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Abstract

Boij–Söderberg theory characterizes syzygies of graded modules and sheaves on projective space. This paper continues earlier work with Sam, extending the theory to the setting of GL_k-equivariant modules and sheaves on Grassmannians. Algebraically, we study modules over a polynomial ring in kn variables, thought of as the entries of a $k \times n$ matrix. We give equivariant analogs of two important features of the ordinary theory: the Herzog–Kühl equations and the pairing between Betti and cohomology tables. As a necessary step, we also extend previous results, concerning the base case of square matrices, to cover complexes other than free resolutions. Our statements specialize to those of ordinary Boij–Söderberg theory when $k = 1$. Our proof of the equivariant pairing gives a new proof in the graded setting: it relies on finding perfect matchings on certain graphs associated to Betti tables and to spectral sequences. As an application, we construct three families of extremal rays on the Betti cone for 2×3 matrices.

1. Introduction

1.1 Boij–Söderberg theory

Let $R = \mathbb{C}[x_1, \ldots, x_n]$ be a polynomial ring and M a graded, finitely generated R-module. The *Betti table* $\beta(M)$ is the list of degrees of the minimal generators, relations, and higher-order relations of M. These numbers encode much of the algebraic structure of M, such as its dimension and whether or not it is Cohen–Macaulay, along with geometric properties of the associated sheaf on $\mathbb{P}(\mathbb{C}^n)$.

Boij–Söderberg theory is a partial structure theory for Betti tables, concerned with describing which tables of numbers β arise as Betti tables of modules. The key early observation was that it is easier to determine which tables arise up to scalar multiple, and so the initial results [BS08, ES09, EFW11] consisted of characterizing the *Boij–Söderberg cone* BS_n of positive scalar multiples of Betti tables:

$$BS_n := \mathbb{Q}_{\geqslant 0} \cdot \{\beta(M) : M \text{ a finitely generated graded } R\text{-module}\}.$$

The theory has rapidly expanded to other settings. More recent work has considered modules over multigraded and toric rings [EE17, BES17], and some homogeneous coordinate rings [BBEG12, GS16, KS15], as well as more detailed homological questions [NS13, BEKS13, EES13]. A good survey of the field is [Flø12].

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In each of these cases, an important feature of the theory is a duality [ES09, EE17] between Betti tables and *cohomology tables* of sheaves on an associated variety. For a coherent sheaf $\mathcal{E}$ on $\mathbb{P}(\mathbb{C}^n)$, the cohomology table is the collection of numbers

$$\gamma_{ij}(\mathcal{E}) := \dim_{\mathbb{C}} H^i(\mathcal{E}(-j)),$$

giving all the sheaf cohomology of all the twists of $\mathcal{E}$. We write ES_n for the cone of such tables (the Eisenbud–Schreyer cone).

For graded modules and projective spaces, the duality takes the form of a bilinear pairing of the cones of Betti and cohomology tables, and it produces a point in the simplest Boij–Söderberg cone BS_1 (technically its derived analog BS_1^D):

$$BS_n \times ES_n \xrightarrow{\langle -,- \rangle} BS_1^D.$$

It says, essentially, that the dot product of a Betti table and cohomology table gives another Betti table – over the smallest graded ring $\mathbb{C}[x_1]$. The inequalities defining BS_1^D, which are simple to describe, therefore pull back to nonnegative bilinear pairings between Betti and cohomology tables. Moreover, these pulled-back inequalities turn out to fully characterize the two cones in the sense that

$$\beta \in BS_n \text{ if and only if } \langle \beta, \gamma \rangle \in BS_1^D \text{ for all } \gamma \in ES_n,$$

and similarly for the dual cone.

In particular, the Boij–Söderberg cone (in the graded setting) is rational polyhedral, and its extremal rays and supporting hyperplanes are explicitly known, with a combinatorial structure related to Young's lattice $\mathbb{Y}$ of partitions with at most k parts. The extremal rays correspond to *pure Betti tables*. These are the simplest possible tables, having only one nonzero entry in each column (that is, for each i, only one β_{ij} is nonzero). Similarly, the supporting hyperplanes come from pairing Betti tables with vector bundles having so-called *supernatural cohomology tables*. In other settings, however, analogous statements on the explicit structure of the Boij–Söderberg cone are not known.

1.2 Grassmannian Boij–Söderberg theory

The goal of this paper is to continue earlier work of the authors, joint with Sam [FLS18], on extending the theory to the setting of Grassmannians $\mathrm{Gr}(k, \mathbb{C}^n)$. On the geometric side, we will be interested in the cohomology of coherent sheaves on $\mathrm{Gr}(k, \mathbb{C}^n)$. On the algebraic side, we consider the polynomial ring in kn variables ($k \leqslant n$),

$$R_{k,n} = \mathbb{C}\left[x_{ij} : \begin{matrix} 1 \leqslant i \leqslant k \\ 1 \leqslant j \leqslant n \end{matrix}\right],$$

thought of as the entries of a $k \times n$ matrix. The group GL_k acts on $R_{k,n}$, and we are interested in (finitely generated) *equivariant modules* M, that is, those with a compatible GL_k-action.

Beyond the inherent interest of understanding sheaf cohomology and syzygies on Grassmannians, there is hope that this setting might avoid some obstacles faced in other extensions of Boij–Söderberg theory (e.g. to products of projective spaces). For example, in the 'base case' of square matrices ($n = k$), the 'irrelevant ideal' is the principal ideal generated by the determinant, and the Boij–Söderberg cone has an especially elegant structure (see below, § 1.3.2).

We define *equivariant Betti tables* $\beta(M)$ using the representation theory of GL_k. We write $\mathbb{S}_\lambda(\mathbb{C}^k)$ for the irreducible GL_k representation of weight λ, and $\mathbb{S}_\lambda$ for the corresponding Schur

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functor. There is a corresponding free module, namely $\mathbb{S}_\lambda(\mathbb{C}^k) \otimes_{\mathbb{C}} R_{k,n}$, and every equivariant free $R_{k,n}$-module is a direct sum of these. Then $\beta(M)$ is the collection of numbers

$$\beta_{i,\lambda}(M) := \#\text{ copies of } \mathbb{S}_\lambda(\mathbb{C}^k) \text{ in the generators of the } i\text{th syzygy module of } M.$$

Thus, by definition, the minimal *equivariant* free resolution of M has the form

$$M \leftarrow F_0 \leftarrow \cdots \leftarrow F_{kn} \leftarrow 0 \quad \text{with } F_i = \bigoplus_\lambda \mathbb{S}_\lambda(\mathbb{C}^k)^{\beta_{i\lambda}} \otimes R_{k,n}.$$

Next, for $\mathcal{E}$ a coherent sheaf on $\mathrm{Gr}(k, \mathbb{C}^n)$, we will define the GL-*cohomology table* $\gamma(\mathcal{E})$, generalizing the usual cohomology table:

$$\gamma_{i,\lambda}(\mathcal{E}) := \dim H^i(\mathcal{E} \otimes \mathbb{S}_\lambda(\mathcal{S})),$$

where $\mathcal{S}$ is the tautological vector bundle on $\mathrm{Gr}(k, \mathbb{C}^n)$ of rank k. As is the case on projective spaces, these numbers detect properties of $\mathcal{E}$ such as global generation and whether $\mathcal{E}$ has a linear locally free resolution, along with more specific geometric applications, such as to deformation theory [Cha99].

We write $\mathbb{BT}_{k,n} := \bigoplus_{i,\lambda} \mathbb{Q}$ for the space of abstract Betti tables. We allow $i \in \mathbb{Z}$ in this definition, since (below) we also consider Betti tables of complexes. However, for a Betti table of a module we will have $0 \leqslant i \leqslant kn$. We write $\mathbb{CT}_{k,n} := \bigoplus_{i=0}^{k(n-k)} \prod_\lambda \mathbb{Q}$ for the space of abstract GL-cohomology tables.

Remark 1.1. The case $k = 1$ reduces to the ordinary Boij–Söderberg theory, since an action of GL_1 is formally equivalent to a grading; the module $R(-j)$ is just $\mathbb{S}_{(j)}(\mathbb{C}) \otimes R$. Note also that $\mathcal{S} = \mathcal{O}(-1)$ on projective space.

The initial questions of Boij–Söderberg theory concerned finite-length graded modules M, that is, those annihilated by a power of the homogeneous maximal ideal, and more generally Cohen–Macaulay modules. Similarly, we restrict our focus (for now!) to the following class of modules, which specializes to finite-length modules when $k = 1$.

Condition 1.2 (The modules of interest). We consider finitely generated GL_k-equivariant Cohen–Macaulay modules M such that $\sqrt{\mathrm{ann}(M)} = P_k$, the ideal of maximal minors of the $k \times n$ matrix.

Viewing $\mathrm{Spec}(R_{k,n}) = \mathrm{Hom}(\mathbb{C}^k, \mathbb{C}^n)$ as the affine variety of $k \times n$ matrices, this means M is set-theoretically supported on the locus of rank-deficient matrices. That is, the sheaf $\widetilde{M}$ associated to M on $\mathrm{Gr}(k, \mathbb{C}^n)$ is zero. (See Remark 2.1.) For this reason, we refer to P_k as the *irrelevant ideal* for this setting. The Cohen–Macaulayness assumption means that

$$\mathrm{pdim}(M) = \dim(R_{P_k}) = n - k + 1,$$

so its minimal free resolution has length $n - k + 1$.

Definition 1.3. We define the *equivariant Boij–Söderberg cone* $BS_{k,n} \subset \mathbb{BT}_{k,n}$ as the positive linear span of Betti tables $\beta(M)$, where M satisfies the assumptions of Condition 1.2. We define the *Eisenbud–Schreyer cone* $ES_{k,n} \subset \mathbb{CT}_{k,n}$ as the positive linear span of GL-cohomology tables of all coherent sheaves $\mathcal{E}$ on $\mathrm{Gr}(k, \mathbb{C}^n)$.

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We wish to understand the cones $BS_{k,n}$ and $ES_{k,n}$ generated by equivariant Betti tables and GL-cohomology tables.

Remark 1.4 (Multiplicities and ranks). The irreducible representations $\mathbb{S}_\lambda(\mathbb{C}^k)$ need not be one-dimensional. As such, the corresponding free modules need not have rank 1. We will write a tilde $\widetilde{\beta}$ to denote the *rank* of the λ-isotypic component (rather than its *multiplicity* β), and likewise write $\widetilde{\mathbb{BT}}_{k,n}$ and $\widetilde{BS}_{k,n}$ for the spaces of rank Betti tables. We may switch between ranks and multiplicities by rescaling, that is, for each λ we have $\widetilde{\beta_{i,\lambda}} = \beta_{i,\lambda} \cdot \dim(\mathbb{S}_\lambda(\mathbb{C}^k))$. Both notions are useful, notably in defining the pairing between Betti and cohomology tables (Definition 1.11).

1.3 Results of this paper

We will generalize two important results from the existing theory on graded modules: the Herzog–Kühl equations and the pairing between Betti and cohomology tables. Along the way, we also extend our existing result on equivariant modules for the square matrices.

1.3.1 *Equivariant Herzog–Kühl equations.* In the graded setting, the Herzog–Kühl equations are n linear conditions satisfied by the Betti tables of finite-length modules M. These conditions say that the Hilbert polynomial of M vanishes identically, that is, each of its coefficients is zero. We give the following equivariant analog.

Theorem 1.5. *Let M be an equivariant $R_{k,n}$-module with Betti table $\beta(M)$. Twisting by $\det(\mathbb{C}^k)$ if necessary (see § 2.2), assume M is generated in positive degree, that is, all representations occurring among the generators of M are indexed by partitions $\lambda \geqslant 0$.*

There is a system of $\binom{n}{k}$ linear conditions on $\beta(M)$, indexed by partitions $\mu \geqslant 0$ that fit inside a $k \times (n-k)$ rectangle, called the equivariant Herzog–Kühl equations. The following are equivalent:

- (i) *$\beta(M)$ satisfies the equivariant Herzog–Kühl equations;*
- (ii) *M is annihilated by a power of the ideal P_k of maximal minors;*
- (iii) *the sheaf $\widetilde{M}$ associated to M on $\mathrm{Gr}(k, \mathbb{C}^n)$ vanishes.*

In particular, the hypotheses of Condition 1.2 are equivalent to the equivariant Herzog–Kühl equations, together with the conditions $\beta_{i,\lambda} = 0$ for all $i > n-k+1$.

An important application is a method to prove that certain 'sparse' Betti tables are extremal on $BS_{k,n}$. See § 6.1 and the rest of that section for examples.

We state and prove the equations in § 3.2, using the combinatorics of standard Young tableaux. Our approach is by equivariant K-theory: namely, that the condition $\widetilde{M} = 0$ on $\mathrm{Gr}(k, \mathbb{C}^n)$ says that the K-theory class of M lies in the kernel of the map

$$K^{\mathrm{GL}_k}(\mathrm{Spec}(R_{k,n})) \to K(\mathrm{Gr}(k, \mathbb{C}^n))$$

induced by restriction to the locus of full-rank matrices and descent.

1.3.2 *The Boij–Söderberg cone for square matrices.* In the graded setting (i.e. when $k = 1$), the case $n = 1$ plays an important role as it is the target of the Boij–Söderberg pairing, and is simple to understand. For general k, we expect the smallest case $n = k$ to play a comparable role. It serves as the base case of the theory and the target of the equivariant Boij–Söderberg pairing (§ 1.3.3). Since the corresponding Grassmannian is just a point, this case is purely algebraic. Rank tables $\widetilde{\beta}$ turn out to be more significant here, so we will state results in terms of the cone $\widetilde{BS}_{k,k}$.

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For square matrices, the modules of interest are Cohen–Macaulay and have $\sqrt{\mathrm{ann}(M)} = (\det)$, the ideal generated by the determinant, so they have projective dimension 1. The cone $\widetilde{BS}_{k,k}$ is fully understood in terms of the extended Young's lattice $\mathbb{Y}_\pm$ of length-k weakly decreasing sequences of integers.

