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THERE ARE NO DENTING POINTS IN THE UNIT BALL OF $\mathcal{P}(^2H)$

SUNG GUEN KIM

For any infinite dimensional real Hilbert space H we show that the unit ball of the space of continuous 2-homogeneous polynomials on H, $\mathcal{P}(^2H)$, has no denting points. Thus the unit ball of $\mathcal{P}(^2H)$ has no strongly exposed points.

Throughout we assume that E is a real Banach space with its dual E^*. Let B_E and S_E be the closed unit ball and the unit sphere of E, respectively. A point $x \in S_E$ is an *extreme point of* B_E if $x = (y+z)/2$ with $y, z \in B_E$ implies $x = y = z$. A point $x \in S_E$ is a *strongly exposed point of* B_E if there is a unit vector $f \in E^*$ so that $f(x) = 1$ and given any sequence (x_k) in B_E with $f(x_k) \to 1$ we can conclude that $x_k \to x$ in norm. A point $x \in S_E$ is said to be a *denting point of* B_E if and only if for every $\varepsilon > 0$ there exist $f \in E^*$ and $0 < \delta < f(x)$ such that $\operatorname{diam} S(B_E, f, \delta) := \operatorname{diam}\{y \in B_E : f(y) > \delta\} < \varepsilon$. It is easy to see that every denting point of B_E is an extreme point, and that every strongly exposed point of B_E is a denting point.

Let H be a real Hilbert space. A mapping $P : H \to \mathbb{R}$ is called a continuous *n-homogeneous polynomial* if there is a continuous n-linear mapping $A : H \times \cdots \times H \to \mathbb{R}$ such that $P(x) = A(x, \ldots, x)$ for each $x \in H$. We let $\mathcal{P}(^nH)$ denote the Banach space of continuous n-homogeneous polynomials of H into $\mathbb{R}$, endowed with the *polynomial norm* $\|P\| = \sup\Big\{\big|P(x)\big| : \|x\| \leqslant 1\Big\}$. See [**1**] for details about the theory of polynomials on an infinite dimensional Banach space.

To establish our result, we need the description of the extreme points of the unit ball of $\mathcal{P}(^2H)$ given in [**2**].

THEOREM 1. (Grecu) *It is true that for a real Hilbert space H, P is an extreme point of the unit ball of $\mathcal{P}(^2H)$ if and only if there exists an orthogonal decomposition of $H = H_1 \bigoplus H_2$ such that $P(x) = \big\|\pi_1(x)\big\|^2 - \big\|\pi_2(x)\big\|^2$, where $\pi_j : H \to H_j$ are the orthogonal projections of H onto H_j $(j = 1, 2)$.*

For an infinite compact set K and for any Banach space E, Rao [**4**] showed that the unit ball of the space of $E-$ valued functions on K that are continuous when E is equipped with the weak topology, has no denting points.

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Recently Kim and Lee [**3**, Theorem 2] showed that if H is an infinite dimensional real Hilbert space, then the unit ball of the space $\mathcal{P}(^2H)$ has no strongly exposed points. In this note we show that for any infinite dimensional real Hilbert space H the unit ball of $\mathcal{P}(^2H)$ has no denting points. Thus the unit ball of $\mathcal{P}(^2H)$ has no strongly exposed points.

Here is our main result.

Theorem 2. *Let H be an infinite dimensional real Hilbert space. Then the unit ball of $\mathcal{P}(^2H)$ has no denting points.*

Proof: It suffices to show that every extreme point of the unit ball of $\mathcal{P}(^2H)$ is not a denting point. Let P be an extreme point of the unit ball of $\mathcal{P}(^2H)$. By Theorem 1 we have

$$P(x) = \sum_{\alpha\in A} \langle x, e_\alpha\rangle^2 - \sum_{\beta\in B} \langle x, t_\beta\rangle^2 \qquad (x \in H)$$

where $\{e_\alpha, t_\beta\}$ forms an orthonormal basis of H.

We claim that $\operatorname{diam} S(B_{\mathcal{P}(^2H)}, f, \delta) = 2$ for each $f \in \mathcal{P}(^2H)^*$ with $f(P) > \delta$ and for each $\delta > 0$. We may assume that A is an infinite set. Note that

$$f(P) = \sum_{\alpha\in A} f\big(\langle\cdot, e_\alpha\rangle^2\big) - \sum_{\beta\in B} f\big(\langle\cdot, t_\beta\rangle^2\big),$$

so $f\big(\langle\cdot, e_\alpha\rangle^2\big) \to 0$ as $\alpha \to \infty$. Choose $\alpha_1 \in A$ such that $2f\big(\langle\cdot, e_{\alpha_1}\rangle^2\big) < f(P) - \delta$.

Let $Q = P - 2\langle\cdot, e_{\alpha_1}\rangle^2$. By Parseval's identity we have

$$Q \in B_{\mathcal{P}(^2H)} \text{ and } f(Q) > \delta,$$

so $Q \in S\big(B_{\mathcal{P}(^2H)}, f, \delta\big)$. So we have

$$2 \geqslant \operatorname{diam} S\big(B_{\mathcal{P}(^2H)}, f, \delta\big) \geqslant \|P - Q\| = \big\|2\langle\cdot, e_{\alpha_1}\rangle^2\big\| = 2.$$

Thus P is not a denting point. $\square$

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Department of Mathematics
Kyungpook National University
Daegu (702-701)
Korea
e-mail: sgk317@knu.ac.kr