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ANOTHER APPROACH TO A NON-ELLIPTIC BOUNDARY PROBLEM

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1. Introduction

Let Ω be a bounded domain in n dimensional Euclidian space $\boldsymbol{R}^n$ $(n \geqq 2)$ with C^∞ boundary Γ of dimension $n-1$ and let there be given two real-valued C^∞-functions α, β on Γ such that $\alpha \geqq 0$, $\beta \geqq 0$ and $\alpha + \beta = 1$ throughout Γ. Then we consider the non-elliptic boundary value problem with $\lambda \geqq 0$ (which is always assumed, and in particular when $\lambda = 0$, we further assume $\beta \not\equiv 0$, throughout this paper):

$$(1) \qquad \begin{cases} (\lambda - \Delta)U = F & \text{in } \Omega \\ \alpha \dfrac{\partial U}{\partial n} + \beta U = 0 & \text{on } \Gamma \end{cases}$$

where $\Delta = (\partial/\partial x_1)^2 + \cdots + (\partial/\partial x_n)^2$ and $\partial U/\partial n$ denotes the exterior normal derivative of U. This kind of problem has been recently discussed from the viewpoint of functional analysis by several authors. In [1], [4] and [5] they used the variational approach in Ω, applying the elliptic regularization, and in [3] and [6] they reduced the problem (1) to a pseudo-differential equation on Γ and applied what is called Melin's theorem.

In this paper, we would like to note that the problem (1) can also be solved without using Melin's theorem in the latter method. Instead of it, we shall apply the classical Riesz-Schauder theory (see Section 7).

2. Operator S

We shall denote by $P\varphi$ the unique solution of the Dirichlet problem

$$(2) \qquad \begin{cases} (\lambda - \Delta)U = 0 & \text{in } \Omega \\ U = \varphi & \text{on } \Gamma\,. \end{cases}$$

Then it follows that the mapping of $C^\infty(\Gamma)$ into itself;

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$$S : \varphi \mapsto \frac{\partial}{\partial n}(P\varphi)|_{\Gamma}$$

is a first order elliptic pseudo-differential operator on Γ and there exist two positive constants c and C such that

$$\mathrm{Re}\,(S\varphi, \varphi) \geqq c\,\|\varphi\|_{1/2}^2 - C\,\|\varphi\|_0^2\,, \qquad \varphi \in C^\infty(\Gamma)\,,$$

where, as well as in the following, $\|\varphi\|_s$ is the norm of φ in the Sobolev space $H_s(\Gamma)$ of order s and (φ, ψ) is the usual inner product in $L^2(\Gamma)$ (see [2]). Hence the first order pseudo-differential operator

$$E = \mathrm{Re}\,S + C\left(\mathrm{Re}\,S = \frac{S + S^*}{2}\right)$$

is formally self-adjoint and positive. By $\hat{E}$, we shall denote the closure of E in $L^2(\Gamma)$. Then it is easily seen that, for any real s, the s powers of $\hat{E}$, $\hat{E}^s$, is a self-adjoint elliptic pseudo-differential operator on Γ of order s and the norms $\|\varphi\|_s$ and $\|\hat{E}^s\varphi\|_0$ are equivalent.

Setting

$$\theta = \hat{E}^{1/2}\,,$$

we have

$$\mathrm{Re}\,S = \theta^2 - C\,. \tag{3}$$

The notation E will be used in the below for the extension of the pseudo-differential operator $\hat{E}$ to the space of distributions $\mathscr{D}'(\Gamma)$.

3. Reduction on Γ

First we shall establish the uniqueness of the problem (1).

PROPOSITION 1. *If $U \in H_2(\Omega)$ and if it satisfies*

$$\begin{cases} (\lambda - \Delta)U = 0 & \text{in } \Omega \\ \alpha\dfrac{\partial U}{\partial n} + \beta U = 0 & \text{on } \Gamma\,, \end{cases}$$

then $U = 0$ in Ω, where $H_s(\Omega)$ is, in general, the Sobolev space of order s.