Theorem 1.6 [FLS18, Theorem 1.2]. *The cone $\widetilde{BS}_{k,k}$ is rational polyhedral. Its supporting hyperplanes are indexed by* order ideals *in the extended Young's lattice $\mathbb{Y}_\pm$. Its extremal rays are indexed by* comparable pairs $\lambda \subsetneq \mu$ *from $\mathbb{Y}_\pm$. These rays correspond to* pure tables *with $\widetilde{\beta_{0,\lambda}} = \widetilde{\beta_{1,\mu}} = 1$ and all other entries zero. Up to scaling, these tables come from free resolutions of the form*

$$M \leftarrow \mathbb{S}_\lambda(\mathbb{C}^k)^{\oplus c_0} \otimes R \leftarrow \mathbb{S}_\mu(\mathbb{C}^k)^{\oplus c_1} \otimes R \leftarrow 0,$$

with all generators in type λ and all syzygies in type μ.

We will need a slightly more general result for the purposes of the equivariant Boij–Söderberg pairing, a derived analog to $\widetilde{BS}_{k,n}$.

Definition 1.7. The *derived Boij–Söderberg cone*, denoted $\widetilde{BS}^D_{k,n}$, is the positive linear span of (rank) Betti tables of bounded minimal complexes $F_\bullet$ of equivariant free modules, such that $F_\bullet$ is exact away from the locus of rank-deficient matrices.

In this definition, we assume only that the homology modules M have $\sqrt{\mathrm{ann}(M)} \supseteq P_k$, not that equality holds. We also do not assume Cohen–Macaulayness. Thus, $\widetilde{BS}^D_{k,k}$ includes, for example, homological shifts of elements of $\widetilde{BS}_{k,k}$, and Betti tables of longer complexes. The simplest tables in the derived cone are *homologically shifted pure tables*, written $\widetilde{\beta}\big[\lambda \xleftarrow{i} \mu\big]$, for $i \in \mathbb{Z}$ and $\lambda \subsetneq \mu$. These are the tables with $\widetilde{\beta_{i,\lambda}} = \widetilde{\beta_{i+1,\mu}} = 1$ and all other entries zero. We show the following theorem.

Theorem 1.8. *The cone $\widetilde{BS}^D_{k,k}$ is rational polyhedral. Its extremal rays are the homological shifts of those of $\widetilde{BS}_{k,k}$, spanned by the tables $\widetilde{\beta}\big[\lambda \xleftarrow{i} \mu\big]$. The supporting hyperplanes are indexed by tuples $(\ldots, S_{-1}, S_1, S_3, \ldots)$ of convex subsets $S_i \subseteq \mathbb{Y}_\pm$, one chosen for every other spot along the complex.*

The key idea in the above theorem is that these Betti tables are characterized by certain perfect matchings. This idea is also crucial in our construction of the pairing between Betti and cohomology tables, so we discuss it now. We introduce a graph-theoretic model of a rank Betti table (this construction was implicit in [FLS18, Lemma 3.6] for free resolutions).

Definition 1.9 (Betti graphs). Let $\widetilde{\beta} \in \widetilde{\mathbb{BT}}_{k,k}$ have nonnegative integer entries. The *Betti graph* $G(\widetilde{\beta})$ is defined as follows:

- the vertex set contains $\widetilde{\beta_{i,\lambda}}$ vertices labeled (i,λ), for each (i,λ);
- the edge set contains, for each i, all possible edges $(i,\lambda) \leftarrow (i+1,\mu)$ with $\lambda \subsetneq \mu$.

Note that this graph is bipartite: every edge connects an even- and an odd-indexed vertex.

Recall that a *perfect matching* on a graph G is a subset of its edges, such that every vertex of G appears on exactly one chosen edge. A perfect matching on $G(\widetilde{\beta})$ is equivalent to a decomposition of $\widetilde{\beta}$ as a positive integer combination of homologically shifted pure tables: an edge $(i,\lambda) \leftarrow (i+1,\mu)$ corresponds to a pure summand $\widetilde{\beta}\big[\lambda \xleftarrow{i} \mu\big]$. Thus, an equivalent characterization of $\widetilde{BS}^D_{k,k}$ is as follows.

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Theorem 1.10. *Let $\widetilde{\beta} \in \mathbb{BT}_{k,k}$ have nonnegative integer entries. Then $\widetilde{\beta} \in \widetilde{BS}^D_{k,k}$ if and only if $G(\widetilde{\beta})$ has a perfect matching.*

Our proof proceeds by exhibiting this perfect matching using homological algebra. The supporting hyperplanes of $\widetilde{BS}^D_{k,k}$ then follow from Hall's matching theorem; see § 4.2 for the precise statement.

1.3.3 *The pairing between Betti tables and cohomology tables.* We now turn to the Boij–Söderberg pairing. This will be a bilinear pairing between abstract Betti tables β and cohomology tables γ, satisfying certain nonnegativity properties when restricted to realizable tables.

Definition 1.11. Let $\beta \in \mathbb{BT}_{k,n}$ and $\gamma \in \mathbb{CT}_{k,n}$ be an abstract Betti table and GL-cohomology table. The *equivariant Boij–Söderberg pairing* is given by

$$\begin{aligned} \widetilde{\Phi} : \mathbb{BT}_{k,n} \times \mathbb{CT}_{k,n} &\to \widetilde{\mathbb{BT}}_{k,k}, \\ (\beta, \gamma) &\mapsto \widetilde{\Phi}(\beta, \gamma), \end{aligned} \tag{1.1}$$

with $\widetilde{\Phi}$ the (derived) *rank* Betti table with entries

$$\widetilde{\varphi_{i,\lambda}}(\beta, \gamma) = \sum_{p-q=i} \beta_{p,\lambda} \cdot \gamma_{q,\lambda}. \tag{1.2}$$

In this definition, recall that the homological index of a complex decreases under the boundary map.

Here is how to read the definition of $\widetilde{\Phi}$. (See Example 1.14 below.) Form a grid in the first quadrant of the plane, whose (p, q) entry is the collection of numbers $\beta_{p,\lambda} \cdot \gamma_{q,\lambda}$ for all λ. Only finitely many of these are nonzero. The line $p - q = i$ is an upward-sloping diagonal through this grid, and $\widetilde{\varphi_{i,\lambda}}$ is the sum of the λ terms along this diagonal.

Remark 1.12. We emphasize that the pairing takes a *multiplicity* Betti table β and a cohomology table γ, and produces a *rank* Betti table $\widetilde{\Phi}$. Intuitively, the entries of γ are dimensions of certain vector spaces (from sheaf cohomology), which, we will see, arise with multiplicities given by β in a certain spectral sequence. In particular, the quantities in (1.2) are again dimensions of vector spaces – that is, they give a rank table.

The final main result of this paper is the nonnegativity of the pairing.

Theorem 1.13 (Pairing the equivariant cones). *The pairing $\widetilde{\Phi}$ restricts to a map of cones,*

$$BS_{k,n} \times ES_{k,n} \to \widetilde{BS}^D_{k,k}.$$

The same is true with $BS_{k,n}$ replaced by $BS^D_{k,n}$ on the source.

In particular, the defining inequalities of the cone $\widetilde{BS}^D_{k,k}$ (which we give explicitly) pull back to nonnegative bilinear pairings of Betti and cohomology tables, and the Betti graph of $\widetilde{\Phi}(\beta, \gamma)$ has a perfect matching. We think of this as a reduction to the base case of square matrices $(k = n)$. A geometric consequence is that each equivariant Betti table induces many interesting linear inequalities constraining sheaf cohomology on $\mathrm{Gr}(k, \mathbb{C}^n)$. The pairing similarly constrains the possible Betti tables (see, for example, Proposition 6.2 and Example 6.6).

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Our proof proceeds by constructing a perfect matching on $\widetilde{\Phi}(\beta, \gamma)$, not by passing to actual modules over $R_{k,k}$. It would be interesting to see a 'categorified' form of the pairing, in the style of Eisenbud and Erman [EE17]. Such a pairing would construct, from a complex $F_\bullet$ of $R_{k,n}$-modules and a sheaf $\mathcal{E}$, a module (or complex) over $R_{k,k}$. Theorem 1.13 would follow from showing that this module is supported along the determinant locus (or that the complex is exact away from the determinant locus). This obstruction to this approach seems to be that GL_k is nonabelian when $k > 1$. The authors welcome any communication or ideas in this direction.

Example 1.14. Let us pair the following tables for $k = 2$, $n = 3$:

$$\begin{array}{c|ccc} \beta_{p,\lambda} & 0 & 1 & 2 \\ \hline \ydiagram{1} & 4 & - & - \\ \ydiagram{2} & - & 1 & - \\ \ydiagram{1,1} & - & 9 & - \\ \ydiagram{2,1} & - & 3 & 3 \\ \ydiagram{3,1} & - & - & 1 \\ \ydiagram{2,2} & - & - & 1 \end{array} \quad \times \quad \begin{array}{c|ccc} \gamma_{q,\lambda} & 0 & 1 & 2 \\ \hline \ydiagram{1} & 3 & 1 & - \\ \ydiagram{2} & - & 3 & - \\ \ydiagram{1,1} & 1 & - & - \\ \ydiagram{2,1} & - & - & - \\ \ydiagram{3,1} & - & 3 & - \\ \ydiagram{2,2} & - & - & 1 \end{array} \tag{1.3}$$

Both are realizable; the cohomology table is for the sheaf $\mathcal{E} = \mathcal{O}(1) \oplus \mathcal{O}(-1)$. We arrange the pairwise products in a first-quadrant grid. The sums along the diagonals $\{p - q = i\}$ result in the rank Betti table $\widetilde{\Phi}$:

$$\begin{array}{c|ccc} & - & - & 1 \cdot \ydiagram{2,2} \\ q & 4 \cdot \ydiagram{1} & 3 \cdot \ydiagram{2} & 3 \cdot \ydiagram{3,1} \\ & 12 \cdot \ydiagram{1} & 9 \cdot \ydiagram{1,1} & - \\ \hline & & p & \end{array} \quad \rightsquigarrow \quad \begin{array}{c|ccc} \widetilde{\varphi_{i,\lambda}} & -1 & 0 & 1 \\ \hline \ydiagram{1} & 4 & 12 & - \\ \ydiagram{2} & - & 3 & - \\ \ydiagram{1,1} & - & - & 9 \\ \ydiagram{3,1} & - & - & 3 \\ \ydiagram{2,2} & - & 1 & - \end{array} \tag{1.4}$$

Finally, we check that $\widetilde{\Phi} \in \widetilde{BS}^D_{k,k}$. The decomposition of $\widetilde{\Phi}$ into pure tables happens to be unique (this is not true in general):

$$\widetilde{\Phi} = 3\, \widetilde{\beta}[\ydiagram{1} \xleftarrow{-1} \ydiagram{2}] + \widetilde{\beta}[\ydiagram{1} \xleftarrow{-1} \ydiagram{2,2}] + 9\, \widetilde{\beta}[\ydiagram{1} \xleftarrow{0} \ydiagram{1,1}] + 3\, \widetilde{\beta}[\ydiagram{1} \xleftarrow{0} \ydiagram{3,1}].$$

This corresponds to an essentially unique perfect matching on $G(\widetilde{\Phi})$.

1.4 Structure of the paper

Section 2 contains background on algebra and representation theory. Sections 3, 4, and 5 respectively establish the equivariant Herzog–Kühl equations, the results on square matrices, and the pairing of Betti and cohomology tables. Finally, § 6 gives examples and applications of our results in the case $k = 2, n = 3$.

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2. Background

2.1 Spaces of interest

Throughout, let V, W be fixed $\mathbb{C}$-vector spaces of dimensions k and n, with $k \leqslant n$. We set

$$X = \operatorname{Hom}(V, W), \quad R_{k,n} = \operatorname{Sym}(\operatorname{Hom}(V, W)^*) \cong \mathbb{C}\left[x_{ij} : \begin{matrix} 1 \leqslant i \leqslant k \\ 1 \leqslant j \leqslant n \end{matrix}\right],$$

so $X = \operatorname{Spec}(R_{k,n})$, the affine variety of $k \times n$ matrices (x_{ij}), and $R_{k,n}$ is the polynomial ring whose variables are the entries of the matrix. We also consider the subvarieties of full-rank and rank-deficient matrices,

$$U = \operatorname{Emb}(V, W) = \{T : \ker(T) = 0\}, \quad X_{k-1} = X - U,$$

which are open and closed, respectively. The locus X_{k-1} is integral and has codimension $n-k+1$. Its prime ideal P_k is generated by the $\binom{n}{k}$ maximal minors Δ_J of the $k \times n$ matrix, one for each k-tuple $J \subset [n]$. Each of the spaces X, $R_{k,n}$, U, and X_{k-1} has an action of $\mathrm{GL}(V)$ and $\mathrm{GL}(W)$; we will primarily care about the $\mathrm{GL}(V)$-action.

2.2 GL-representation theory

A good introduction to these notions is [Ful96]. The irreducible algebraic representations of $\mathrm{GL}(V)$ are indexed by weakly decreasing integer sequences $\lambda = (\lambda_1 \geqslant \cdots \geqslant \lambda_k)$, where $k = \dim(V)$. We write $\mathbb{S}_\lambda(V)$ for the corresponding representation, and $d_\lambda(k)$ for its dimension. We call $\mathbb{S}_\lambda$ a *Schur functor*. If λ has all nonnegative parts, we write $\lambda \geqslant 0$ and say λ is a *partition*. In this case, $\mathbb{S}_\lambda(V)$ is functorial for linear transformations $V \to W$. If λ has negative parts, $\mathbb{S}_\lambda$ is only functorial for isomorphisms $V \xrightarrow{\sim} W$.

We often represent partitions by their Young diagrams:

$$\lambda = (3,1) \longleftrightarrow \lambda = \ydiagram{3,1}.$$

We write $\boxplus$ for the rectangular partition $(n-k)^k$, with k rows and $n-k$ columns. We partially order partitions and integer sequences by containment:

$$\lambda \subseteq \mu \quad \text{if } \lambda_i \leqslant \mu_i \text{ for all } i.$$

We write $\mathbb{Y}$ for the poset of partitions with at most k parts, with this ordering, called *Young's lattice*. We write $\mathbb{Y}_\pm$ for the set of length-k weakly decreasing integer sequences; we call it the *extended Young's lattice*. Schur functors include symmetric and exterior powers:

$$\lambda = d\left\{\ydiagram{1,1,1,1}\right. \iff \mathbb{S}_\lambda(V) = \bigwedge^d(V),$$

$$\lambda = \overbrace{\ydiagram{4}}^{d} \iff \mathbb{S}_\lambda(V) = \operatorname{Sym}^d(V).$$

We will write $\det(V)$ for the one-dimensional representation $\bigwedge^{\dim(V)}(V) = \mathbb{S}_{1^k}(V)$. We may always twist a representation by powers of the determinant:

$$\det(V)^{\otimes a} \otimes \mathbb{S}_{\lambda_1,\ldots,\lambda_k}(V) = \mathbb{S}_{\lambda_1+a,\ldots,\lambda_k+a}(V)$$

for any integer $a \in \mathbb{Z}$. This operation is invertible and can sometimes be used to reduce to considering the case when λ is a partition.

