Canad. J. Math. Vol. **65** (3), 2013 pp. 702–720
http://dx.doi.org/10.4153/CJM-2012-014-3

Regularity of Standing Waves on Lipschitz Domains

Michael Taylor

Abstract. We analyze the regularity of standing wave solutions to nonlinear Schrödinger equations of power type on bounded domains, concentrating on Lipschitz domains. We establish optimal regularity results in this setting, in Besov spaces and in Hölder spaces.

1 Introduction

Let Ω be a relatively compact open subset of $\mathbb{R}^n$, or more generally a connected, n-dimensional Riemannian manifold M, with smooth metric tensor. We take $n \geq 2$. We assume that Ω is connected and $M \setminus \overline{\Omega} \neq \varnothing$. Let Δ denote the Laplace–Beltrami operator on M, and place the Dirichlet condition on $\partial\Omega$. A standing wave solution to the nonlinear Schrödinger equation

$$i\partial_t v + \Delta v = -K|v|^{p-1}v \tag{1.1}$$

on $\mathbb{R} \times \Omega$ is a solution of the form $v(t,x) = e^{i\lambda t}u(x)$. Such a function solves (1.1) if and only if u solves

$$(-\Delta + \lambda)u = K|u|^{p-1}u. \tag{1.2}$$

It is well known (*cf.* [9]) that, as long as

$$\operatorname{Spec}(-\Delta) \subset [\alpha, \infty), \quad \lambda > -\alpha, \quad p \in \Big(1, \frac{n+2}{n-2}\Big), \quad K > 0, \tag{1.3}$$

one can find a solution $u \in H_0^1(\Omega)$ to (1.2), by the process of minimizing

$$F_\lambda(u) = \|\nabla u\|_{L^2}^2 + \lambda\|u\|_{L^2}^2 \tag{1.4}$$

over $u \in H_0^1(\Omega)$, subject to the constraint

$$I_p(u) = \int_\Omega |u|^{p+1}\, dV = A, \tag{1.5}$$

Received by the editors August 30, 2011.
Published electronically 0, 0000.
This work was partially supported by NSF grant DMS-0758320.
AMS subject classification: **35J25**, 35J65.
Keywords: standing waves, elliptic regularity, Lipschitz domain.

where $A \in (0, \infty)$ is specified. Such a solution is called a ground state solution to (1.2). We briefly recall the argument. The Sobolev embedding theorem gives

$$H_0^1(\Omega) \subset L^r(\Omega), \quad \forall\, r \in \Big(1, \frac{2n}{n-2}\Big],$$

with the right endpoint omitted when $n = 2$, and the inclusion is compact if $r < 2n/(n-2)$. The condition on p in (1.3) is equivalent to $2 < p+1 < 2n/(n-2)$. Thus a sequence (u_ν) minimizing (1.4), subject to (1.5), has a subsequence converging to $u \in H_0^1(\Omega)$ weak*, hence in L^{p+1}-norm. The limit satisfies (1.5) and hence is a minimizer. Since, for each $w \in H_0^1(\Omega)$,

$$\begin{aligned} \frac{d}{d\tau} F_\lambda(u+\tau w)\Big|_{\tau=0} &= \operatorname{Re}(\nabla u, \nabla w) + \lambda \operatorname{Re}(u, w) \\ &= \operatorname{Re}\big((-\Delta+\lambda)u, w\big) \end{aligned}$$

and

$$\frac{d}{d\tau} I_p(u+\tau w)\Big|_{\tau=0} = (p+1)\operatorname{Re}(|u|^{p-1}u, w),$$

it follows that there exists a real constant K_0 such that

$$(-\Delta+\lambda)u = K_0 |u|^{p-1} u.$$

Note that

$$\|\nabla u\|_{L^2}^2 + \lambda \|u\|_{L^2}^2 = \big((-\Delta+\lambda)u, u\big) = K_0 \int_\Omega |u|^{p+1}\, dV = AK_0,$$

so $K_0 > 0$. Replacing u by $u_a = au$ gives a solution to (1.2) with $K = |a|^{1-p} K_0$, yielding "ground state" solutions to (1.2) for arbitrary $K > 0$, when (1.3) holds.

Our goal in this paper is to establish regularity results for solutions $u \in H_0^1(\Omega)$ to (1.2), particularly regularity up to the boundary for Lipschitz domains, *i.e.*, domains whose boundary is given locally as the graph of a Lipschitz function. One phenomenon influencing the results is the fact that

$$f(u) = K|u|^{p-1}u \tag{1.6}$$

is not smooth at $u = 0$, unless $p - 1$ is an even integer. In this context, note that the condition on p in (1.3) requires

$$\begin{gathered} 1 < p < \infty \ \text{ for } \ n = 2, \quad 1 < p < 5 \ \text{ for } \ n = 3, \\ 1 < p < 3 \ \text{ for } \ n = 4, \quad 1 < p < \frac{7}{3} \ \text{ for } \ n = 5, \\ 1 < p < 2 \ \text{ for } \ n = 6. \end{gathered}$$

We start off with the following result.

