b) \qquad - \qquad - \qquad (5)$$

Then dividing throughout by $(a)_n$ we obtain

$$1 - n\frac{(a-b)_n}{a_n} + \frac{n.n-1}{2}\ \frac{(a-2b)_n}{(a)_n} - \ \ldots\ldots$$
$$+(-1)^r\frac{n.n-1\ \ldots\ n-r+1}{r!}\ \frac{(a-rb)_n}{(a)_n} = \frac{f(nb)}{(a)_n} \qquad (6)$$

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The following equations show fundamental properties of the function $(a)_n$

$$(a)_n \times (a-n)_m = (a)_{m+n} = (a)_m \times (a-m)_n$$

from which we obtain

$$\frac{(a-rb)_n}{(a)_n} = \frac{(a-n)_{rb}}{(a)_{rb}} \quad - \quad - \quad - \quad - \quad (\alpha)$$

$$\frac{(a)_{rb}}{(a)_{rb+1}} = \frac{1}{a-rb} \quad - \quad - \quad - \quad - \quad - \quad (\beta)$$

$$\frac{(a)_{rb}}{(a)_{rb+2}} = \frac{1}{(a-rb)(a-rb-1)} \quad - \quad - \quad - \quad (\gamma)$$

$$\cdots$$
$$\cdots$$

$$\cdots$$
$$\cdots$$

By means of the relation (α). The series (6) may be transformed into

$$1 - n\frac{(a-n)_b}{(a)_b} + \frac{n.n-1}{2!}\,\frac{(a-n)_{2b}}{(a)_{2b}} - \;.\;.\; + (-1)^r\frac{(n)_r}{r!}\,\frac{(a-n)_{rb}}{(a)_{rb}} + \;.\;.$$
$$= \frac{f(nb)}{(a)_n} \qquad (7)$$

For convenience in subsequent work, change $a-n$ to c, then

$$1 - n\frac{(c)_b}{(c+n)_b} + \frac{n.n-1}{2!}\,\frac{(c)_{2b}}{(c+n)_{2b}} - \;.\;.\; + (-1)^r\frac{(n)_r}{r!}\,\frac{(c)_{rb}}{(c+n)_{rb}} + \;.\;.$$
$$= \frac{f(nb)}{(c+n)_n} \qquad (8)$$

Now $$\frac{(c)_{sb}}{(c+n)_{sb}} = \frac{\Pi(c)\Pi(c+n-sb)}{\Pi(c-sb)\Pi(c+n)}$$

$$= 1 - \frac{n}{1!}\,\frac{sb}{c+1} + \frac{n.n-1}{2!}\,\frac{sb.sb+1}{c+1.c+2} - \;.\;.\;.\;.$$

(subject to conditions for convergence).

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On replacing each term of the series on the left side of (8) by an infinite series we have the expression

$$1-\frac{(n)_1}{1!}\left[1-\frac{(n)_1}{1!}\;\frac{b}{c+1}+\frac{(n)_2}{2!}\;\frac{b\,.\,b+1}{c+1\,.\,c+2}-\;\ldots\ldots\;+(-1)^r\frac{(n)_r}{r!}\;\frac{b\,.\,b+1\;\ldots\ldots\;b+r-1}{c+1\,.\,c+2\;\ldots\ldots\;c+r}+\;\ldots\ldots\right]$$

$$+\frac{(n)_2}{2!}\left[1-\frac{(n)_1}{1!}\;\frac{2b}{c+1}+\frac{(n)_2}{2!}\;\frac{2b\,.\,2b+1}{c+1\,.\,c+2}-\;\ldots\ldots\;+(-1)^r\frac{(n)_r}{r!}\;\frac{2b\,.\,2b+1\;\ldots\ldots\;2b+r-1}{c+1\,.\,c+2\;\ldots\ldots\;c+r}+\;\ldots\ldots\right]$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$+(-1)^s\frac{(n)_s}{s!}\left[1-\frac{(n)_1}{1!}\;\frac{sb}{c+1}+\frac{(n)_2}{2!}\;\frac{sb\,.\,sb+1}{c+1\,.\,c+2}-\;\ldots\ldots\;+(-1)^r\frac{(n)_r}{r!}\;\frac{sb\,.\,sb+1\;\ldots\ldots\;sb+r-1}{c+1\,.\,c+2\;\ldots\ldots\;c+r}+\;\ldots\ldots\right]$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