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2.3 Equivariant rings and modules

If R is a $\mathbb{C}$-algebra with an action of $\mathrm{GL}(V)$ and S is any $\mathrm{GL}(V)$-representation, then $S \otimes_{\mathbb{C}} R$ is an *equivariant free R-module*; it has the universal property

$$\mathrm{Hom}_{\mathrm{GL}(V),R}(S \otimes_{\mathbb{C}} R, M) \cong \mathrm{Hom}_{\mathrm{GL}(V)}(S, M)$$

for all equivariant R-modules M. The basic examples will be the modules $\mathbb{S}_\lambda(V) \otimes R$.

Let $R = R_{k,n}$ be the polynomial ring defined above. Its structure as a $\mathrm{GL}(V) \times \mathrm{GL}(W)$ representation is known as the *Cauchy identity*:

$$R_{k,n} = \mathrm{Sym}^\bullet(\mathrm{Hom}(V, W)^*) \cong \bigoplus_{\lambda \geqslant 0} \mathbb{S}_\lambda(V) \otimes \mathbb{S}_\lambda(W^*).$$

Note that the prime ideal P_k and the maximal ideal $\mathfrak{m} = (x_{ij})$ of the zero matrix are $\mathrm{GL}(V)$- and $\mathrm{GL}(W)$-equivariant.

Let M be a finitely generated $\mathrm{GL}(V)$-equivariant R-module. The module $\mathrm{Tor}^i_R(R/\mathfrak{m}, M)$ naturally has the structure of a finite-dimensional $\mathrm{GL}(V)$-representation. We define the *equivariant Betti number* $\beta_{i,\lambda}(M)$ as the multiplicity of the Schur functor $\mathbb{S}_\lambda(V)$ in this Tor module, that is,

$$\mathrm{Tor}^i_R(R/\mathfrak{m}, M) \;\cong\; \bigoplus_\lambda \mathbb{S}_\lambda(V)^{\oplus \beta_{i,\lambda}(M)} \quad \text{(as } \mathrm{GL}(V)\text{-representations)}.$$

By semisimplicity of $\mathrm{GL}(V)$-representations, any minimal free resolution of M can be made equivariant, so we may instead define $\beta_{i,\lambda}$ as the multiplicity of the equivariant free module $\mathbb{S}_\lambda(V) \otimes R$ in the ith step of an equivariant minimal free resolution of M:

$$M \leftarrow F_0 \leftarrow F_1 \leftarrow \cdots \leftarrow F_d \leftarrow 0 \quad \text{where } F_i = \bigoplus_\lambda \mathbb{S}_\lambda(V)^{\beta_{i,\lambda}(M)} \otimes R.$$

All other notation on Betti tables is as defined in § 1.2.

Remark 2.1 (Descending from $\mathrm{Hom}(V, W)$ to $\mathrm{Gr}(k, W)$). There is an exact functor from $\mathrm{GL}(V)$-equivariant R-modules to quasicoherent sheaves on $\mathrm{Gr}(k, W)$. For $k = 1$, it is the well-known tilde construction, which turns a graded module into a sheaf on $\mathbb{P}^n$. For $k > 1$, the construction is analogous: given M as defined above, we consider, for each minor Δ_J of the matrix, the module $R[1/\Delta_J] \otimes M$. The submodule of invariants $(R[1/\Delta_J] \otimes M)^{\mathrm{GL}(V)}$ lives on the chart $\{\Delta_J \neq 0\}$ of $\mathrm{Gr}(k, W)$, and it is straightforward to check that these modules glue to form a sheaf, denoted $\widetilde{M}$. (This is also a particularly simple case of geometric invariant theory [MF82], using the line bundle $\det(V^*) \times \mathrm{Hom}(V, W)$.) Note that $\widetilde{V \otimes R} = \mathcal{S}$, the tautological rank-$k$ bundle. The assignment $M \mapsto \widetilde{M}$ commutes with tensor operations; in particular, $\widetilde{\mathbb{S}_\lambda(V) \otimes R} = \mathbb{S}_\lambda(\mathcal{S})$.

3. The equivariant Herzog–Kühl equations

In this section we derive the equivariant analog of the Herzog–Kühl equations. This will be a system of linear conditions on the entries of an equivariant Betti table. It will detect when the resolved module M is supported only along the locus of rank-deficient matrices. An important application of these equations is a method to prove that certain Betti tables are extremal on $BS_{k,n}$ (see § 6.1 for discussion and several examples).

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3.1 K-theory rings

For background on equivariant K-theory, we refer to the original paper by Thomason [Tho87]; a more recent discussion is [Mer05].

Excision in equivariant K-theory [Tho87, Theorem 2.7] gives the right-exact sequence of abelian groups

$$K^{\mathrm{GL}(V)}(X_{k-1}) \xrightarrow{i_*} K^{\mathrm{GL}(V)}(\mathrm{Hom}(V,W)) \xrightarrow{j^*} K^{\mathrm{GL}(V)}(U) \to 0.$$

The pullback j^*, induced by the open inclusion $j : U \hookrightarrow X$, is a map of rings. The pushforward i_*, induced by the closed embedding $i : X_{k-1} \hookrightarrow X$, is only a map of abelian groups. Its image is the ideal I generated by the classes of modules supported along the rank-deficient locus X_{k-1}.

We do not attempt to describe the first term. For the second term, we have ([Tho87, Theorem 4.1] or [Mer05, Example 2 and Corollary 12])

$$K^{\mathrm{GL}(V)}(\mathrm{Hom}(V,W)) \cong \mathbb{Z}[t_1^{\pm}, \ldots, t_k^{\pm}]^{S_k},$$

the ring of symmetric Laurent polynomials in k variables, essentially the representation ring of $\mathrm{GL}(V)$. Here, the class of the equivariant R-module $\mathbb{S}_\lambda(V) \otimes_{\mathbb{C}} R$ is identified with the Schur polynomial $s_\lambda(t_1, \ldots, t_k)$. We note that the exterior powers $\bigwedge^d(V) \otimes_{\mathbb{C}} R$ correspond to the elementary symmetric polynomials $e_d(t)$, while symmetric powers $\mathrm{Sym}^d(V) \otimes_{\mathbb{C}} R$ correspond to homogeneous symmetric polynomials $h_d(t)$.

If M is a finitely generated R-module, its equivariant minimal free resolution expresses the K-class $[M]$ as a finite alternating sum of Schur polynomials. In other words, the equivariant Betti table determines the K-class

$$[M] = \sum_{i,\lambda} (-1)^i \beta_{i,\lambda}(M) s_\lambda(t).$$

An equivalent approach is to write

$$M \cong \bigoplus_\lambda \mathbb{S}_\lambda(V)^{c_\lambda(M)} \quad \text{as a } \mathrm{GL}(V)\text{-representation},$$

and define the equivariant Hilbert series of M,

$$\begin{aligned} H_M(t) &= \sum_\lambda c_\lambda(M) s_\lambda(t) \\ &= \frac{f(t)}{\prod_{i=1}^k (1-t_i)^n} \end{aligned}$$

for some symmetric function $f(t)$. Then $f(t)$ is the K-theory class of M. (If we forget the $\mathrm{GL}(V)$-action and remember only the grading of M, we recover the usual Hilbert series.)

To see that these definitions agree, note that the second definition is additive in short exact sequences, hence is well-defined on K-classes. Replacing M by an equivariant minimal free resolution, it suffices to consider indecomposable free modules $M = \mathbb{S}_\lambda(V) \otimes_{\mathbb{C}} R$. This tensor product multiplies the entire series by $s_\lambda(t)$ and does the same to $f(t)$, so it suffices to consider the case $M = R$. For this case, we observe

$$R = \mathrm{Sym}(V \otimes W) \cong \mathrm{Sym}(\overbrace{V \oplus \cdots \oplus V}^{n \text{ times}}) = \mathrm{Sym}(V)^{\otimes n},$$

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so the equivariant Hilbert series of R is the nth power of the series for $\mathrm{Sym}(V)$, which is

$$\sum_{d \geqslant 0} h_d(t) = \prod_{i=1}^{k} \frac{1}{1-t_i},$$

where $h_d(t)$ is the dth homogeneous symmetric polynomial.

Remark 3.1. It will be convenient in this section to restrict attention to modules M generated in positive degree, that is, modules for which $\beta_{i,\lambda}(M) \neq 0$ implies $\lambda \geqslant 0$. In this case, the class of M is a polynomial, not a Laurent polynomial. We write $K_+^{\mathrm{GL}(V)}(\mathrm{Hom}(V,W))$ for this subring.

Finally, we have for the third term (cf. [Tho87, Proposition 6.2])

$$K^{\mathrm{GL}(V)}(U) \cong K(U/\mathrm{GL}(V)) = K(\mathrm{Gr}(k,W)),$$

because the action of $\mathrm{GL}(V)$ is free on U. Note that, under this correspondence, j^* sends the module M to the class of the induced sheaf $\widetilde{M}$ (see Remark 2.1). Notably, $j_*[\mathbb{S}_\lambda(V) \otimes R] = [\mathbb{S}_\lambda(\mathcal{S})]$. The ring structure of $K(\mathrm{Gr}(k,W))$ is well-known from K-theoretic Schubert calculus (e.g. [KK90] or [Buc02]). We will only need to know the following: it is a free abelian group with an additive basis consisting of $\binom{n}{k}$ generators, indexed by partitions μ fitting inside a $k \times (n-k)$ rectangle, that is, $\mu \subseteq \boxplus$. These correspond to the classes $[\mathcal{O}_\mu]$ of structuresheaves of Schubert varieties. It is easy to check that $K_+^{\mathrm{GL}(V)}(\mathrm{Hom}(V,W)) \to K(\mathrm{Gr}(k,W))$ is also surjective (because, for example, matrix Schubert varieties are generated in positive degree).

3.2 Modules on the rank-deficient locus and the equivariant Herzog-Kühl equations

From the surjection $K_+^{\mathrm{GL}(V)}(\mathrm{Hom}(V,W)) \to K(\mathrm{Gr}(k,W))$, we see that the ideal

$$I' := I \cap K_+^{\mathrm{GL}(V)}(\mathrm{Hom}(V,W)),$$

as a linear subspace, has co-rank $\binom{n}{k}$. We wish to find exactly this many linear equations cutting out the ideal, indexed appropriately by partitions. That is, given a K-class written in the Schur basis,

$$f = \sum_{\lambda \geqslant 0} a_\lambda s_\lambda \in K_+^{\mathrm{GL}(V)}(\mathrm{Hom}(V,W)),$$

we wish to have coefficients $b_{\lambda\mu}$ for each $\mu \subseteq \boxplus$, such that

$$f \in I' \text{ if and only if } \sum_{\lambda \geqslant 0} a_\lambda b_{\lambda\mu} = 0 \text{ for all } \mu \subseteq \boxplus .$$

We will then apply these equations in the case where f is the class of a module M, and

$$a_\lambda = \sum_i (-1)^i \beta_{i,\lambda}(M)$$

comes from the equivariant Betti table of M. Our approach is to prove the following. We write $1-t$ as shorthand for the tuple $(1-t_1, \ldots, 1-t_k)$ and we show the following theorem.

THEOREM 3.2. *We have* $I' = \mathrm{span}_{\mathbb{C}}\{s_\lambda(1-t) : \lambda \not\subseteq \boxplus\}$.

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We will prove Theorem 3.2 in the next section. Here is how it leads to the desired equations. Let $b_{\lambda\mu}$ be the change-of-basis coefficients defined by sending $t_i \mapsto 1 - t_i$. So, by definition,

$$s_\lambda(1-t) = \sum_\mu b_{\lambda\mu} s_\mu(t).$$

Note that we have, equivalently,

$$s_\lambda(t) = \sum_\mu b_{\lambda\mu} s_\mu(1-t).$$

Thus

$$f = \sum_\lambda a_\lambda s_\lambda(t) = \sum_{\lambda,\mu} a_\lambda b_{\lambda\mu} s_\mu(1-t).$$

The polynomials $s_\mu(1-t)$ for all $\mu \geqslant 0$ form an additive basis for the $K_+^{\mathrm{GL}(V)}(\mathrm{Hom}(V,W))$. Thus, $f \in I'$ if and only if the coefficient of $s_\mu(1-t)$ is 0 for all $\mu \subseteq \boxplus$. That is,

$$0 = \sum_\lambda a_\lambda b_{\lambda\mu} \quad \text{for all } \mu \subseteq \boxplus.$$

The following description of $b_{\lambda\mu}$ is due to Stanley. Recall that, if $\mu \subseteq \lambda$ are partitions, the *skew shape* λ/μ is the Young diagram of λ with the squares of μ deleted. A *standard Young tableau* is a filling of a (possibly skew) shape by the numbers $1, 2, \ldots, t$ (with t boxes in all), such that the rows increase from left to right, and the columns increase from top to bottom. We write f^σ for the number of standard Young tableaux of shape σ.