Proof. By Green's formula, it follows that

$$0 = \int_\Omega (\lambda - \Delta)U \cdot \bar{U} dx = \int_\Omega \left(\sum_{j=1}^n \left|\frac{\partial U}{\partial x_j}\right|^2 + \lambda\,|U|^2\right)dx - \int_\Gamma \frac{\partial U}{\partial n} \cdot \bar{U} dS\,.$$

Since $\alpha \geqq 0$, $\beta \geqq 0$ and $\alpha + \beta = 1$ on Γ, we have

$$\frac{\partial U}{\partial n}\cdot \bar{U} = \alpha\frac{\partial U}{\partial n}\cdot \bar{U} + \frac{\partial U}{\partial n}\cdot \beta\bar{U} = -\beta\,|U|^2 - \alpha\left|\frac{\partial U}{\partial n}\right|^2 \leqq 0$$

on Γ, and hence $U = 0$ in Ω. Q.E.D.

Now we shall reduce the problem (1) to that on Γ, that is

PROPOSITION 2. *If U is contained in $H_{s+2}(\Omega)$ $(s \geqq 0)$ and if it satisfies*

$$(4) \qquad \begin{cases} (\lambda - \Delta)U = 0 & \text{in } \Omega \\ \alpha\dfrac{\partial U}{\partial n} + \beta U = f & \text{on } \Gamma\,. \end{cases}$$

then the Dirichlet data of U on Γ, $u = U|_\Gamma$, is contained in $H_{s+3/2}(\Gamma)$ and satisfies

$$(4') \qquad (\alpha S + \beta)u = f \qquad \text{on } \Gamma\,.$$

Conversely, if $u \in H_{s+3/2}(\Gamma)$ and if it satisfies (4'), *then the solution $U = Pu$ of Dirichlet problem* (2) *with $\varphi = u$ satisfies* (4).

Moreover the inequality

$$(5) \qquad \|U\|_{s+2,\Omega} \leqq \text{const.}\ \|u\|_{s+3/2}$$

holds for any $u \in H_{s+3/2}(\Gamma)$, where $\|U\|_{s,\Omega}$ denotes the norm of U in $H_s(\Omega)$.

We shall omit the simple proof.

4. Main theorems

To solve the problem (1), the following theorem is fundamental.

THEOREM 1. *Let f be in $H_s(\Gamma)$ $(s \geqq 3/2)$. Then there exists one and only one $u \in H_s(\Gamma)$ satisfying* (4'). *Moreover there exists a positive constant c_s such that the inequality*

$$(6) \qquad \|u\|_s \leqq c_s\,\|f\|_s$$

holds for every $f \in H_s(\Gamma)$.

The proof will be given in the following sections. Before doing so, we shall solve the problem (1), by using Propositions 1, 2 and Theorem 1, that is, THEOREM 2. *Let F be given in $H_s(\Omega)$ $(s \geqq 0)$. Then we can find one and only one $U \in H_{s+2}(\Omega)$ satisfying* (1). *Moreover there exists a positive constant C_s such that*

$$\|U\|_{s+2,\Omega} \leqq C_s \|F\|_{s,\Omega} . \tag{7}$$

Proof. We shall prove by the similar argument as in [3]. Let V be the unique solution in $H_{s+2}(\Omega)$ of the elliptic boundary value problem

$$\begin{cases} (\lambda - \Delta)V = F & \text{in } \Omega \\ \dfrac{\partial V}{\partial n} + V = 0 & \text{on } \Gamma . \end{cases}$$

Then $v = V|_\Gamma \in H_{s+3/2}(\Gamma)$ and the inequality

$$\|v\|_{s+3/2} \leqq \text{const.}\, \|V\|_{s+2} \leqq \text{const.}\, \|F\|_{s,\Omega} \tag{8}$$

holds for every $F \in H_s(\Omega)$.

By Theorem 1, we can find a unique solution $w \in H_{s+3/2}(\Gamma)$ of the equation $(\alpha S + \beta)w = -(\beta - \alpha)v$ on Γ. It then follows from Propositions 1 and 2 that $W = Pw$ is a unique solution in $H_{s+2}(\Omega)$ of the problem

$$\begin{cases} (\lambda - \Delta)W = 0 & \text{in } \Omega \\ \alpha\dfrac{\partial W}{\partial n} + \beta W = -(\beta - \alpha)v & \text{on } \Gamma . \end{cases}$$

Furthermore, we have by (5) and (6)

$$\|W\|_{s+2,\Omega} \leqq \text{const.}\, \|w\|_{s+3/2} \leqq \text{const.}\, \|v\|_{s+3/2} . \tag{9}$$

If we define as $U = V + W$, it is obvious that the U is the desired solution. The inequality (7) is immediately obtained from (8) and (9).