For convenience denote b by $-\beta$ then

$$(-1)^r b\,.\,b+1\,.\,b+2\;\ldots\ldots\;b+r-1=(\beta)_r$$

and the expression (9) may be written, after splitting up the terms into partial fractions

$$-\frac{(n)_1}{1!}\left[1+\frac{(n)_1(\beta)_1}{1!}\left\{\frac{1}{c+1}\right\}+\frac{(n)_2(\beta)_2}{2!}\left\{\frac{1}{c+1}-\frac{1}{c+2}\right\}+\,.\,+\frac{(n)_r(\beta)_r}{r!}\left\{\frac{1}{r-1!\,0!\,c+1}-\frac{1}{r-2!\,1!\,c+2}+\,.\,+(-1)^{r-1}\frac{1}{0!\,r-1!\,c+r}\right\}+\,.\right]$$

$$+\frac{(n)_2}{2!}\left[1+\frac{(n)_1(2\beta)_1}{1!}\left\{\frac{1}{c+1}\right\}+\frac{(n)_2(2\beta)_2}{2!}\left\{\frac{1}{c+1}-\frac{1}{c+2}\right\}+\,.\,+\frac{(n)_r(2\beta)_r}{r!}\left\{\frac{1}{r-1!\,0!\,c+1}-\frac{1}{r-2!\,1!\,c+2}+\,.\,+(-1)^{r-1}\frac{1}{0!\,r-1!\,c+r}\right\}+\,.\right]$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$(-1)^s\frac{(n)_s}{s!}\left[\qquad \text{A series similar to the above in terms of } s\beta \qquad\right]$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad\qquad (10)$$

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The expression (9) may now be written

$$P^0 + P'\frac{1}{c+1} + P''\frac{1}{c+2} + \;.\;.\;.\;.\; + P^{(r)}\frac{1}{c+r} + \;.\;.\;.\;. \qquad (11)$$

where
$$P^0 = 1 - \frac{(n)_1}{1!} + \frac{(n)_2}{2!} - \;.\;.\;.\;.$$

and
$$(-1)^r P^{(r)} \equiv \frac{(n)_1}{1!}\left[\frac{(n)_r(\beta)_r}{r!0!r-1!} + \frac{(n)_{r+1}(\beta)_{r+1}}{r+1!1!r-1!} + \frac{(n)_{r+2}(\beta)_{r+2}}{r+2!2!r-1!} + \;.\;.\right]$$
$$- \frac{(n)_2}{2!}\left[\quad \text{Similar series to above in } 2\beta \quad\right]$$
$$+ \frac{(n)_3}{3!}\left[\quad \text{Similar series in } 3\beta \quad\right]$$
$$\cdots \qquad \cdots \qquad \cdots$$

By means of the coefficients $P^0 \,.\, P' \;.\;.\;.\; P^{(r)} \;.\;.\;.$ we can obtain an expansion of b^n

For the series
$$1 - n\frac{(c)_b}{(c+n)_b} + \frac{n.n-1}{2!}\;\frac{(c)_{2b}}{(c+n)_{2b}} - \;.\;.\;.\;. \qquad (12)$$

has been reduced to the form

$$P^0 + P'\frac{1}{c+1} + \;.\;.\;.\;.\; + P^{(r)}\frac{1}{c+r} + \;.\;.\;.\;. \qquad (13)$$

in which the coefficients P are functions of n and b only.

When n is a positive integer the series (12) $\equiv \dfrac{n!b^n}{(c+n)_n}$

$$\equiv \frac{\Pi(n)\Pi(c)}{\Pi(c+n)}b^n$$

$$\equiv b^n\left[1 - \frac{c}{c+1}\cdot\frac{(n)_1}{1!} + \frac{c}{c+2}\;\frac{(n)_2}{2!} - \;.\;.\; + (-1)^r\frac{c}{c+r}\;\frac{(n)_r}{r!} - \;.\;.\right]$$

$$\equiv b^n\left[1 - \frac{(n)_1}{1!} + \frac{(n)_2}{2!} - \;.\;.\;.\;.\right.$$
$$\left. + \frac{1}{c+1}\cdot\frac{(n)_1}{1!} - \frac{2}{c+2}\;\frac{(n)_2}{2!} + \;.\;.\; - (-1)^r\;\frac{r}{c+r}\;\frac{(n)_r}{r!} + \;.\;.\right]$$

This series must be identical with (13). Equating the coefficients of

$$\frac{1}{c+1}\quad\frac{1}{c+2}\;\cdots\cdots$$

we get

$$b^n\frac{(n)_1}{0!}=\mathrm{P}'$$

$$-b^n\frac{(n)_2}{1!}=\mathrm{P}''$$

$$\cdots$$

$$-(-1)^r b^n\frac{(n)_r}{r-1!}=\mathrm{P}^{(r)}$$

$$\cdots$$

Therefore since $b=-\beta$

$$(-1)^{n+1}\beta^n\frac{(n)_r}{r-1!}=\frac{(n)_1}{1!}\left[\frac{(n)_r(\beta)_r}{r!0!r-1!}+\frac{(n)_{r+1}(\beta)_{r+1}}{r+1!1!r-1!}+\;\cdots\cdots\right]$$