Proposition 3.3 [Sta99]. *If $\mu \not\subseteq \lambda$ then $b_{\lambda\mu} = 0$. If $\mu \subseteq \lambda$, then*

$$b_{\lambda\mu} = (-1)^{|\mu|} \frac{f^{\lambda/\mu} f^\mu}{f^\lambda} \binom{|\lambda|}{|\mu|} \frac{d_\lambda(k)}{d_\mu(k)}.$$

An equivalent formulation is

$$b_{\lambda\mu} = (-1)^{|\mu|} \frac{f^{\lambda/\mu}}{|\lambda/\mu|!} \prod_{(i,j)\in\lambda/\mu} (k + j - i).$$

Corollary 3.4 (Equivariant Herzog–Kühl equations). *Let M be an equivariant R-module with equivariant Betti table $\beta_{i,\lambda}$. Assume M is generated in positive degree.*

The set-theoretic support of M is contained in the rank-deficient locus if and only if,

$$\text{for each } \mu \subseteq \boxplus, \quad \sum_{i,\lambda\supseteq\mu} (-1)^i \underbrace{\beta_{i,\lambda} d_\lambda(k)}_{(=\widetilde{\beta_{i,\lambda}})} \frac{f^{\lambda/\mu} f^\mu}{f^\lambda} \binom{|\lambda|}{|\mu|} = 0. \tag{3.1}$$

Note that $\beta_{i,\lambda}$ is the multiplicity *of the λ-isotypic component of the resolution of M (in cohomological degree i), whereas $\beta_{i,\lambda} d_\lambda(k) = \widetilde{\beta_{i,\lambda}}$ is the* rank *of this isotypic component.*

Proof. ($\Rightarrow$) The only thing to note is that, for simplicity, we have rescaled the μ-indexed equation by $(-1)^{|\mu|} d_\mu(k)$.

($\Leftarrow$) If the equations are satisfied, then M maps to the trivial K-theory class on $\mathrm{Gr}(k, \mathbb{C}^n)$. Since the Grassmannian is projective, the induced sheaf $\widetilde{M}$ must be zero. (Explicitly, $\chi(\widetilde{M} \otimes \mathcal{O}(d)) = 0$ for all d. By ampleness, for $d \gg 0$, we get that $\widetilde{M} \otimes \mathcal{O}(d)$ is globally generated but has no H^0, so $\widetilde{M} = 0$.) This implies the support restriction. □

Example 3.5. The equation for $\mu = \emptyset$ is just $\sum_{i,\lambda} (-1)^i \widetilde{\beta_{i,\lambda}} = 0$, saying that the alternating sum of the ranks of the free modules vanishes (i.e. that M is torsion).

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Example 3.6 (Projective spaces). When $k=1$, the equivariant Herzog–Kühl equations specialize to the usual equations: they are indexed by the row partitions $(j): \square, \square\square, \square\square\square, \dots$ for $j=1,\dots,n$. We have $f^{\lambda/\mu}=f^{\mu}=f^{\lambda}=d_{\lambda}(1)=1$ (since the partitions are single rows), so the equation for $\mu=(j)$ just says

$$\sum_{i,d\geqslant j}(-1)^i\beta_{i,d}\binom{d}{j}=\sum_{i,d\geqslant j}(-1)^i\beta_{i,d}\left(\frac{1}{j!}d^j+\cdots\right)=0.$$

These are upper-triangular to the familiar Herzog–Kühl equations for graded modules,

$$\sum_{i,d}(-1)^i\beta_{i,d}\ d^j=0.$$

Example 3.7. See § 6.1 for the smallest new case, $k=2, n=3$.

The coefficient in equation (3.1) has the following interpretation. Consider a uniformly random filling T of the shape λ by the numbers $1,\dots,|\lambda|$. Say that T *splits along* $\mu\sqcup\lambda/\mu$ if the numbers $1,\dots,|\mu|$ lie in the subshape μ. Then:

$$\frac{f^{\lambda/\mu}f^{\mu}}{f^{\lambda}}\binom{|\lambda|}{|\mu|}=\frac{\mathrm{Prob}(T\text{ splits along }\mu\sqcup\lambda/\mu\mid T\text{ is standard})}{\mathrm{Prob}(T\text{ splits along }\mu\sqcup\lambda/\mu)}. \tag{3.2}$$

3.3 Proof of Theorem 3.2

First, we recall the following fact about K-theory of Grassmannians.

Proposition 3.8 [FS12, p. 21]. *The following identity holds of formal power series in the variable u over $K(\mathrm{Gr}(k,W))$:*

$$\left(\sum_p\left[\bigwedge{}^p\mathcal{S}\right]u^p\right)\cdot\left(\sum_q\left[\bigwedge{}^q\mathcal{Q}\right]u^q\right)=(1+u)^n.$$

It is a consequence of the tautological exact sequence of vector bundles on $\mathrm{Gr}(k,W)$,

$$0\to\mathcal{S}\to W\to\mathcal{Q}\to 0.$$

We rearrange the above identity as

$$\left(\sum_q\left[\bigwedge{}^q\mathcal{Q}\right]u^q\right)=(1+u)^n\cdot\frac{1}{(\sum_p[\bigwedge^p\mathcal{S}]u^p)}.$$

Recall that $\bigwedge^p\mathcal{S}$ is the sheaf on $\mathrm{Gr}(k,W)$ induced by the equivariant free module $\bigwedge^p(V)\otimes R$. Thus, viewing $K(\mathrm{Gr}(k,W))$ as a quotient of the ring of symmetric functions, $\bigwedge^p\mathcal{S}$ comes from the pth elementary symmetric polynomial $e_p(t)$. So, we consider the coefficients f_ℓ of u^ℓ in the analogous expression over $\mathbb{Z}[t_1^{\pm},\dots,t_k^{\pm}]^{S_k}$:

$$\begin{aligned}\sum_{\ell\geqslant 0}f_\ell u^\ell&=(1+u)^n\cdot\frac{1}{(\sum_p e_p(t)u^p)}\\&=(1+u)^n\cdot\prod_{i=1}^k\frac{1}{1+ut_i}\\&=(1+u)^n\sum_p(-1)^p h_p(t)u^p,\end{aligned}$$

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where h_p is the pth homogeneous symmetric polynomial. The key observation is that, in $K(\mathrm{Gr}(k,W))$, this expression becomes a polynomial in u of degree $n-k$: all higher terms vanish because $\mathcal{Q}$ has rank $n-k$. In particular, $f_\ell \in I'$ for $\ell > n-k$.

We compute the coefficient f_ℓ. We have

$$\begin{aligned}\sum_\ell f_\ell u^\ell &= (1+u)^n \sum_p (-1)^p h_p u^p \\ &= \sum_{q=0}^{n}\sum_{p=0}^{\infty} u^{p+q}(-1)^p h_p \binom{n}{q} \\ &= \sum_{\ell=0}^{\infty} u^\ell \sum_{p=\ell-n}^{\ell} (-1)^p h_p \binom{n}{\ell-p},\end{aligned}$$

so our desired coefficients are

$$f_\ell = \sum_{p=\ell-n}^{\ell} (-1)^p h_p \binom{n}{\ell-p},$$

where in the last two lines we use the convention $h_p = 0$ for $p<0$. We next show the following lemma.

Lemma 3.9. *The ideal I' contains the ideal $(h_{n-k+1}(1-t),\dots,h_n(1-t))$.*

Proof. Equivalently, we change basis $t \mapsto 1-t$, calling the (new) ideal J, and we show

$$J \supseteq (h_{n-k+1},\dots,h_n).$$

We consider the elements $f_{n-k+i}(1-t) \in J$ for $i=1,\dots,k$:

$$f_{n-k+i}(1-t) = \sum_{p=-k+i}^{n-k+i} (-1)^p h_p(1-t)\binom{n}{n-k+i-p}.$$

Since $i \leqslant k$, we have

$$= \sum_{p=0}^{n-k+i} (-1)^p h_p(1-t)\binom{n}{n-k+i-p}.$$

We apply the second formula from Proposition 3.3. Note that all terms are single-row partitions, $\lambda=(p)$ and $\mu=(s)$, with $s \leqslant p$, so $f^{\lambda/\mu}=1$ and the change of basis is

$$b_{\lambda\mu} = (-1)^s \frac{f^{\lambda/\mu}}{|\lambda/\mu|!}\cdot (k+s)\cdots(k+p-1) = (-1)^s\binom{k+p-1}{k+s-1}.$$

Hence,

$$\begin{aligned}f_{n-k+i}(1-t) &= \sum_{p=0}^{n-k+i}\sum_{s=0}^{p} (-1)^{p+s}\binom{n}{n-k+i-p}\binom{k+p-1}{k+s-1} h_s \\ &= \sum_{s=0}^{n-k+i}(-1)^s h_s \sum_{p=s}^{n-k+i}(-1)^p \binom{n}{n-k+i-p}\binom{k+p-1}{k+s-1}.\end{aligned}$$

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We reindex, sending $p \mapsto n-k+i-p$, and reverse the order of the inner sum:

$$= (-1)^{n-k+i} \sum_{s=0}^{n-k+i} (-1)^s h_s \sum_{p=0}^{n-k+i-s} (-1)^p \binom{n}{p} \binom{n+i-p-1}{k+s-1}.$$

The terms h_s for $s \leqslant n-k$. First, we show that all the lower terms h_s, with $s \leqslant n-k$, vanish. For these terms, we view the large binomial coefficient as a polynomial function of p. It has degree $k+s-1$, with zeros at $p = (n-k+i-s)+1, \ldots, n+i-1$, so we may freely include these terms in the inner sum. It is convenient to extend the inner sum only as far as $p = n$, obtaining

$$\sum_{p=0}^{n} (-1)^p \binom{n}{p} \binom{n-i-p-1}{k+s-1}.$$

Recall from the theory of finite differences that

$$\sum_{p=0}^{d} (-1)^p \binom{d}{p} g(p) = 0$$

whenever g is a polynomial of degree less than d. Since the above sum has degree $k+s-1 \leqslant n-1$, it vanishes. Thus, dropping the lower terms, we are left with

$$f_{n-k+i}(1-t) = (-1)^i \sum_{s=1}^{i} (-1)^s h_{n-k+s} \sum_{p=0}^{i-s} (-1)^p \binom{n}{p} \binom{n+i-p-1}{n+s-1}. \tag{3.3}$$

Showing $h_{n-k+i} \in J$ for $i = 1, \ldots, k$. From equation (3.3), we see directly that the coefficient of h_{n-k+i} in $f_{n-k+i}(1-t)$ is 1. This is the leading coefficient, so the claim follows by induction on i. $\square$

Corollary 3.10. *We have*

$$J = (h_i : i > n-k) = \operatorname{span}_{\mathbb{C}}\{s_\lambda : \lambda \not\subseteq \boxplus\}.$$

Proof. The equality of ideals

$$(h_{n-k+1}, \ldots, h_n) = (h_i : i > n-k)$$

follows from Newton's identities and induction. The equality

$$(h_i : i > n-k) = \operatorname{span}_{\mathbb{C}}\{s_\lambda : \lambda \not\subseteq \boxplus\}$$

follows from the Pieri rule (for $\subseteq$) and the Jacobi–Trudi formula (for $\supseteq$). See [Ful96] for these identities. This shows that J *contains* this linear span. But then quotienting by J leaves at most $\binom{n}{k}$ classes. This is already the rank of $K(\mathrm{Gr}(k, W))$, so we must have equality. $\square$

Changing bases $t \mapsto 1-t$ a final time completes the proof of Theorem 3.2.

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4. Square matrices and perfect matchings

Remark 4.1. In this section, rank Betti tables play a more significant role than multiplicity tables. As such, we will state results in terms of the cones $\widetilde{BS}_{k,k}$ and $\widetilde{BS}^D_{k,k}$.

We now describe the Boij–Söderberg cone in the base case of square matrices; thus we set $n = k$ for the remainder of this section. The corresponding Grassmannian is a point, so there is no dual geometric picture or cone. We will recall the description of $\widetilde{BS}_{k,k}$ due to [FLS18]; we then describe the derived cone $\widetilde{BS}^D_{k,k}$.

When $k = 1$, the ring is just $\mathbb{C}[t]$, and its torsion graded modules are essentially trivial to describe. See [EE17, § 4] for a short, complete description of both cones. For $k > 1$, however, the cones are algebraically and combinatorially interesting, although simpler than the general case.

4.1 Prior work on $\widetilde{BS}_{k,k}$ [FLS18]

The rank-deficient locus $\{\det(T) = 0\} \subset \mathrm{Hom}(V, W)$ is codimension 1. Thus, modules satisfying Condition 1.2 have free resolutions of length 1,

$$M \leftarrow F^0 \leftarrow F^1 \leftarrow 0.$$

There is only one equivariant Herzog–Kühl equation, labeled by the empty partition $\mu = \emptyset$:

$$\sum_{\lambda} \widetilde{\beta_{0,\lambda}} = \sum_{\lambda} \widetilde{\beta_{1,\lambda}}, \quad \text{that is, } \mathrm{rank}(F^0) = \mathrm{rank}(F^1). \tag{4.1}$$

Algebraically, this simply says that M is a torsion module.

The extremal rays and supporting hyperplanes of $\widetilde{BS}_{k,k}$ are as follows.

Definition 4.2 (Pure tables). Fix $\lambda, \mu \in \mathbb{Y}_\pm$ with $\lambda \subsetneq \mu$. The *pure table* $\widetilde{\beta}(\lambda \subsetneq \mu)$ is defined by setting

$$\widetilde{\beta_{0,\lambda}} = \widetilde{\beta_{1,\mu}} = 1$$

and all other entries 0.

It is nontrivial to show that each pure table $\widetilde{\beta}(\lambda \subsetneq \mu)$ is realizable up to scalar multiple [FLS18, Theorem 4.1]. Any such table generates an extremal ray of $\widetilde{BS}_{k,k}$. The construction relies on an adapted version of Weyman's geometric technique. We do not reproduce the proof, but see Example 6.1 for a similar construction in the case $k = 2, n = 3$.

It is, by contrast, easy to establish the following inequalities on $\widetilde{BS}_{k,k}$. Recall that a subset S of a poset $(P, \preceq)$ is *downwards closed* if, whenever $x \in S$ and $y \preceq x$, it follows that $y \in S$.

Definition 4.3 (Antichain inequalities). Let $S \subseteq \mathbb{Y}_\pm$ be a downwards closed set. Let

$$\Gamma = \{\lambda : \lambda \subsetneq \mu \text{ for some } \mu \in S\}.$$

For any rank Betti table $(\widetilde{\beta_{i,\lambda}})$, the *antichain inequality* (*for* S) is then

$$\sum_{\lambda \in \Gamma} \widetilde{\beta_{0,\lambda}} \geqslant \sum_{\lambda \in S} \widetilde{\beta_{1,\lambda}}. \tag{4.2}$$

(The terminology of antichains is due to [FLS18], where the inequality (4.2) is stated in terms of the maximal elements of S, which form an antichain in $\mathbb{Y}_\pm$.)

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These conditions follow directly from minimality of the underlying maps of modules: the summands corresponding to S in F_1 must map into those corresponding to Γ in F_0.