5. Spaces $\mathscr{U}_s$ and $\mathscr{F}_s$

We introduce the Banach spaces $\mathscr{U}$ and $\mathscr{F}$ obtained by the completion of $C^\infty(\Gamma)$ with respect to the norms $|||\cdot|||$ and $|||\cdot|||'$, respectively, defined by

$$|||u||| = (\|\sqrt{\alpha}\,\theta u\|_0^2 + \|u\|_0^2)^{1/2} ,$$

$$|||f|||' = \sup_{\substack{u \in \mathscr{U} \\ u \neq 0}} \frac{|(f, u)|}{|||u|||} .$$

More generally, for any real s, we define two Banach spaces

$$\mathscr{U}_s = \{u \in \mathscr{D}'(\Gamma) ; E^s u \in \mathscr{U}\}$$

and

$$\mathscr{F}_s = \{f \in \mathscr{D}'(\Gamma) ; E^s f \in \mathscr{F}\}$$

with the norms defined by

$$|||u|||_s = |||E^s u||| \quad \text{and} \quad |||f|||'_s = |||E^s f|||'$$

respectively. Particularly, we have $|||u|||_0 = |||u|||$ and $|||f|||'_0 = |||f|||'$.

PROPOSITION 3. *We have*

$$H_{s-1/2}(\Gamma) \supset \mathscr{F}_s \supset H_s(\Gamma) \supset \mathscr{U}_s \supset H_{s+1/2}(\Gamma)$$

for all s, and

$$\mathscr{U}_s \subset \mathscr{U}_{s'}, \qquad \mathscr{F}_s \subset \mathscr{F}_{s'}$$

for all s, s' such that $s' + 1/2 \leqq s$. Moreover the injections are all continuous.

Proof. For any $u \in C^\infty(\Gamma)$, we have by definition

(10) $$\text{const.}\, \|u\|_s \leqq |||u|||_s \leqq \text{const.}\, \|u\|_{s+1/2}\,.$$

For any $f \in C^\infty(\Gamma)$, we have obviously

$$|||f|||'_s = \sup_{0 \neq u \in \mathscr{U}} \frac{|(E^s f, u)|}{|||u|||} \leqq \|E^s f\|_0 \leqq \text{const.}\, \|f\|_s$$

and on the other hand, noting that $E = \theta^2$,

$$\begin{aligned} |||f|||'_s = \sup_{0 \neq u \in C^\infty(\Gamma)} \frac{|(E^s f, u)|}{|||u|||} &= \sup \frac{|(E^{s-1/2} f, \theta u)|}{\|\theta u\|_0} \frac{\|\theta u\|_0}{|||u|||} \\ &\geqq \text{const.}\, \|E^{s-1/2} f\|_0 \geqq \text{const.}\, \|f\|_{s-1/2}\,, \end{aligned}$$

from which and (10), it follows that if $s' + 1/2 \leqq s$,

$$|||u|||_{s'} \leqq \text{const.}\, |||u|||_s \quad \text{and} \quad |||f|||_{s'} \leqq \text{const.}\, |||f|||_s\,.$$

Thus the proof is completed. Q.E.D.

6. Operator $\alpha S + \beta + H$

Let H be a pseudo-differential operator on Γ defined by

(11) $$H = \alpha + \alpha C - \frac{[[\alpha, \theta], \theta]}{2} - \frac{[\alpha, S - S^*]}{4}\,,$$

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where $[A, B] = AB - BA$. Clearly H is of order zero. Let Q be a quadratic form

$$Q[u, v] = ((\alpha S + \beta + H)u, v)$$

and, for any ε such that $0 < \varepsilon \leqq 1$, we put

$$Q_\varepsilon[u, v] = Q[u, v] + \varepsilon((S + C)u, v) \, .$$

PROPOSITION 4. *For every $u \in H_{1/2}(\Gamma)$, we have*

$$\operatorname{Re} Q_\varepsilon[u, u] = |||u|||^2 + \varepsilon \, \|\theta u\|_0^2 \, .$$

Proof. Since

$$\operatorname{Re} (\alpha S) = \alpha \operatorname{Re} S + \tfrac{1}{2}[S^*, \alpha] \, ,$$

we have, by (3),

$$\operatorname{Re} ((S + C)u, u) = (Eu, u) = (\theta u, \theta u)$$

and

$$\begin{aligned} \operatorname{Re} (\alpha S u, u) &= (\alpha\theta^2 u, u) - (\alpha C u, u) + \operatorname{Re} \left(\frac{[S^*, \alpha]}{2} u, u\right) \\ &= (\alpha\theta u, \theta u) + \operatorname{Re} ([\alpha, \theta]\theta u, u) - (\alpha C u, u) + \operatorname{Re} \left(\frac{[S^*, \alpha]}{2} u, u\right) \\ &= (\alpha\theta u, \theta u) + \left(\left(\frac{[[\alpha, \theta], \theta]}{2} + \frac{[\alpha, S - S^*]}{4} - \alpha C\right)u, u\right) . \end{aligned}$$

