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SOME REMARKS ON SYMMETRY FOR A MONOIDAL CATEGORY

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It is shown that, for a monoidal category $\mathcal{V}$, not every commutation is a symmetry and also that a commutation does not suffice to define the tensor product $\mathcal{A} \otimes \mathcal{B}$ of $\mathcal{V}$-categories $\mathcal{A}$ and $\mathcal{B}$. Moreover, it is shown that every symmetry can be transported along a monoidal equivalence.

1.

Let $\mathcal{V} = (\mathcal{V}_0, \otimes, I, a, l, r)$ be a monoidal category. By a *symmetry* c for $\mathcal{V}$ is meant ([2], p. 512) a natural isomorphism $c = c_{AB} : A \otimes B \to B \otimes A$ in $\mathcal{V}_0$ rendering commutative the diagrams

(i)

$$\begin{array}{ccc} A \otimes B & \xrightarrow{\;c\;} & B \otimes A \\ & \searrow^{1} & \downarrow c \\ & & A \otimes B \, , \end{array}$$

(ii)

$$\begin{array}{ccccc} (A \otimes B) \otimes C & \xrightarrow{\;a\;} & A \otimes (B \otimes C) & \xrightarrow{\;c\;} & (B \otimes C) \otimes A \\ \downarrow c \otimes 1 & & & & \downarrow a \\ (B \otimes A) \otimes C & \xrightarrow[\;a\;]{} & B \otimes (A \otimes C) & \xrightarrow[\;1 \otimes c\;]{} & B \otimes (C \otimes A) \, . \end{array}$$

By the coherence-theorem of Mac Lane [5], along with the observations of

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[3], all "proper" diagrams made from a, l, r, c commute in the presence of (i) and (ii).

On the other hand, Bénabou [1] has defined a *commutation* for a monoidal category V to be a natural isomorphism $c : A \otimes B \to B \otimes A$ satisfying (i) and such that $(1_V, c, 1_I) : V \to V_{\text{rev}}$ is a monoidal functor; here V_{rev} is V_0 with the evident "reversed" monoidal structure where $A \otimes_{\text{rev}} B = B \otimes A$. In other words, a commutation is an isomorphism c rendering commutative (i) and the diagrams

(iii)

$$\begin{array}{ccc} (A \otimes B) \otimes C & \xrightarrow{\ a\ } & A \otimes (B \otimes C) \\ {\scriptstyle c \otimes 1}\downarrow & & \downarrow{\scriptstyle 1 \otimes c} \\ (B \otimes A) \otimes C & & A \otimes (C \otimes B) \\ {\scriptstyle c}\downarrow & & \downarrow{\scriptstyle c} \\ C \otimes (B \otimes A) & \xleftarrow[\ a\ ]{} & (C \otimes B) \otimes A \end{array} ,$$

(iv)

$$\begin{array}{ccc} I \otimes A & \xrightarrow{\ c\ } & A \otimes I \\ & {\scriptstyle l}\searrow \quad \swarrow{\scriptstyle r} & \\ & A & \end{array}$$

By the coherence theorem, every symmetry is clearly a commutation. It is natural to ask whether every commutation is also a symmetry.

PROPOSITION 1. *Not every commutation is a symmetry, even if the monoidal category is closed.*

Proof. Take for V the category of positively-graded abelian groups, with its usual closed monoidal structure. Define $c : A \otimes B \to B \otimes A$ as follows: for homogeneous elements $x \in A_n$ and $y \in B_m$ set $c(x \otimes y) = y \otimes x$ if $mn = 0$, $c(x \otimes y) = -y \otimes x$ otherwise. Then (i), (iii) and (iv) commute, but not (ii).

2.

Eilenberg and Kelly make essential use of a symmetry c on the monoidal category V for two purposes: to define the dual A^{op} of a V-category A ([2], p. 514), and to define the tensor product $A \otimes B$ of

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V-categories A and B ([2], p. 518). The question arises whether the weaker notion of a commutation would serve for these purposes.

For the first it clearly does. The diagram they need to commute on their p. 515, to ensure the associativity of composition in A^{op}, uses precisely (iii); and the similar diagrams for the unit-laws need precisely (iv). Indeed this is predictable: the monoidal functor $(1_V, c, 1_I) : V \to V_{rev}$ induces, from the V-structure on A, a V_{rev}-structure on A, and hence a V-structure on A^{op}.

For the second they use coherence to infer the commutativity of the diagram

(v)
$$\begin{array}{ccc}
((A \otimes X) \otimes (B \otimes Y)) \otimes (C \otimes Z) & \xrightarrow{a} & (A \otimes X) \otimes ((B \otimes Y) \otimes (C \otimes Z)) \\
\downarrow{\scriptstyle m \otimes 1} & & \downarrow{\scriptstyle 1 \otimes m} \\
((A \otimes B) \otimes (X \otimes Y)) \otimes (C \otimes Z) & & (A \otimes X) \otimes ((B \otimes C) \otimes (Y \otimes Z)) \\
\downarrow{\scriptstyle m} & & \downarrow{\scriptstyle m} \\
((A \otimes B) \otimes C) \otimes ((X \otimes Y) \otimes Z) & \xrightarrow[a \otimes a]{} & (A \otimes (B \otimes C)) \otimes (X \otimes (Y \otimes Z))
\end{array}$$

where m is the "middle-four interchange", given by

(vi)
$$\begin{array}{ccc}
(A \otimes X) \otimes (B \otimes Y) & \xrightarrow{m} & (A \otimes B) \otimes (X \otimes Y) \\
\downarrow{\scriptstyle a} & & \downarrow{\scriptstyle a} \\
A \otimes (X \otimes (B \otimes Y)) & & A \otimes (B \otimes (X \otimes Y)) \\
\downarrow{\scriptstyle 1 \otimes a^{-1}} & & \downarrow{\scriptstyle 1 \otimes a^{-1}} \\
A \otimes ((X \otimes B) \otimes Y) & \xrightarrow[1 \otimes (c \otimes 1)]{} & A \otimes ((B \otimes X) \otimes Y) .
\end{array}$$