Finally, we recall the graph-theoretic model of $\widetilde{\beta}$ introduced in § 1.3.2. This construction was implicit in [FLS18, Lemma 3.6]. It is especially simple in this case.

DEFINITION 4.4. The *Betti graph* $G(\widetilde{\beta})$ is the directed bipartite graph with left vertices L and right vertices R, defined as follows:

- the set L (respectively, R) contains $\widetilde{\beta_{0,\lambda}}$ (respectively, $\widetilde{\beta_{1,\lambda}}$) vertices labeled λ, for each λ;
- the edge set contains all possible edges $\lambda \leftarrow \mu$, from R to L, for $\lambda \subsetneq \mu$.

The Boij–Söderberg cone $\widetilde{BS}_{k,k}$ is characterized as follows.

THEOREM 4.5 [FLS18, Theorem 3.8]. *The cone $\widetilde{BS}_{k,k} \subseteq \widetilde{\mathbb{BT}}_{k,k}$ is defined by the rank equation (4.1), the conditions $\widetilde{\beta_{i,\lambda}} \geqslant 0$ for $i = 0, 1$ and $\widetilde{\beta_{i,\lambda}} = 0$ for $i \neq 0, 1$, and the antichain inequalities (4.2). Its extremal rays are the pure tables $\widetilde{\beta}(\lambda \subsetneq \mu)$, for all choices of $\lambda \subsetneq \mu$ in $\mathbb{Y}_{\pm}$.*

Moreover, if $\widetilde{\beta} \in \widetilde{\mathbb{BT}}_{k,k}$ is an abstract rank table with nonnegative integer entries, then $\widetilde{\beta} \in \widetilde{BS}_{k,k}$ if and only if the Betti graph $G(\widetilde{\beta})$ has a perfect matching.

A perfect matching on $G(\widetilde{\beta})$ expresses $\widetilde{\beta}$ as a positive integer sum of pure tables: an edge $\lambda \leftarrow \mu$ corresponds to a summand

$$\widetilde{\beta} = \cdots + \widetilde{\beta}(\lambda \subsetneq \mu) + \cdots .$$

It is easy to see that the cone spanned by the pure tables is contained in the cone defined by the antichain inequalities. The fact that these cones agree follows from Hall's matching theorem for bipartite graphs.

THEOREM 4.6 (Hall's matching theorem). *Let G be a bipartite graph with left vertices L and right vertices R, with $|L| = |R|$. For each subset $S \subseteq R$ or L, let $\Gamma(S)$ be the set of vertices adjacent to S. Then G has a perfect matching if and only if $|\Gamma(S)| \geqslant |S|$ for all subsets $S \subseteq R$ (equivalently, for all subsets $S \subseteq L$).*

In the antichain inequality (4.2), S corresponds to a set of vertex labels on the right-hand side of the Betti graph $G(\widetilde{\beta})$. The set Γ consists of the labels of vertices adjacent to S. The numbers of such vertices are then the right- and left-hand sides of the inequality. (The structure of $G(\widetilde{\beta})$ implies easily that it suffices to consider inequalities from downwards-closed sets S.)

4.2 The derived cone

We now generalize Theorem 4.5 to describe the derived cone $\widetilde{BS}^D_{k,k}$. We are interested in bounded free equivariant complexes

$$\cdots \leftarrow F_i \leftarrow F_{i+1} \leftarrow F_{i+2} \leftarrow \cdots ,$$

all of whose homology modules are torsion.

The supporting hyperplanes of $\widetilde{BS}^D_{k,k}$ are quite complicated and we do not prove directly that they hold. We instead generalize the descriptions in terms of extremal rays and perfect matchings, which remain fairly simple. We then deduce the inequalities from Hall's theorem.

The extremal rays of $\widetilde{BS}^D_{k,k}$ will be homological shifts of those of $\widetilde{BS}_{k,k}$.

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Definition 4.7 (Homologically shifted pure tables). Fix $i \in \mathbb{Z}$ and $\lambda, \mu \in \mathbb{Y}_\pm$ with $\lambda \subsetneq \mu$. We define the *homologically shifted pure table* $\widetilde{\beta}[\lambda \xleftarrow{i} \mu]$ by setting

$$\widetilde{\beta_{i,\lambda}} = \widetilde{\beta_{i+1,\mu}} = 1$$

and all other entries 0.

The supporting hyperplanes will be defined by the following inequalities. Recall that a *convex subset* S of a poset P is the intersection of an upwards-closed set with a downwards-closed set.

Definition 4.8 (Convexity inequalities). For each odd i, let $S_i \subseteq \mathbb{Y}_\pm$ be any choice of convex set. For each even i, define

$$\Gamma_i = \{\lambda : \mu \subsetneq \lambda \text{ for some } \mu \in S_{i-1}\} \cup \{\lambda : \lambda \subsetneq \mu \text{ for some } \mu \in S_{i+1}\}.$$

For any rank Betti table $(\widetilde{\beta_{i,\lambda}})$, the *convexity inequality* (*for the* S_i) is then

$$\sum_{i \text{ even}} \sum_{\lambda \in \Gamma_i} \widetilde{\beta_{i,\lambda}} \geqslant \sum_{i \text{ odd}} \sum_{\lambda \in S_i} \widetilde{\beta_{i,\lambda}}. \tag{4.3}$$

(We may, if we wish, switch 'even' and 'odd' in this definition. We will see that either collection of inequalities yields the same cone.)

We recall the general definition of the Betti graph.

Definition 4.9 (Betti graphs for complexes). Let $\widetilde{\beta} \in \widetilde{\mathbb{BT}}_{k,k}$ have nonnegative integer entries. The *Betti graph* $G(\widetilde{\beta})$ is defined as follows:

- the vertex set contains $\widetilde{\beta_{i,\lambda}}$ vertices labeled (i, λ), for each (i, λ);
- the edge set contains, for each i, all possible edges $(i, \lambda) \leftarrow (i+1, \mu)$ with $\lambda \subsetneq \mu$.

Note that this graph is bipartite: every edge connects an even-indexed and an odd-indexed vertex.

Each segment S_i of Definition 4.8 corresponds to a set of vertex labels in $G(\widetilde{\beta})$. The set Γ_i then contains the labels of vertices adjacent to S_{i-1} and S_{i+1}. The numbers of vertices counted this way give the right- and left-hand sides of inequality (4.3). Thus, the inequality will follow (using Hall's theorem) from the existence of a perfect matching on $G(\widetilde{\beta})$. Note that if the S_i were not convex, we could replace them by their convex hulls without changing the Γ_i.

We now characterize the derived Boij–Söderberg cone $\widetilde{BS}^D_{k,k}$.

Theorem 4.10 (The derived Boij–Söderberg cone, for square matrices). *Let $\widetilde{\beta}$ be an abstract rank Betti table. Without loss of generality, assume the entries of $\widetilde{\beta}$ are nonnegative integers. The following are equivalent:*

(i) $\widetilde{\beta} \in \widetilde{BS}^D_{k,k}$;

(ii) $\widetilde{\beta}$ *satisfies all the convexity inequalities (for all choices of tuple (S_i)), together with the rank condition*

$$\sum_{i,\lambda} (-1)^i \widetilde{\beta_{i,\lambda}} = 0;$$

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(iii) $\widetilde{\beta}$ *is a positive integral linear combination of homologically shifted pure tables;*

(iv) *the Betti graph* $G(\widetilde{\beta})$ *has a perfect matching.*

Remark 4.11. It is clear that (iv) $\Rightarrow$ (iii): each edge of a perfect matching indicates a pure table summand for $\widetilde{\beta}$. We have (iii) $\Rightarrow$ (ii) since the conditions (ii) hold for each homologically shifted pure table individually. Hall's matching theorem gives the statement (ii) $\Leftrightarrow$ (iv) and shows that we may exchange 'even' and 'odd' in the definition of the convexity inequalities. Homologically shifted pure tables are realizable, hence (iii) $\Rightarrow$ (i). We will complete the proof by exhibiting a perfect matching on any realizable Betti graph, so that (i) $\Rightarrow$ (iv). We postpone the proof until § 5.1.1 (Corollary 5.10).

5. The pairing between Betti tables and cohomology tables

In this section, we establish the numerical pairing between Betti tables and cohomology tables. We recall that the pairing is defined as follows (Definition 1.11):

$$\begin{aligned} \widetilde{\Phi} : \mathbb{BT}_{k,n} \times \mathbb{CT}_{k,n} &\to \widetilde{\mathbb{BT}}_{k,k}, \\ (\beta,\gamma) &\mapsto \widetilde{\Phi}(\beta,\gamma), \end{aligned} \tag{5.1}$$

with $\widetilde{\Phi}$ the (derived) rank Betti table with entries

$$\widetilde{\varphi_{i,\lambda}}(\beta,\gamma) = \sum_{p-q=i} \beta_{p,\lambda} \cdot \gamma_{q,\lambda}. \tag{5.2}$$

Recall also that the convention is that homological degree (p and i) decreases under the boundary map of the complex.

The remainder of this section is devoted to the proof of the following theorem.

Theorem 5.1 (Pairing the equivariant Boij–Söderberg cones). *The pairing* $\widetilde{\Phi}$ *restricts to a pairing of cones,*

$$BS^D_{k,n} \times ES_{k,n} \to \widetilde{BS}^D_{k,k}.$$

In light of our description (Theorem 4.10) of the derived cone $\widetilde{BS}^D_{k,k}$, the goal will be to exhibit a perfect matching on the Betti graph of $\widetilde{\Phi}(\beta,\gamma)$. Along the way, we will also complete the proof of Theorem 4.10 itself, showing that $\widetilde{BS}^D_{k,k}$ is characterized by the existence of such matchings (Corollary 5.10).

In § 6, we will use the pairing to establish constraints on Betti tables in the case $k = 2, n = 3$, by pairing with careful choices of GL-cohomology table. See, for example, Proposition 6.2 and Example 6.6.

Remark 5.2. The pairing is based on the hypercohomology spectral sequence for a complex of sheaves $\mathscr{F}^\bullet$. The proof, however, requires an explicit realization of this spectral sequence via a double complex (taking an injective resolution of $\mathscr{F}^\bullet$, then taking sections). See Remark 5.15 and Example 5.16 for the necessity of this approach. Any injective resolution that is functorial in the underlying maps of sheaves will do; we use the Čech complex.

Proof of Theorem 5.1. Let $\beta = \beta(F^\bullet)$ be the Betti table of a minimal free equivariant complex $F^\bullet$ of finitely generated R-modules, with $R = R_{k,n}$ the coordinate ring of the $k \times n$ matrices.

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Assume $F^\bullet$ is exact away from the locus of rank-deficient matrices, so descending $F^\bullet$ to $\mathrm{Gr}(k,\mathbb{C}^n)$ (as in Remark 2.1) gives an exact sequence of vector bundles $\mathscr{F}^\bullet$:

$$F^\bullet = \bigoplus_\lambda \mathbb{S}_\lambda(V)^{\beta_{\bullet,\lambda}} \otimes R \quad \xrightarrow{\text{descends to}} \quad \mathscr{F}^\bullet = \bigoplus_\lambda \mathbb{S}_\lambda(\mathcal{S})^{\beta_{\bullet,\lambda}}, \tag{5.3}$$

with $\mathcal{S}$ the tautological subbundle on $\mathrm{Gr}(k,\mathbb{C}^n)$. Let $\gamma = \gamma(\mathcal{E})$ be the GL-cohomology table of a coherent sheaf $\mathcal{E}$ on $\mathrm{Gr}(k,n)$. Observe that $\mathcal{E}\otimes\mathscr{F}^\bullet$ is again exact.

We study the hypercohomology spectral sequence. Explicitly, we take the Čech resolution of $\mathcal{E}\otimes\mathscr{F}^\bullet$, an exact double complex of sheaves. Let $E^{\bullet,\bullet}$ be the result of taking global sections: a double complex of vector spaces, with $\mathrm{Tot}(E^{\bullet,\bullet})$ exact (since, by exactness of $\mathcal{E}\otimes\mathscr{F}^\bullet$, the spectral sequence abuts to 0). By functoriality of the Čech complex, each term splits as a direct sum $E^{\bullet,\bullet} = \bigoplus_\lambda E^{\bullet,\bullet,\lambda}$, while the differentials satisfy

$$d_v(E^{\bullet,\bullet,\lambda}) \subset E^{\bullet,\bullet+1,\lambda} \quad \text{and} \quad d_h(E^{\bullet,\bullet,\lambda}) \subset \bigoplus_{\mu \subsetneq \lambda} E^{\bullet-1,\bullet,\mu}.$$

(The strict inequality in the statement about d_h is from the minimality of the original complex $F^\bullet$.) Note that the labels λ formally refer only to the $\mathbb{S}_\lambda(\mathcal{S})$ summands used to construct the Čech complex – after descending to $\mathrm{Gr}(k,W)$, our complex no longer carries any $\mathrm{GL}(V)$ action. We run the sequence beginning with the vertical maps, giving the E_1 page

$$E_1^{p,q} = \bigoplus_\lambda H^q(\mathcal{E}\otimes\mathbb{S}_\lambda(\mathcal{S}))^{\beta_{p,\lambda}}.$$

Observe that the λ summand has dimension $\beta_{p,\lambda}(F^\bullet)\gamma_{q,\lambda}(\mathcal{E})$. The (i,λ) coefficient produced in the Boij–Söderberg pairing, $\widetilde{\varphi_{i,\lambda}}(F^\bullet,\mathcal{E})$, is the sum of this quantity along the diagonal $\{p-q=i\}$. That is, $\widetilde{\Phi}$ is akin to a Betti table for $\mathrm{Tot}(E_1)$:

$$\widetilde{\varphi_{i,\lambda}} = \dim_{\mathbb{C}} \mathrm{Tot}(E_1)_{i,\lambda}.$$

We emphasize, however, that there is no actual GL_k-action on $\mathrm{Tot}(E_1)$, nor an $R_{k,k}$-module structure. Instead, we will show by homological techniques that, for a wide class of double complexes including $E^{\bullet,\bullet}$, there is a perfect matching on a graph associated to $\mathrm{Tot}(E_1)$; in our setting, this will give the desired perfect matching on the Betti graph of $\widetilde{\Phi}$. $\square$

The key properties of the double complex $E^{\bullet,\bullet}$ constructed above are that:

(1) each term $E^{p,q}$ has a direct sum decomposition labeled by a poset P;
(2) the vertical maps d_v are label-preserving;
(3) the horizontal maps are strictly label-decreasing.