Hence

$$\begin{aligned} \operatorname{Re} Q_\varepsilon[u, u] &= (\alpha\theta u, \theta u) + (u, u) + \varepsilon(\theta u, \theta u) \\ &= |||u|||^2 + \varepsilon \, \|\theta u\|_0^2 \, , \end{aligned}$$

where we used the fact $\alpha + \beta = 1$ on Γ. Q.E.D.

LEMMA 1. *Let $a(x)$ be in $C^\infty(\boldsymbol{R}^m)$ and P be a pseudo-differential operator on $\boldsymbol{R}^m$ of order t. Then there exist the pseudo-differential operators P_j $(j = 1, \cdots, m)$ and Q on $\boldsymbol{R}^m$ of order $t - 1$ and $t - 2$, respectively, such that*

$$[a, P] = \sum_{j=1}^m \frac{\partial a}{\partial x_j} P_j + Q \, .$$

This is immediately obtained by the generalized Leibniz formula.

THEOREM 3. *Let $s \geqq 1/2$. For every $f \in \mathscr{F}_s$, we can find one and only one $u \in \mathscr{U}_s$ satisfying the equation*

$$(\alpha S + \beta + H)u = f \qquad \text{on } \Gamma$$

and the inequality

$$|||u|||_s \leqq c_s |||f|||'_s$$

with a positive constant c_s independent of f.

Proof. First we assume $f \in C^\infty(\Gamma)$. The Lax-Milgram theorem guarantees the existence of $u_\varepsilon \in H_{1/2}(\Gamma)$ such that

$$Q_\varepsilon[u, v] = (f, v) , \qquad v \in H_{1/2}(\Gamma) ,$$

for the equality in Proposition 4 and the inequality $|Q_\varepsilon[u, v]| \leq$ const. $\|u\|_{1/2} \|v\|_{1/2}$ are valid for every $u, v \in H_{1/2}(\Gamma)$. Substituting $v = u_\varepsilon$ in the above and using Proposition 4, we have

$$(12) \qquad |||u_\varepsilon||| \leqq |||f|||' .$$

Since u_ε is a weak solution of the elliptic equation $(\alpha_\varepsilon S + \beta + H + \varepsilon C)u = f$, we can assert $u_\varepsilon \in C^\infty(\Gamma)$, where we put $\alpha_\varepsilon = \alpha + \varepsilon$.

By Proposition 4 again, we have for all real s

$$\begin{aligned}
|||u_\varepsilon|||_s^2 + \varepsilon \|E^{s+1/2}u_\varepsilon\|_0^2 &= \mathrm{Re}\, Q_\varepsilon[E^s u_\varepsilon, E^s u_\varepsilon] = \mathrm{Re}\, (\alpha_\varepsilon S + \beta + H + \varepsilon C)E^s u_\varepsilon, E^s u_\varepsilon) \\
&= \mathrm{Re}\, Q_\varepsilon[u_\varepsilon, E^{2s}u_\varepsilon] + \mathrm{Re}\, ([\alpha_\varepsilon S + \beta + H, E^s]u_\varepsilon, E^s u_\varepsilon) \\
&= \mathrm{Re}\, (f, E^{2s}u_\varepsilon) + \mathrm{Re}\, ([\beta + H, E^s]u_\varepsilon, E^s u_\varepsilon) + \mathrm{Re}\, ([\alpha_\varepsilon S, E^s]u_\varepsilon, E^s u_\varepsilon) \\
&\leqq |||f|||'_s\, |||u_\varepsilon|||_s + \text{const.}\, \|u_\varepsilon\|_{s-1/2}^2 + |([\alpha_\varepsilon S, E^s]u_\varepsilon, E^s u_\varepsilon)| .
\end{aligned}$$