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Theorem 1.1 *When Ω is smoothly bounded, each solution $u \in H_0^1(\Omega)$ to (1.2), with p satisfying (1.3), has the property*

$$u \in C^{p+2}(\overline{\Omega}) \tag{1.7}$$

if $p \notin \mathbb{N}$. If p is an odd integer, $u \in C^\infty(\overline{\Omega})$. If p is an even integer, $u \in C^s(\overline{\Omega})$ for all $s < p+2$. If p is an even integer and $u \geq 0$ on Ω, $u \in C^\infty(\overline{\Omega})$.

This is established in Section 2. The sign of K does not matter for this regularity result, nor for that of Theorem 1.2. The proof of Theorem 1.1 is relatively straightforward. We include this material mainly to set the stage for the case of Lipschitz domains, where the analysis is much less straightforward. In Section 2 we also establish analogous interior regularity of solutions to (1.2), valid for arbitrary open $\Omega \subset M$. In Sections 3 and 4, we establish the following theorem, which is the main result of this paper.

Theorem 1.2 *When $\Omega \subset M$ is a bounded Lipschitz domain, each solution $u \in H_0^1(\Omega)$ to (1.2), with p satisfying (1.3), has the following properties. First, for some $\varepsilon = \varepsilon(\Omega) >$ 0,*

$$u \in B_{q,q}^{1+1/q}(\Omega), \quad \forall q \in \big[2, 2+\varepsilon(\Omega)\big). \tag{1.8}$$

Next, for some $s > 0$,

$$u \in C^s(\overline{\Omega}). \tag{1.9}$$

If $n = \dim \Omega = 2$, (1.9) holds for some $s > 1/2$.

We describe the function spaces arising in (1.7)–(1.9). For $0 < s < 1$, $C^s(\overline{\Omega})$ denotes the space of functions on $\overline{\Omega}$ that are Hölder continuous of exponent s (where $C^0(\overline{\Omega}) = C(\overline{\Omega})$). For $s = k + \sigma,\ k \in \mathbb{Z}^+,\ \sigma \in [0,1)$, $C^s(\overline{\Omega})$ consists of functions whose derivatives of order $\leq k$ belong to $C^\sigma(\overline{\Omega})$.

For $s > 0,\ p \in (1,\infty),\ q \in [1,\infty]$, $B_{p,q}^s(\Omega)$ is a Besov space, a relative of the L^p-Sobolev space $H^{s,p}(\Omega)$. One characterization is

$$B_{p,q}^s(\Omega) = \big(L^p(\Omega), H^{k,p}(\Omega)\big)_{\theta,q}, \quad s = \theta k,\ k \in \mathbb{N},\ \theta \in (0,1), \tag{1.10}$$

where $(X,Y)_{\theta,q}$ is the real interpolation functor. Similarly, if M is compact, without boundary,

$$B_{p,q}^s(M) = (L^p(M), H^{kp}(M))_{\theta,q}, \quad s = \theta k,\ k \in \mathbb{N},\ \theta \in (0,1), \tag{1.11}$$

and it is the case that

$$B_{p,q}^s(\Omega) = \{u\big|_\Omega : u \in B_{p,q}^s(M)\}. \tag{1.12}$$

In fact, (1.12) follows from (1.10)–(1.11) and the use of Calderon's extension operator, which works for Lipschitz domains. See [5], [7], and [11] for more on this, in the context of Lipschitz domains. We mention that

$$B_{2,2}^s(\Omega) = H^{s,2}(\Omega) = H^s(\Omega),$$

so the case $q = 2$ of (1.8) gives $u \in H^{3/2}(\Omega)$.

We find it both pedagogically and logically useful to tackle Theorem 1.2 in stages, starting with dimension $n = 2$. We then move in Section 3 to $n = 3,\ p \le 4$, which is a little more complicated, and then to $n = 3,\ 4 < p < 5$, which involves a further twist. The case $n \ge 4$ requires a somewhat more intricate bootstrap argument. This is treated in Section 4.

In connection with Theorem 1.2, we mention the work of Dindos and Mitrea [3] on semilinear equations on Lipschitz domains, of the form

$$\Delta u - a(x, u)u = f, \quad u\big|_{\partial\Omega} = 0.$$

The authors there work with $a(x, u)$ satisfying

$$a(x, u) \ge 0, \tag{1.13}$$

in concert with certain upper bounds. Techniques include a careful study of $(\Delta - V)^{-1}$ for certain $V \ge 0$. Note that the nonlinearity in (1.2), for $K > 0$, has exactly the opposite sign from (1.13), so we require different arguments in Sections 3 and 4 of this paper.