By (1) and (2), the E_1 page (the homology of d_v) again has a direct sum decomposition labeled by P, $E_1^{p,q} = \bigoplus_{\lambda\in P} E_1^{p,q,\lambda}$. We define the following graph.

Definition 5.3. The E_1 *graph* $G = G(E^{\bullet,\bullet})$ is the following directed graph:

- the vertex set has $\dim(E_1^{p,q,\lambda})$ vertices labeled (p,q,λ), for each $p,q\in\mathbb{Z}$ and $\lambda\in P$;
- the edge set includes all possible edges $(p,q,\lambda)\to(p',q',\lambda')$ whenever $\lambda' \precneqq \lambda$ and $(p',q') = (p-r, q-r+1)$ for some $r>0$.

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The edges of G thus respect the strictly-decreasing-P-labels condition, and are shaped like the higher-order differentials of the associated spectral sequence.

We show the following theorem.

Theorem 5.4. *Let $E^{\bullet,\bullet}$ be a double complex of vector spaces satisfying (1)–(3). If* $\mathrm{Tot}(E^{\bullet,\bullet})$ *is exact, its E_1 graph has a perfect matching.*

We think of this theorem as a combinatorial analog of the fact that the associated spectral sequence (beginning with the homology of d_v) converges to zero. We explore this idea further in § 5.1.

Finishing the proof of Theorem 5.1. With $E^{\bullet,\bullet}$ as above, we identify the vertices of the E_1 graph and the Betti graph of $\widetilde{\Phi}(\beta,\gamma)$; for any such identification, the edges of the E_1 graph become a subset of the Betti graph's edges. (We may recover the missing edges by allowing $r \leqslant 0$ in Definition 5.3.) Hence, the perfect matching produced by Theorem 5.4 is valid for the Betti graph, completing the proof of Theorem 5.1. □

5.1 Perfect matchings in linear and homological algebra

Our approach uses linear maps to produce perfect matchings. The starting point is the following construction.

Definition 5.5. Let $T : V \to W$ be a map of vector spaces, having specified bases $\mathcal{V}, \mathcal{W}$. The *coefficient graph* G of T is the directed bipartite graph with vertex set $\mathcal{V} \sqcup \mathcal{W}$ and edges

$$E = \{v \to w : T(v) \text{ has a nonzero } w\text{-coefficient}\}.$$

Note that the adjacency matrix of G is T with all nonzero coefficients replaced by 1s.

Proposition 5.6. *For finite-dimensional vector spaces, the coefficient graph of an isomorphism admits a perfect matching.*

We will say the corresponding bijection $\mathcal{V} \leftrightarrow \mathcal{W}$ is *compatible with* T, a combinatorial analog of the fact that T is an isomorphism. The proof of existence is simple, but essentially nonconstructive in practice. Here are two ways to do it.

(i) (*All at once.*) Since $\det(T) \neq 0$, some monomial term of $\det(T)$ is nonzero. This exhibits the perfect matching.

(ii) (*By induction, using the Laplace expansion.*) Expand $\det(T)$ along a row or column; some term $a_{ij} \cdot$ (complementary minor) is nonzero, and so on.

Method (ii) actually satisfies a slightly stronger condition: the resulting matching is compatible with both T and T^{-1} (since, up to scaling by $\det(T)$, the complementary minors are the entries of the inverse matrix).

Similarly, if T is merely assumed to be injective or surjective, we may produce a maximal matching in this way (choose some nonvanishing maximal minor).

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We generalize Proposition 5.6 to the setting of homological algebra in three ways: to infinite-dimensional vector spaces, to long exact sequences, and to double complexes (motivated by spectral sequences).

Proposition 5.7. *For vector spaces of arbitrary dimension, the coefficient graph of an isomorphism admits a perfect matching.*

We will not need Proposition 5.7 (which uses the axiom of choice) for our proof of Theorem 5.4, so we prove it in the appendix. See Remark 5.14 for additional discussion on our usage of the axiom of choice.

5.1.1 *Long exact sequences and the proof of Theorem 4.10.* We generalize to the case of long exact sequences. Let

$$\cdots \leftarrow V_i \overset{\delta}{\leftarrow} V_{i+1} \leftarrow \cdots$$

be a long exact sequence, with $\mathcal{V}_i$ a fixed basis for V_i. (The vector spaces may be finite- or infinite-dimensional.)

Definition 5.8. The *coefficient graph* G for $(V_\bullet, \delta)$ (with respect to $\mathcal{V}_\bullet$) is the directed graph with vertex set $\bigsqcup_i \mathcal{V}_i$ and an edge $v \to v'$ whenever $\delta(v)$ has a nonzero v'-coefficient.

Proposition 5.9. *The coefficient graph of a long exact sequence has a perfect matching.*

Proof. Choose subsets $\mathcal{F}_i \subset \mathcal{V}_i$ descending to bases of $\operatorname{im}(\delta) \subset V_{i-1}$, using Zorn's lemma in the infinite case. Let $\mathcal{G}_i = \mathcal{V}_i - \mathcal{F}_i$, and let $F_i = \operatorname{span}(\mathcal{F}_i)$ and $G_i = \operatorname{span}(\mathcal{G}_i)$. The composition $\tilde{\delta} : F_{i+1} \hookrightarrow V_{i+1} \overset{\delta}{\to} V_i \twoheadrightarrow G_i$ is an isomorphism and has the same coefficients as δ, restricted to $\mathcal{F}_{i+1}$ and $\mathcal{G}_i$. Thus Proposition 5.6 (or 5.7 in the infinite case) yields a matching of $\mathcal{F}_{i+1}$ with $\mathcal{G}_i$. $\square$

At this point, we complete the proof of Theorem 4.10, characterizing the derived Boij–Söderberg cone $\widetilde{BS}^D_{k,k}$ of the square matrices.

Corollary 5.10. *If $\widetilde{\beta} \in \widetilde{BS}^D_{k,k}$, then the Betti graph $G(\widetilde{\beta})$ has a perfect matching.*

Proof. Let $\widetilde{\beta}$ be the Betti table of a minimal free equivariant complex $(F^\bullet, \delta)$ of R-modules, with $R = R_{k,k}$ the coordinate ring of the $k \times k$ matrices, and $F^\bullet \otimes R[1/\det]$ exact.

Choose, for each F^i, a $\mathbb{C}$-basis of each copy of $\mathbb{S}_\lambda(V)$ occurring in F^i. Label the corresponding basis elements x_λ. It follows from minimality that each $\delta(x_\lambda)$ is an R-linear combination of basis elements labeled by partitions $\lambda' \subsetneq \lambda$.

Since the homology modules are torsion, $F_\bullet \otimes \operatorname{Frac}(R)$ is an exact sequence of $\operatorname{Frac}(R)$-vector spaces, with bases given by the x_λ chosen above. By the previous proposition, its coefficient graph has a perfect matching. This graph has the same vertices as the Betti graph $G(\widetilde{\beta})$, and its edges are a subset of $G(\widetilde{\beta})$'s edges. $\square$

Remark 5.11. Rather than tensoring with $\operatorname{Frac}(R)$, we may instead specialize to any convenient invertible $k \times k$ matrix $T \in \operatorname{Hom}(\mathbb{C}^k, \mathbb{C}^k)$, such as the identity matrix. This approach is useful for computations, since the resulting exact sequence consists of finite-dimensional $\mathbb{C}$-vector spaces.

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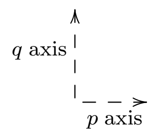
5.1.2 *Double complexes and the proof of Theorem 5.1.* Finally, we generalize to the setting of double complexes and spectral sequences. Let $(E^{\bullet,\bullet}, d_v, d_h)$ be a double complex of vector spaces, with differentials pointing up and to the left:



$$\begin{tikzcd} E^{p-1,q+1} & E^{p,q+1} \arrow[l, "d_h"'] \\ E^{p-1,q} \arrow[u, "d_v"] & E^{p,q} \arrow[l, "d_h"] \arrow[u, "d_v"'] \end{tikzcd}$$

We assume the squares anticommute, so the total differential is

$$d_{\mathrm{tot}} = d_h + d_v, \quad \text{where } d_h d_v + d_v d_h = 0.$$

We will always assume the total complex $\mathrm{Tot}(E^{\bullet,\bullet})$ has a finite number of columns. Note that we do *not* assume a basis has been specified for each $E^{\bullet,\bullet}$.

We recall the complexes $E^{\bullet,\bullet}$ of interest:

(1) each term $E^{p,q}$ has a direct sum decomposition

$$E^{p,q} = \bigoplus_{\lambda \in P} E^{p,q,\lambda},$$

with labels λ from a poset P;

(2) the vertical differential d_v is *graded* with respect to this labeling; and

(3) the horizontal differential d_h is *downwards filtered*.

Conditions (2) and (3) mean that

$$d_v(E^{p,q,\lambda}) \subseteq E^{p,q+1,\lambda} \quad \text{and} \quad d_h(E^{p,q,\lambda}) \subseteq \bigoplus_{\lambda' \prec \lambda} E^{p-1,q,\lambda'},$$

so the vertical differential preserves the label and the horizontal differential strictly decreases it.

We are interested in the homology of the vertical map d_v. Since d_v is P-graded, so is its homology $E_1^{p,q,\lambda} = H(d_v)^{p,q,\lambda}$. We recall that the E_1 *graph* $G(E^{\bullet,\bullet})$ is defined as follows:

– the vertex set contains $\dim(E_1^{p,q,\lambda})$ vertices labeled (p, q, λ), for each p, q and each $\lambda \in P$;

– the edge set includes all possible edges $(p, q, \lambda) \to (p', q', \lambda')$ whenever $\lambda' \prec \lambda$ and $(p', q') = (p - r, q - r + 1)$ for some $r > 0$.

The edges of G respect the downwards filtered condition on P-labels, and are shaped like higher-order differentials of the associated spectral sequence, that is, they point downwards and leftwards. We wish to show the following theorem.

Theorem 5.12. *If $\mathrm{Tot}(E^{\bullet,\bullet})$ is exact, the E_1 graph of $E^{\bullet,\bullet}$ has a perfect matching.*

Remark 5.13. Consider summing the E_1 page along diagonals. Call the resulting complex $\mathrm{Tot}(E_1)$. If it were exact, the matching would exist by Proposition 5.9, and in fact would only use the edges corresponding to $r = 1$. Since $\mathrm{Tot}(E_1)$ is not exact in general, the proof works by modifying its maps to make it exact.

Explicitly, we will exhibit a quasi-isomorphism from $\mathrm{Tot}(E^{\bullet,\bullet})$ to a complex with the same terms as $\mathrm{Tot}(E_1)$, but different maps – whose nonzero coefficients are only in the spots permitted by the E_1 graph. Since $\mathrm{Tot}(E^{\bullet,\bullet})$ is exact, so is the new complex, so we will be done by Proposition 5.9.

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Remark 5.14 (The role of the axiom of choice). In our intended usage (Theorem 5.1), the terms $E_1^{p,q,\lambda}$ are all finite-dimensional, so we do not need the axiom of choice to produce the perfect matching from Proposition 5.9. The reduction to E_1 does require choices (on the E_0 page). For the Čech complex used in the proof of Theorem 5.1, the axiom of countable choice suffices. In general the proof relies on a sufficiently strong choice axiom.

Proof of Theorem 5.12. First, we split all the vertical maps: for each p, q, λ, we define subspaces $B, H, B^* \subseteq E$ (suppressing the indices) as follows. We put $B = \mathrm{im}(d_v)$; we choose H to be linearly disjoint from B and such that $B + H = \ker(d_v)$; then we choose B^* linearly disjoint from $B + H$, such that $B + H + B^* = E$.

In particular, d_v maps the subspace B^* isomorphically to the subsequent subspace B, and the space H descends isomorphically to $H(d_v)$, the E_1 term. The picture of a single column of the double complex looks like the following:

$$\begin{tikzcd}[row sep=small, column sep=small]
{} & \vdots & \\
B & H & B^* \arrow[llu, "\sim"'] \\
B & H & B^* \arrow[llu, "\sim"'] \\
 & H & B^* \arrow[llu, "\sim"']
\end{tikzcd} \qquad \text{(note that } d_v(B) = d_v(H) = 0\text{)}$$

As for d_h, we have $d_h(B) \subset B$ and $d_h(H) \subset B + H$, and the poset labels λ strictly decrease.

Our goal will be to choose bases carefully, so as to match the H basis elements to one another, in successive diagonals, while decreasing the poset labels.

We first choose an arbitrary basis of each H and B^* space. We descend the basis of B^* to a basis of the subsequent B using d_v. Note that every basis element has a position (p, q) and a label λ. We will write x_λ if we wish to emphasize that a basis vector x has label λ.

We now change basis on the entire diagonal $E^i := \bigoplus_{p-q=i} E^{p,q}$. We leave the H and B^* bases untouched, but replace all the B basis vectors, as follows. Let $b_\lambda \in B^{p,q,\lambda}$ and let $b_\lambda^* = d_v^{-1}(b_\lambda) \in (B^*)^{p,q-1,\lambda}$ be its 'twin'. We define

$$\tilde{b}_\lambda := d_{\mathrm{tot}}(b_\lambda^*) = b_\lambda + d_h(b_\lambda^*).$$

We replace b_λ by $\tilde{b}_\lambda$, formally labeling the new basis vector by (p, q, λ). We write $\widetilde{B}^{p,q,\lambda}$ for the span of the vectors $\tilde{b}$, so, in particular, $\widetilde{B}^{p,q,\lambda} := d_{\mathrm{tot}}((B^*)^{p,q-1,\lambda})$.

It is clear that $\widetilde{B}, H, B^*$ collectively gives a new basis for the entire diagonal, unitriangular in the old basis. Notice also that the old basis element $b_\lambda \in B^{p,q,\lambda}$ becomes, in general, a linear combination of $\widetilde{B}, H, B^*$ elements in all positions down and left of p, q, with leading term $\tilde{b}_\lambda$:

$$b_\lambda = \tilde{b}_\lambda + \sum_{i>0} x^{p-i,q-i}, \quad \text{with } x^{p-i,q-i} \in \bigoplus_{\lambda' \subsetneq \lambda} E^{p-i,q-i,\lambda'}.$$

The lower terms have strictly smaller labels $\lambda' \subsetneq \lambda$. (In fact, slightly more is true: if a label λ' occurs in the ith term, the poset P contains a chain of length at least i from λ' to λ.)