The commutativity of (v) ensures ([2], p. 518) the commutativity of the diagram expressing the associativity of the composition in $A \otimes B$. But, if V has initial object 0 preserved by the bifunctor $\otimes$, the commutativity of (v) is also *necessary* for this associativity. Take for A the free V-category on the V-graph G with four objects P, Q, R, S and with $G(P, Q) = C$, $G(Q, R) = B$, $G(R, S) = A$, and all other $G(U, V)$ equal to the initial object 0. That is, $A(P, Q) = C$, $A(P, R) = B \otimes C$, $A(P, S) = A \otimes B \otimes C$, and so on, and the relevant components of the

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composition in $\mathcal{A}$ are identities. Take for $\mathcal{B}$ the $\mathcal{V}$-category similarly defined in relation to X, Y, Z . Then the associativity condition for the composition in $\mathcal{A} \otimes \mathcal{B}$ actually reduces to (v).

Now in fact a commutation is *not* enough to ensure the commutativity of (v); for

PROPOSITION 2. *In the presence of* (i) *and* (iv), (v) *implies* (ii).

Proof. Set $A = B = Z = I$ in (v) and simplify.

3.

That being so, it is relevant to ask whether a symmetry for $\mathcal{V}$ can be transported along an equivalence $\eta, \varepsilon : \Phi \dashv \Psi : \mathcal{V}' \to \mathcal{V}$ in the 2-category of the monoidal categories, to give a symmetry on $\mathcal{V}'$. In fact more than this is true: it suffices that $\mathcal{V}'$ be equivalent to a "full monoidally-reflective monoidal subcategory" of $\mathcal{V}$. Precisely:

PROPOSITION 3. *Let* $\eta, \varepsilon : \Phi \dashv \Psi : \mathcal{V}' \to \mathcal{V}$ *be an adjunction in the 2-category of monoidal categories, with* ε *an isomorphism; and let* c *be a symmetry for* $\mathcal{V}$. *Then there is exactly one symmetry* c' *for* $\mathcal{V}'$ *such that the above adjunction becomes one in the 2-category of symmetric monoidal categories.*

Proof. By [4], Theorem 1.5, the left adjoint $\Phi : \mathcal{V} \to \mathcal{V}'$ is a *strong* monoidal functor, so that $\tilde{\phi} : \phi A \otimes \phi B \to \phi(A \otimes B)$ is an isomorphism. If Φ is to be a *symmetric* monoidal functor, it must satisfy the condition MF4 of [2], p. 513, asserting that $\tilde{\phi}$ commutes with the symmetries. From this and the naturality of c' , we must have commutativity in

(vii)
$$\begin{array}{ccc}
A \otimes B & \xrightarrow{\quad c' \quad} & B \otimes A \\
{\scriptstyle \varepsilon^{-1} \otimes \varepsilon^{-1}}\downarrow & & \downarrow{\scriptstyle \varepsilon^{-1} \otimes \varepsilon^{-1}} \\
\phi\psi A \otimes \phi\psi B & & \phi\psi B \otimes \phi\psi A \\
{\scriptstyle \tilde{\phi}}\downarrow & & \downarrow{\scriptstyle \tilde{\phi}} \\
\phi(\psi A \otimes \psi B) & \xrightarrow[\quad \phi c \quad]{} & \phi(\psi B \otimes \psi A)
\end{array} \quad ;$$

and this suffices to define c' , since ε and $\tilde{\phi}$ are isomorphisms. That $(c')^2 = 1$ is immediate since $c^2 = 1$. It remains to verify (ii).

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Imagine a hexagonal prism, whose top face is the diagram (ii) for A, B, C, a' and c', and whose bottom face is ϕ applied to the diagram (ii) for ψA, ψB, ψC, a and c ; so that the bottom commutes. The vertical edges are all isomorphisms of the form

$\tilde{\phi}(\tilde{\phi} \otimes 1)\big((\varepsilon^{-1} \otimes \varepsilon^{-1}) \otimes \varepsilon^{-1}\big)$ or $\tilde{\phi}(1 \otimes \tilde{\phi})\big(\varepsilon^{-1} \otimes (\varepsilon^{-1} \otimes \varepsilon^{-1})\big)$, as appropriate. The desired result that the top commutes will follow if all the vertical faces commute. For five of them this follows easily from (vii), the axiom MF3 for monoidal functors on p. 473 of [2], and naturality. For the sixth - the one with c' at the top and ϕc at the bottom - it is necessary to observe that $\varepsilon^{-1} : B \otimes C \to \phi\psi(B \otimes C)$ is, by the axiom MN2 of [2], p. 474, for the monoidal natural transformation $\varepsilon : \Phi\Psi \to 1$, equal to $\phi\tilde{\psi}\cdot\tilde{\phi}\cdot(\varepsilon^{-1} \otimes \varepsilon^{-1})$; and also that the maps involving $\tilde{\psi}$ are isomorphisms because everything else in the diagram is. With that said, the reader will have no trouble drawing the necessary diagrams, which it would be tedious to print.

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