The last term is calculated as follows. Since

$$[\alpha_\varepsilon S, E^s] = \alpha_\varepsilon [S, E^s] + [\alpha, E^s]S ,$$

we have

$$([\alpha_\varepsilon S, E^s]u_\varepsilon, E^s u_\varepsilon) = ([S, E^s]u_\varepsilon, \alpha_\varepsilon E^s u_\varepsilon) + ([\alpha, E^s]S u_\varepsilon, E^s u_\varepsilon) .$$

Applying Lemma 1 and the partition of unity to the last term of the above, we have

$$|([\alpha_\varepsilon S, E^s]u_\varepsilon, E^s u_\varepsilon)| \leqq \text{const.}\, (\|u_\varepsilon\|_s \|\sqrt{\alpha}\,\theta E^{s-1/2}u_\varepsilon\|_0 + \|u_\varepsilon\|_{s-1/2}^2 + \varepsilon \|u_\varepsilon\|_s^2) ,$$

where we used Lemma A.1 of [4] (i.e., $|\partial\alpha/\partial x_j| \leqq$ const. $\sqrt{\alpha}$) and denote by const. a constant not depending on ε, $0 < \varepsilon \leqq 1$, as well as in the below. Hence we have, for every ε,

$$|||u_\varepsilon|||_s^2 + \varepsilon\,\|E^{s+1/2}u_\varepsilon\|_0^2 \leqq |||f|||'_s\,|||u_\varepsilon|||_s$$
$$+ \text{const.}\,(|||u_\varepsilon|||_s\,|||u_\varepsilon|||_{s-1/2} + \|u_\varepsilon\|_{s-1/2}^2 + \varepsilon\,\|u_\varepsilon\|_s^2)\,,$$

from which it follows that

$$|||u_\varepsilon|||_s \leqq |||f|||'_s + \text{const.}\,|||u_\varepsilon|||_{s-1/2}\,, \tag{13}$$

where we used the interpolation inequality; for any $\delta > 0$, there exists a constant $C_\delta > 0$ such that

$$\|u\|_s^2 \leqq \delta\,\|u\|_{s+1/2}^2 + C_\delta\,\|u\|_{s-1/2}^2\,, \qquad u \in C^\infty(\Gamma)\,.$$

Substituting $s - 1/2$ in the place of s in (13), we have

$$|||u_\varepsilon|||_{s-1/2} \leqq |||f|||'_{s-1/2} + \text{const.}\,|||u_\varepsilon|||_{s-1}\,. \tag{14}$$

By (12), Proposition 3 and the interpolation inequality, it follows that for any $\delta > 0$ there exists a positive constant C'_δ such that

$$|||u_\varepsilon|||_{s-1} \leqq \delta\,|||u_\varepsilon|||_s + C'_\delta\,|||f|||'\,,$$

from which together with (13) and (14) we obtain

$$|||u_\varepsilon|||_s \leqq \text{const.}\,(|||f|||'_s + |||f|||'_{s-1/2} + |||f|||')\,.$$

By Proposition 3, we finally obtain, for every $s \geqq 1/2$,

$$|||u_\varepsilon|||_s \leqq c_s\,|||f|||'_s$$

with some positive constant c_s not depending on ε. Hence we can choose a sequence $\varepsilon_1 > \varepsilon_2 > \cdots \to 0$ such that u_{ε_j} converges in $C^\infty(\Gamma)$ and the limit function u satisfies

$$(\alpha S + \beta + H)u = f \quad \text{and} \quad |||u|||_s \leqq c_s\,|||f|||'_s$$

for every real $s \geqq 1/2$.

Now let $f \in \mathscr{F}_s$ $(s \geqq 1/2)$ and let $f_j \in C^\infty(\Gamma)$ $(j = 1, 2, \cdots)$ such that $f_j \to f$ in $\mathscr{F}_s$ as $j \to \infty$. We have just proved that, for each f_j, there exists $u_j \in C^\infty(\Gamma)$ such that

$$(\alpha S + \beta + H)u_j = f_j \quad \text{and} \quad |||u_j|||_s \leqq c_s\,|||f_j|||'_s\,.$$

It then follows that $u_j \to u$ in $\mathscr{U}_s$ as $j \to \infty$ and the u is contained in $\mathscr{U}_s$ and satisfies

$$(\alpha S + \beta + H)u = f \quad \text{and} \quad |||u|||_s \leqq c_s\,|||f|||'_s\,.$$

Finally, we shall show the uniqueness of such u. Suppose that $u \in \mathscr{U}_s$ and $(\alpha S + \beta + H)u = 0$. Since $u \in H_{1/2}(\Gamma)$, we have, by Proposition 4,

$$0 = \mathrm{Re}\,((\alpha S + \beta + H)u, u) = \mathrm{Re}\, Q[u, u] = |||u||| ,$$

from which we can conclude $u = 0$. Q.E.D.