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We now inspect the coefficients of $(\mathrm{Tot}(E^{\bullet,\bullet}), d_{\mathrm{tot}})$ in the new basis. We have

$$\begin{aligned} d_{\mathrm{tot}}(b^*_\lambda) &= \tilde{b}_\lambda, \\ d_{\mathrm{tot}}(\tilde{b}_\lambda) &= 0 \ (= d^2_{\mathrm{tot}}(\tilde{b}^*_\lambda)), \end{aligned}$$

so the B^* elements map one by one onto the $\widetilde{B}$ elements, with the same λ labels; the latter elements then map to 0.

Next, for a basis element $h_\lambda \in H$, the coefficients change but remain 'filtered'. If $d_{\mathrm{tot}}(h_\lambda)$ included (in the old basis) some nonzero term $t \cdot b_\mu$, then in the new basis we have

$$d_{\mathrm{tot}}(h_\lambda) = d_h(h_\lambda) = \cdots + t \cdot (\tilde{b}_\mu - d_h(b^*_\mu)) + \cdots .$$

Since t is nonzero, we have $\mu \subsetneq \lambda$; and the additional terms coming from $d_h(b^*_\mu)$ all have labels $\mu' \subsetneq \mu$. Thus all labels occurring in $d_{\mathrm{tot}}(h_\lambda)$ in the new basis are, again, strictly smaller than λ. We note that $d_{\mathrm{tot}}(h_\lambda)$ is a linear combination of $\widetilde{B}, H, B^*$ elements in all positions below and to the left of h_λ along the subsequent diagonal:

$$\begin{tikzcd}
 & & & \widetilde{B}, H & H \arrow[l] \arrow[dl] \arrow[dddllll] \\
 & & & \widetilde{B}, H, B^* & \\
 & \iddots & & & \\
\widetilde{B}, H, B^* & & & &
\end{tikzcd}$$

Finally, we observe that the spaces $\widetilde{B} + B^*$ collectively span a subcomplex of $\mathrm{Tot}(E)$, so we have a short exact sequence of complexes

$$0 \to \mathrm{Tot}(\widetilde{B} + B^*) \to \mathrm{Tot}(E) \to \mathrm{Tot}(H) \to 0.$$

By construction, $\mathrm{Tot}(\widetilde{B} + B^*)$ is exact, so $\mathrm{Tot}(E) \to \mathrm{Tot}(H)$ is a quasi-isomorphism. Note that $\mathrm{Tot}(H)$ and $\mathrm{Tot}(E_1)$ have 'the same' terms, but different maps, as desired. Since $\mathrm{Tot}(E)$ is exact, so is $\mathrm{Tot}(H)$. The desired matching therefore exists by Proposition 5.9. $\square$

Remark 5.15. Our initial attempts to establish the Boij–Söderberg pairing (Theorems 5.1 and 5.4) used the higher differentials on the $E_1, E_2, \ldots$ pages, rather than the E_0 page as above – aiming to systematize 'chasing cohomology of the underlying sheaves'. The following example shows that such an approach fails on general double complexes.

Example 5.16 (A cautionary example). Consider the following double complex:

$$\begin{tikzcd}
 & & \ydiagram{2} & \ydiagram{3} \arrow[l] \\
 & \ydiagram{2,1} \oplus \ydiagram{1} & \ydiagram{3,1} \oplus \ydiagram{2} \arrow[l, "f"'] \arrow[u] & \\
\ydiagram{1,1} & \ydiagram{2,1} \arrow[l] \arrow[u] & &
\end{tikzcd}$$

Each partition denotes a single basis vector with that label. The vertical map d_v preserves labels and the horizontal map d_h decreases labels. The unlabeled arrows denote maps taking one indicated basis vector to another, and the map f is given by

$$f(\ydiagram{2}) = \ydiagram{1}\,, \quad f\left(\ydiagram{3,1}\right) = \ydiagram{2,1} - \ydiagram{1}\,.$$

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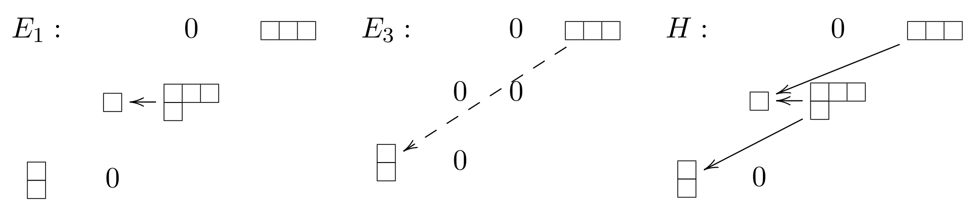
Note that the rows are exact, so the total complex is exact as well, and the spectral sequence abuts to zero. The only nonzero higher differentials are on the E_1 and E_3 pages. These pages, and (for contrast) the complex H constructed in Theorem 5.4, are as follows.



All the arrows are coefficients of ± 1. In particular, *no* combination of the E_1 and E_3 differentials gives a valid matching (the E_3 arrow violates the P-filtered condition). In contrast, H finds the (unique) valid matching $\{\ydiagram{1} \leftarrow \ydiagram{3}\,,\ \ydiagram{1,1} \leftarrow \ydiagram{3,1}\}$.

6. Extremal Betti tables on 2×3 matrices

To demonstrate our results, and as a source of interesting examples, we construct three families of extremal equivariant Betti tables in the case of 2×3 matrices. We do not know if all extremal rays of $BS_{2,3}$ are of this form (see Example 6.3). We also show how the equivariant Boij–Söderberg pairing rules out certain otherwise plausible tables.

In this setting, the modules of interest (as in Condition 1.2) are Cohen–Macaulay and of projective dimension 2. We will assume all modules are generated in positive degree. All Betti tables in this section will be multiplicity Betti tables. We write $R = R_{2,3}$, so $\operatorname{Spec}(R) = \operatorname{Hom}(V, W)$ is the space of 2×3 matrices.

6.1 Extremal tables and the Herzog–Kühl equations

There are three equivariant Herzog–Kühl equations, corresponding to $\mu = \emptyset$, $\square$, and $\ydiagram{1,1}$, which may be simplified to

$$
\begin{aligned}
\mu = \emptyset: &\quad 0 = \sum_{i,\lambda} (-1)^i \beta_{i,\lambda} d_\lambda(2), \\
\mu = \square: &\quad 0 = \sum_{i,\lambda} (-1)^i \beta_{i,\lambda} d_\lambda(2) \cdot (\lambda_1 + \lambda_2), \\
\mu = \ydiagram{1,1}: &\quad 0 = \sum_{i,\lambda} (-1)^i \beta_{i,\lambda} d_\lambda(2) \cdot \tfrac{1}{2}(\lambda_1 + 1)\lambda_2.
\end{aligned}
$$

Note that $d_\lambda(2) = 1 + \lambda_1 - \lambda_2$.

The key observation is the following. Since there are three Herzog–Kühl equations, if we allow exactly *four* entries in the Betti table to be nonzero, we expect the equations to pick out one dimension's worth of valid tables. That is, the resulting table β will be unique up to scalar multiple. By the same reasoning, such a table cannot be decomposed into a nontrivial positive combination of other valid tables. Thus, if realizable, β is automatically an extremal ray of $BS_{2,3}$. In certain cases, the equations will be redundant, and three nonzero entries will suffice. We will call the result a *three-term* or *four-term table* accordingly. (Three entries are required or the resolution will be too short.)

The observation above underpins the characterization of pure tables for graded modules, where every choice of increasing degree sequence results in a unique table (up to scaling) with

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exactly $n+1$ nonzero entries, one in each column. Unlike in the graded case, we see that our extremal tables need not be 'pure': each four-term table will have a column with two distinct entries.

6.2 Three-term tables

These tables closely resemble the extremal Betti tables in the square-matrix and graded settings. Each column has one nonzero entry, so the resolution is 'pure':

$$M \leftarrow \mathbb{S}_\lambda(V)^{\beta_{0,\lambda}} \otimes R \leftarrow \mathbb{S}_\mu(V)^{\beta_{1,\mu}} \otimes R \leftarrow \mathbb{S}_\nu(V)^{\beta_{2,\nu}} \otimes R \leftarrow 0$$

for some triple of partitions $\lambda \subsetneq \mu \subsetneq \nu$.

Example 6.1 (An example with border strips). We realize the table

$\beta_{i,\lambda}$	0	1	2
$\ydiagram{2}$	8		
$\ydiagram{2,1}$		15	
$\ydiagram{4,3}$			3

We sketch the construction, a version of Weyman's geometric technique similar to that used in [FLS18, Theorem 4.6]. Let $X = \mathbb{P}(V)$, with tautological line subbundle $\mathcal{O}(-1)$ and quotient line bundle $\mathcal{Q}$ (we distinguish between $\mathcal{Q}$ and the dual of $\mathcal{O}(-1)$ since we are tracking the whole $\mathrm{GL}(V)$-action). By the Borel–Weil–Bott theorem, $\mathcal{Q}^{\otimes a} \otimes \mathcal{O}(-b)$ has cohomology

$$\begin{cases} H^0 = \mathbb{S}_{a,b}(V), H^1 = 0 & \text{if } a \geqslant b, \\ H^0 = H^1 = 0 & \text{if } a = b-1, \\ H^0 = 0, H^1 = \mathbb{S}_{b-1,a+1}(V) & \text{if } a \leqslant b-2. \end{cases}$$

Eisenbud *et al.* [EFW11] have constructed, over any polynomial ring $S = \mathrm{Sym}(E)$ with E a vector space of dimension 3, a $\mathrm{GL}(E)$-equivariant free resolution

$$\mathbb{S}_{(1,0,-1)}(E) \otimes S \leftarrow \mathbb{S}_{(2,0,-1)}(E) \otimes S \leftarrow \mathbb{S}_{(2,2,-1)}(E) \otimes S \leftarrow \mathbb{S}_{(2,2,1)}(E) \otimes S \leftarrow 0,$$

resolving a module of finite length. (In the graded setting, this gives a pure resolution for the degree sequence $(0,1,3,5)$.) Since the resolution is equivariant, it makes sense for families of vector spaces, so we may replace E by the rank-3 vector bundle $\mathcal{E} := W^* \otimes \mathcal{O}(-1)$ over X. The resulting locally free resolution has terms

$$\mathbb{S}_\alpha(\mathcal{E}) \otimes \mathrm{Sym}(\mathcal{E}) = \mathbb{S}_\alpha(W^*) \otimes \mathcal{O}(-|\alpha|) \otimes \mathrm{Sym}(\mathcal{E}).$$

Next, we base-change the resolution along the flat inclusion of $\mathcal{O}_X$-algebras

$$\mathrm{Sym}(\mathcal{E}) \hookrightarrow \mathrm{Sym}(V \otimes W^*) \otimes \mathcal{O}_X = R \otimes \mathcal{O}_X$$

(locally an inclusion of polynomial rings) induced by the inclusion $\mathcal{O}(-1) \hookrightarrow V \otimes \mathcal{O}_X$. We twist through by $\mathcal{Q}^{\otimes 2}$ and take hypercohomology (the relevant terms are $\mathcal{Q}^{\otimes 2} \otimes \mathcal{O}(-b)$ for $b = 0,1,3,5$). Note that the resolved sheaf was supported set-theoretically on

$$\{(\ell, T) : T(\ell) = 0\} \subset \mathbb{P}(V) \times \mathrm{Hom}(V, W),$$

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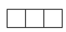
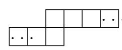
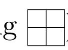
so the resulting R-module is automatically supported on the locus of rank-deficient matrices. The spectral sequence collapses at E_3 and yields the free resolution

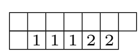
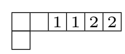
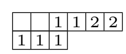
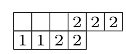
$$\begin{aligned} \mathbb{S}_{(2,0)}(V) \otimes \mathbb{S}_{(1,0,-1)}(W^*) \otimes R &\cong \mathbb{S}_{(2,0)}(V)^8 \otimes R \\ \leftarrow \mathbb{S}_{(2,1)}(V) \otimes \mathbb{S}_{(2,0,-1)}(W^*) \otimes R &\cong \mathbb{S}_{(2,1)}(V)^{15} \otimes R \\ \leftarrow \mathbb{S}_{(4,3)}(V) \otimes \mathbb{S}_{(2,2,1)}(W^*) \otimes R &\cong \mathbb{S}_{(3,2)}(V)^3 \otimes R. \end{aligned}$$

The last arrow is from the E_2 page. This realizes the desired Betti table.

The construction above produces (based on the input from [EFW11]) the three-term tables, up to scalar multiple, for the triples (λ, μ, ν) for which the skew shapes μ/λ and ν/μ are *border strips*:

 or  (i.e. connected shapes not containing ),

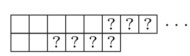
and where the second border strip is adjacent and to the right of the first. Examples of such triples are:

Here λ consists of the empty squares, μ contains the additional squares marked 1, and ν contains the squares marked 1, 2. We can rule out almost all other possible triples. Indeed, we have the following proposition.

Proposition 6.2. *Let β be a three-term table with entries $\lambda \subsetneq \mu \subsetneq \nu$. Then $\nu_2 \leqslant \lambda_1 + 1$.*

That is, the shapes μ/λ and ν/μ are *contained in* the border strip formed by the squares along the outer edge of λ:



We do not know if they must be connected or adjacent (see Example 6.3). We prove Proposition 6.2 using the equivariant pairing.

Proof. We pair with the cohomology table $\gamma = \gamma(\mathcal{O}(d))$ for appropriate choice of d. Note that $\lambda_1 \leqslant \mu_1 \leqslant \nu_1$ and that $\nu_2 \leqslant \nu_1$; we assume that $\nu_2 > \lambda_1 + 1$.