$$-\frac{(n)_2}{2!}\left[\frac{(n)_r(2\beta)_r}{r!0!r-1!}+\frac{(n)_{r+1}(2\beta)_{r+1}}{r+1!1!r-1!}+\;\cdots\cdots\right]\quad(14)$$

$$\cdots\qquad\cdots\qquad\cdots$$

$$-(-)^s\frac{(n)_s}{s!}\left[\frac{(n)_r(sb)_r}{r!0!r-1!}+\frac{(n)_{r+1}(s\beta)_{r+1}}{r+1!1!r-1!}+\;\cdots\cdots\right]$$

$$\cdots\qquad\cdots\qquad\cdots$$

Removing the factor $\frac{(n)_r}{r-1!}$ which is common to both sides of the above equation we obtain

$$\left.\begin{aligned}(-1)^{n+1}\beta^n=\frac{(n)_1}{1!}&\left[\frac{(\beta)_r}{0!r!}+\frac{(n-r)(\beta)_{r+1}}{1!r+1!}+\frac{(n-r)_2(\beta)_{r+2}}{2!r+2!}+\;\cdots\right]\\-\frac{(n)_2}{2!}&\left[\frac{(2\beta)_r}{0!r!}+\frac{(n-r)(2\beta)_{r+1}}{1!r+1!}+\frac{(n-r)_2(2\beta)_{r+2}}{2!r+2!}+\;\cdots\right]\\&\cdots\qquad\cdots\qquad\cdots\end{aligned}\right\}\quad(15)$$

in which r is any positive integer, this is the same as (1).
Putting $r=1$ we have

$$(-1)^{n+1}\beta^n=\frac{(n_1)}{1!}\left[\frac{\beta}{0!1!}+\frac{n-1.\beta.\beta-1}{1!2!}+\frac{n-1.n-2.\beta.\beta-1.\beta-2}{2!3!}+\cdots\right]\quad(16)$$

$$-\frac{(n)_2}{2!}\left[\frac{2\beta}{0!1!}+\frac{n-1.2\beta.2\beta-1}{1!2!}+\frac{n-1.n-2.2\beta.2\beta-1.2\beta-2}{2!3!}+\cdots\right]$$

$+$ **Similar series.**

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When n is a positive integer. Expression (16) will consist of n series each containing n terms. Thus if $n=3$ we have

$$\left.\begin{aligned}+\beta^3 = &\frac{3}{1!}\left[\frac{\beta}{0!1!}+2.\frac{\beta.\beta-1}{1!2!}+\frac{2.1.\beta.\beta-1.\beta-2}{2!3!}\right]\\ &-\frac{3.2}{2!}\left[\frac{2\beta}{0!1!}+2.\frac{2\beta.2\beta-1}{1!2!}+\frac{2.1.2\beta.2\beta-1.2\beta-2}{2!3!}\right]\\ &+\frac{3.2.1}{3!}\left[\frac{3\beta}{0!1!}+2.\frac{3\beta.3\beta-1}{1!2!}+\frac{2.1.3\beta.3\beta-1.3\beta-2}{2!3!}\right]\end{aligned}\right\} \quad (17)$$

Other expansions of β^3 may be obtained by putting $r=2$ $r=3$ namely

$$\left.\begin{aligned}+\beta^3 = &\frac{3}{1!}\left[\frac{\beta.\beta-1}{0!2!}+\frac{1.\beta.\beta-1.\beta-2}{1!3!}\right]=\frac{3}{1!}\left[\frac{\beta.\beta-1.\beta-2}{0!3!}\right]\\ &-\frac{3.2}{2!}\left[\frac{2\beta.2\beta-1}{0!2!}+\frac{1.2\beta.2\beta-1.2\beta-2}{1!3!}\right]-\frac{3.2}{2!}\left[\frac{2\beta.2\beta-1.2\beta-2}{0!3!}\right]\\ &+\frac{3.2.1}{3!}\left[\frac{3\beta.3\beta-1}{0!2!}-\frac{1.3\beta.3\beta-1.3\beta-2}{1!3!}\right]+\frac{3.2.1}{3!}\left[\frac{3\beta.3\beta-1.3\beta-2}{0!3!}\right]\end{aligned}\right\} \quad (18)$$

Similarly β^4 may be obtained in 4 different forms
and β^n in n different forms by giving r the values $1.2.3\ldots n$ in the series 16.

The series (15) and (16) are perfectly general in form although we have proved the expansion only when n is a positive integer

If any proof exists that subject to conditions for convergence

$$1-n\frac{(c)_b}{(c+n)_b}+\frac{n.n-1}{2!}\;\frac{(c)_{2b}}{(c+n)_{2b}}-\;\ldots\ldots=b^n\frac{\Pi(n)\Pi(c)}{\Pi(c+n)} \quad . \quad (19)$$

when n is unrestricted, then the expansions (15) (16) *will hold generally subject to convergence.*