7. Proof of Theorem 1

In this section, we shall prove Theorem 1, which was proved, in [3] and [6], by applying Melin's theorem.

LEMMA 2. *Let γ be a first order differential operator on Γ with C^∞-coefficients. If $u \in H_t(\Gamma)$ (t: real), then $\gamma(\alpha)u \in \mathscr{F}_{t+1/2}$ and the estimate*

$$|||\gamma(\alpha)u|||'_{t+1/2} \leqq \text{const. } \|u\|_t$$

holds.

Proof. For any $v \in \mathscr{U}$, we have

$$\begin{aligned}(E^{t+1/2}\gamma(\alpha)u, v) &= (\theta E^t\gamma(\alpha)u, v)\\ &= (\theta\gamma(\alpha)E^t u, v) + (\theta[E^t, \gamma(\alpha)]u, v)\\ &= (\mathrm{E}^t u, \gamma(\alpha)\theta v) + (\theta[E^t, \gamma(\alpha)]u, v) .\end{aligned}$$

Hence

$$\begin{aligned}|(E^{t+1/2}\gamma(\alpha)u, v)| &\leqq \text{const. } (\|u\|_t \|\sqrt{\alpha}\,\theta v\|_0 + \|u\|_{t-1/2}\|v\|_0)\\ &\leqq \text{const. } \|u\|_t\, |||v||| ,\end{aligned}$$

which completes the proof. Q.E.D.

In Theorem 3, if we write $u = Kf$, or

$$(\alpha S + \beta + H)K = 1 \qquad \text{on } \mathscr{F}_s ,$$

then K is a continuous mapping of $\mathscr{F}_s$ into $\mathscr{U}_s$ ($s \geqq 1/2$). It follows from Proposition 3 that K is a continuous mapping of $H_s(\Gamma)$ into itself. Let $f \in H_s(\Gamma)$ and assume that $u \in H_s(\Gamma)$ satisfy

$$(1 - KH)u = Kf .$$

Then the u satisfies $(\alpha S + \beta)u = f$. This follows easily by operating $\alpha S + \beta + H$ to both sides of the above equation.

LEMMA 3. *The operator KH is a continuous mapping of $H_s(\Gamma)$ ($s \geqq 0$) into $H_{s+1/2}(\Gamma)$, and so a compact operator on $H_s(\Gamma)$.*

Proof. The operator H is given by (11). First we note that if $u \in H_s(\Gamma)$, then $Qu \in H_{s+1}(\Gamma)$ for every pseudo-differential operator Q of order -1. It follows from Lemmas 1 and 2 that if $u \in H_s(\Gamma)$, then αu and $[\alpha, S - S^*]u$ belong to $\mathscr{F}_{s+1/2}$. Hence, for every $u \in H_s(\Gamma)$, we have $Hu \in \mathscr{F}_{s+1/2}$ and so $KHu \in \mathscr{U}_{s+1/2}$ by Theorem 3. Thus the operator KH is a continuous mapping of $H_s(\Gamma)$ into $H_{s+1/2}(\Gamma)$. Q.E.D.

Now we shall apply the Riesz-Schauder theory to the operator KH. Let $s \geqq 3/2$. Then the only solution in $H_s(\Gamma)$ of the equation $(1 - KH)u = 0$ is null solution. In fact, the u satisfies also the equation $(\alpha S + \beta)u = 0$ and hence the solution U of the Dirichlet problem (2) with $\varphi = u$ is belonging to $H_{s+1/2}(\Omega)$. It follows from Proposition 2 that U satisfies (4) with $f = 0$. Therefore we can assert, by Proposition 1, $U = 0$ or $u = 0$, since $s + 3/2 \geqq 2$. The Riesz-Schauder theorey guarantees that, for every $f \in H_s(\Gamma)$ $(s \geqq 3/2)$, there exists one and only one $u \in H_s(\Gamma)$ such that

$$(1 - KH)u = Kf$$

(note that $Kf \in H_s(\Gamma)$) and that the correspondence $f \to u$ is a continuous mapping of $H_s(\Gamma)$ into itself. Thus we can conclude that the u is the only solution in $H_s(\Gamma)$ of the equation $(\alpha S + \beta)u = f$. This completes the proof of Theorem 1.

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