Suppose first that $\lambda_1 + 1 < \nu_2 \leqslant \mu_1 + 1$. Then we put $d = \nu_2 - 2$. By the Borel–Weil–Bott theorem, we have $\gamma_{0,\lambda} \neq 0$, at most one of $\gamma_{1,\mu} \neq 0$ or $\gamma_{2,\mu} \neq 0$, and all ν entries and $\gamma_{0,\mu}$ zero, giving

$$\langle \beta, \gamma(\mathcal{O}(d)) \rangle = \begin{array}{c|cc} & -1 & 0 \\ \hline \lambda & - & \beta_{0,\lambda}\gamma_{0,\lambda} \\ \mu & \beta_{1,\mu}\gamma_{2,\mu} & \beta_{1,\mu}\gamma_{1,\mu} \\ \nu & - & - \end{array}$$

Since $\lambda \subsetneq \mu$, there is no valid perfect matching regardless of whether $\gamma_{1,\mu}, \gamma_{2,\mu}$ are zero.

Suppose instead that $\mu_1 + 1 < \nu_2$ $(< \nu_1 + 1)$. Then we put $d = \mu_1 - 1$. This time $\gamma_{2,\nu} \neq 0$ and possibly $\gamma_{0,\lambda} \neq 0$, but all μ entries are zero. But then $\langle \beta, \gamma \rangle$ has only one column:

$$\langle \beta, \gamma(\mathcal{O}(d)) \rangle = \begin{array}{c|c} & 0 \\ \hline \lambda & \beta_{0,\lambda}\gamma_{0,\lambda} \\ \mu & - \\ \nu & \beta_{2,\nu}\gamma_{2,\mu} \end{array}$$

Since $\beta_{2,\nu}\gamma_{2,\nu} \neq 0$, there is no perfect matching. ∎

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The next example shows, however, that not all three-term solutions to the equivariant Herzog–Kühl equations are built from connected border strips.

Example 6.3 (Exceptional three-term tables). Consider the tables

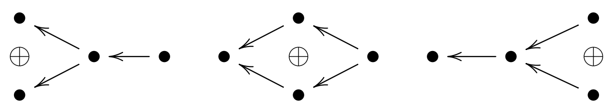
	0	1	2
$\ydiagram{4}$	2		
$\ydiagram{5,2}$		5	
$\ydiagram{7,3}$			2

	0	1	2
$\ydiagram{4}$	14		
$\ydiagram{5,1}$		21	
$\ydiagram{8,2}$			5

We do not know whether the tables above (part of an infinite family) are realizable, but we remark that they pair positively with the cohomology tables $\gamma(\mathcal{O}(d))$ for all $d \in \mathbb{Z}$ (these are all the tables $\gamma(\mathbb{S}_\mu(\mathcal{Q}))$ on $\mathrm{Gr}(2, 3) \cong \mathbb{P}^2$, since $\mathcal{Q} \cong \mathcal{O}(1)$). The corresponding equivariant Hilbert series are also Schur-positive.

6.3 Four-term tables

In each four-term table, one of the columns has two nonzero entries. We will say the table is *diamond-shaped* when the middle column has two nonzero entries and *Y-shaped* otherwise:



We will construct resolutions of these forms starting from modules with three-term resolutions, obtaining diamond-shaped tables as (co)kernels and Y-shaped tables as extensions. For brevity, we will write the Young diagram for λ to mean the free module $\mathbb{S}_\lambda(V) \otimes R$.

Example 6.4 (Y-shaped tables via extensions). Consider the following Y-shaped table:

	0	1	2
$\ydiagram{1}$	1		
$\ydiagram{2,1}$		3	
$\ydiagram{3,1}$			1
$\ydiagram{2,2}$			1

We obtain this table as an extension of the following two three-term resolutions:

$$
\begin{array}{ccccccc}
M & \leftarrow & \ydiagram{1} & \xleftarrow{f} & \ydiagram{1,1}^{\oplus 3} & \xleftarrow{e} & \ydiagram{2,2}, \\
N & \xleftarrow{h} & \ydiagram{1,1}^{\oplus 3} & \xleftarrow{g} & \ydiagram{2,1}^{\oplus 3} & \leftarrow & \ydiagram{3,1}.
\end{array}
$$

Note that the term $\ydiagram{1,1}^{\oplus 3}$ appears in both resolutions, and that all the other terms appear in exactly the desired positions for the resolution we wish to construct. We claim that a map $t : \ydiagram{2,2} \to \ydiagram{2,1}^{\oplus 3}$ exists such that the following diagram commutes:

$$
\begin{tikzcd}
\ydiagram{1} & \ydiagram{1,1}^{\oplus 3} \arrow[l, "f"'] \arrow[d, "\cong"'] & \ydiagram{2,2} \arrow[l, "e"'] \arrow[d, dashed, "t"'] & \\
& \ydiagram{1,1}^{\oplus 3} & \ydiagram{2,1}^{\oplus 3} \arrow[l, "g"'] & \ydiagram{3,1} \arrow[l]
\end{tikzcd}
$$

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Consequently, the module E resolved by the total complex above has (after removing the two $\ydiagram{1,1}^{\oplus 3}$ terms) the desired resolution. To show that t exists, it is enough to show $h \circ e = 0$, and so by exactness $\operatorname{im}(e) \subset \operatorname{im}(g)$. Then t exists because the module $\ydiagram{2,2}$ is free, in particular projective. The key fact, which is straightforward to compute, is that the $\ydiagram{2,2}$ coefficient is zero in the equivariant Hilbert series of N:

$$H_N(t_1, t_2) = \frac{3t_1t_2 - 3(t_1^2t_2 + t_1t_2^2) + (t_1^3t_2 + t_1^2t^2 + t_1t_2^3)}{(1-t_1)^3(1-t_2)^3}.$$

Thus the generators of $\ydiagram{2,2}$ must map to zero in N.

The construction above relies only on properties of M and N that are observable from their Betti tables and invariant under scaling. Namely, M and N have three-term resolutions with a term in common, and the appropriate s_λ coefficient in $H_N(t)$ is zero. Also, since the constructed module E is Cohen–Macaulay, its dual $\operatorname{Ext}^2(E, R)$ has the dual four-term resolution, which is Y-shaped facing the other way.

We construct diamond-shaped tables in a similar way.

Example 6.5 (Diamond shaped tables via (co)kernels). Consider the table

	0	1	2
$\ydiagram{1}$	3		
$\ydiagram{2}$		1	
$\ydiagram{1,1}$		9	
$\ydiagram{2,1}$			3

We obtain (a multiple of) this table by constructing the following double complex:

$$\begin{tikzcd}
\emptyset^{\oplus 3} \arrow[d, "\cong"'] & \ydiagram{1,1}^{\oplus 9} \arrow[l] \arrow[d, dashed, "t_1"'] & \ydiagram{2,1}^{\oplus 3} \arrow[l] \arrow[d, dashed, "t_2"'] \\
\emptyset^{\oplus 3} & \ydiagram{1}^{\oplus 3} \arrow[l] & \ydiagram{2}^{\oplus 1} \arrow[l]
\end{tikzcd}$$

As in Example 6.4, the rows are three-term resolutions as constructed up to scalar multiple in Example 6.1. (The first row is just three copies of the Eagon–Northcott complex.) Here $\emptyset$ denotes R as a module over itself. The existence of t_1 follows again from an equivariant Hilbert series computation (in fact the same computation as in the previous example, only twisted by $\det(\mathbb{C}^2)$), showing that the module resolved by the second row has no $\ydiagram{1,1}$-isotypic part. Then t_2 existsautomatically, in fact uniquely since the map $\ydiagram{2} \to \ydiagram{1}^{\oplus 3}$ is injective. The total complex is the desired minimal free resolution (after dropping the $\emptyset^{\oplus 3}$ terms).

As with the three-term tables, we do not know if all four-term tables can be constructed in this way (moreover, it would be natural to construct certain four-term tables from the conjectural three-term tables of Example 6.3). Nonetheless, we can rule out some otherwise plausible four-term tables using the equivariant pairing.

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Example 6.6 (A nonrealizable Y-shaped table). Consider the following table:

$$\beta = \begin{array}{c|ccc} \beta_{p,\lambda} & 0 & 1 & 2 \\ \hline \varnothing & 1 & & \\ \ydiagram{2} & 3 & & \\ \ydiagram{2,1} & & 8 & \\ \ydiagram{3,2} & & & 3 \end{array}$$

Despite satisfying the Herzog–Kühl equations and resembling the unobjectionable examples above, this table is not in $BS_{2,3}$. This follows from the numerical pairing: when paired with $\mathcal{O}(d)$, the result is in $\widetilde{BS}_{2,2}$ for every d except $d = 0$, for which the output is

$$\langle \beta, \gamma(\mathcal{O}) \rangle = \begin{array}{c|cc} & -1 & 0 \\ \hline \varnothing & & 1 \\ \ydiagram{2} & 9 & \\ \ydiagram{2,1} & & 8 \end{array}$$

Clearly, this table has no perfect matching. We note in this example that the *graded* Betti table obtained by forgetting the GL_2-action *is* realizable:

$$\begin{array}{c|ccc} \beta_{i,d} & 0 & 1 & 2 \\ \hline d=0 & 1 & & \\ d=2 & 9 & & \\ d=3 & & 16 & \\ d=5 & & & 6 \end{array} = 15 \cdot \begin{array}{c|ccc} \beta_{i,d} & 0 & 1 & 2 \\ \hline d=0 & 1/15 & & \\ d=2 & - & & \\ d=3 & & 1/6 & \\ d=5 & & & 1/10 \end{array} + 27 \cdot \begin{array}{c|ccc} \beta_{i,d} & 0 & 1 & 2 \\ \hline d=0 & - & & \\ d=2 & 1/3 & & \\ d=3 & & 1/2 & \\ d=5 & & & 1/6 \end{array}$$

The latter two tables are for the degree sequences $(0, 3, 5)$ and $(2, 3, 5)$. Thus, in this case the equivariant pairing detects that such a table cannot be realized with the additional GL_2-structure given above.

We end with a phenomenon that does not appear in the square-matrix or graded cases.

Example 6.7 ('Stably realizable' Betti tables). Let β be the nonrealizable table of Example 6.6. Consider tensoring β with $\mathbb{S}_1 V$. (That is, write the Betti table that would result from tensoring such a resolution, if it existed, with $\mathbb{S}_1 V$.) The result is as follows:

$$\begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline \ydiagram{1} & 1 & & \\ \ydiagram{3} & 3 & & \\ \ydiagram{2,1} & 3 & & \\ \ydiagram{3,1} & & 8 & \\ \ydiagram{2,2} & & 8 & \\ \ydiagram{4,2} & & & 3 \\ \ydiagram{3,3} & & & 3 \end{array}$$

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This table in fact *is* realizable, a linear combination of the following realizable tables:

A	0	1	2
(3)	1		
$(3,1)$		2	
$(3,3)$			2

B	0	1	2
$(2,1)$	3		
$(3,1)$		1	
$(2,2)$		6	
$(4,2)$			1

C	0	1	2
(3)	1		
$(2,1)$	1		
$(3,1)$		3	
$(4,2)$			1

D	0	1	2
(1)	1		
$(2,2)$		5	
$(4,2)$			1

Specifically, the large table above is $\frac{3}{2}A + \frac{1}{2}B + \frac{3}{2}C + D$.

This example suggests that it might be easier to study 'stably realizable' Betti tables, that is, Betti tables that become realizable after tensoring with some $\mathrm{GL}(V)$-representation.

Acknowledgements

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Finally, computations of free resolutions and Betti tables in Macaulay2 [GS], and facets and rays of cones in Sage [Sage16] (via SageMathCloud) have been, and continue to be, instrumental.

Appendix A. The proof of Proposition 5.7

We give the proof of Proposition 5.7, which uses the axiom of choice. Throughout, let V, W be vector spaces of arbitrary dimension, with specified bases $\mathcal{V}, \mathcal{W}$. Let $T : V \to W$ be an isomorphism.

Proposition A.1. *The coefficient graph of T has a perfect matching.*

The bulk of this proof, notably the reduction to countable-dimensional spaces, is due to Lampert [Lam16].

Proof. We first reduce to $\mathcal{V}, \mathcal{W}$ countable. Let $b \in \mathcal{V}$ be arbitrary. Then $T(b)$ involves only finitely many basis elements, say $\mathcal{B}_1 \subset \mathcal{W}$. For each $s \in \mathcal{B}_1$, $T^{-1}(s)$ only involves finitely many basis elements; let $\mathcal{A}_2 \subset \mathcal{V}$ contain these new elements, together with b. Repeat this construction, building two sequences of coordinate subspaces, writing $A_i = \mathrm{span}(\mathcal{A}_i)$ and $B_i = \mathrm{span}(\mathcal{B}_i)$,

$$(b) = A_1 \subset A_2 \subset A_3 \subset \cdots \subset V, \quad B_1 \subset B_2 \subset B_3 \subset \cdots \subset W,$$

such that, for each i, $T(A_i) \subset B_i$ and $T^{-1}(B_i) \subset A_{i+1}$. Let A_∞, B_∞ be the unions and let A', B' be spanned by the remaining vectors. It now follows that $T(A_\infty) = B_\infty$ and, moreover, the composition $A' \hookrightarrow A \to B \twoheadrightarrow B'$ is again an isomorphism, with the same coefficients as T. (To see that $A' \to B'$ is injective: if $a \mapsto 0$, we get $T(a) \in B_\infty$, but $T^{-1}(B_\infty) = A_\infty$. For surjectivity, given $b \in B$, decompose $T^{-1}(b) = a_\infty + a'$, with $a_\infty \in A_\infty$ and $a' \in A'$. Then $T(a') = b$ modulo B_∞ since $B_\infty = T(A_\infty)$.) By transfinite induction, we reduce to the countable case.

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We now assume $\mathcal{V}, \mathcal{W}$ are countable and build the matching inductively. Fix a basis vector $v \in \mathcal{V}$ and write $T(v) = \sum a_i w_i$, and assume every w_i in the sum has $a_i \neq 0$. Equivalently,

$$v = \sum a_i T^{-1}(w_i),$$

so some $T^{-1}(w_i)$ contributes a nonzero v-coefficient. Fix one such w; we match $v \leftrightarrow w$. Note that this choice is compatible with both T and T^{-1}. Let

$$C = \operatorname{span}(\mathcal{V} \backslash \{v\}), \quad D = \operatorname{span}(\mathcal{W} \backslash \{w\}).$$

By a similar argument to the above, $C \hookrightarrow V \to W \twoheadrightarrow D$ is an isomorphism (with the same coefficients as T, but with v and w removed). We continue, alternating between V and W, always choosing the first unmatched basis vector on each side to ensure that every basis vector gets matched. Note that the construction is symmetric with respect to T and T^{-1}. □

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