We end this introduction with a positivity result, bearing on the last assertion in Theorem 1.1, and also of relevance to the interior regularity for the ground states, established in Section 2. First, suppose u_0 is a minimizer of F_λ, given by (1.4), subject to the constraint (1.5), with fixed $A > 0$, and set $u = |u_0|$. Clearly $\|\nabla u\|^2_{L^2} \le \|\nabla u_0\|^2_{L^2}$, while $\|u\|^2_{L^2} = \|u_0\|^2_{L^2}$ and $\|u\|_{L^{p+1}} = \|u_0\|_{L^{p+1}}$. Hence u is also minimizing, so it satisfies (1.3) (with $K = K_0$). We have the following additional property.

Proposition 1.3 *If $u \in H^1_0(\Omega) \cap C(\Omega)$ solves (1.2), with $K > 0$ and (1.3) holding, and if $u \ge 0$ on Ω, then*

$$u > 0 \ \text{ on } \ \Omega. \tag{1.14}$$

Proof Write (1.2) as

$$(-\Delta + \lambda)u = \varphi, \quad \varphi \ge 0,\ \varphi \ne 0.$$

Then

$$u(x) = \int_0^\infty e^{t(\Delta-\lambda)}\varphi(x)\, dt.$$

Well known positivity properties of the heat semigroup $e^{t\Delta}$ then imply (1.14). ■

2 Regularity on Smoothly Bounded Domains and Interior Regularity

Throughout this section, except at the end where we discuss interior regularity, we assume that Ω has smooth boundary $\partial\Omega$. We collect some tools that will be useful. First, we have embedding theorems,

$$H^{s,q}(\Omega) \subset L^{nq/(n-sq)}(\Omega), \ \text{ for } \ s > 0,\ q \in (1, \infty),\ sq < n, \tag{2.1}$$

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where $n = \dim \Omega$, and

(2.2) $$H^{s,q}(\Omega) \subset C^{\sigma}(\overline{\Omega}), \quad \text{for } q \in (1, \infty),\ s = \frac{n}{q} + \sigma,\ 0 < \sigma < 1.$$

These results also hold when Ω is a Lipschitz domain, and we will make use of them in Sections 3 and 4. We next bring in higher-order linear elliptic regularity results, which require smoothness of $\partial\Omega$. Namely, the solution operator

$$G = (-\Delta + \lambda)^{-1} \colon H^{-1}(\Omega) \longrightarrow H_0^1(\Omega)$$

has the following behavior on L^p-Sobolev spaces:

(2.3) $$G \colon H^{s,q}(\Omega) \longrightarrow H^{s+2,q}(\Omega), \quad s \geq 0,\ 1 < q < \infty.$$

Furthermore,

(2.4) $$G \colon C^{r}(\overline{\Omega}) \longrightarrow C^{r+2}(\overline{\Omega}), \quad r \in (0, \infty) \setminus \mathbb{N}.$$

Proofs of (2.1)–(2.2) can be found in [10, Chapter 13]. A proof of (2.3) is given in [4], for $s \in \mathbb{Z}^+$, and the result for general $s \in \mathbb{R}^+$ follows by interpolation. The result (2.4) is also proved in [4].

As a warm-up for the proof of Theorem 1.1, we first treat the case $n = 2$. In such a case, we have

$$u \in H_0^1(\Omega) \subset L^q(\Omega), \quad \forall\, q < \infty.$$

Then, with $f(u)$ given by (1.6), *i.e.*,

(2.5) $$f(u) = K|u|^{p-1}u,$$

we have $f(u) \in L^q(\Omega), \quad \forall\, q < \infty$. One application of (2.3) yields

$$u = Gf(u) \in H^{2,q}(\Omega), \quad \forall\, q < \infty.$$

Hence,

(2.6) $$u \in C^{s}(\overline{\Omega}), \quad \forall\, s < 2.$$

Examining (2.5), we see that

(2.7) $$f(u) \in C^{s}(\overline{\Omega}), \quad \forall\, s \in (0, 2) \cap (0, p],$$

unless p is an integer. If p is an odd integer, this holds for all $s \in (0, 2)$, and if p is an even integer, it holds for all $s \in (0, 2) \cap (0, p)$, if u changes sign. If $u \geq 0$ on $\overline{\Omega}$, then $f(u) = Ku^p$, and it again holds for all $s \in (0, 2)$. Now an application of (2.4) yields

(2.8) $$u = Gf(u) \in C^{s}(\overline{\Omega}), \quad \forall\, s \in (2, 4) \cap (2, p + 2],$$

with the same comments about the endpoint case, which we will not repeat. This in turn yields

$$f(u) \in C^s(\overline{\Omega}), \quad \forall s \in (0,4) \cap (0,p] \tag{2.9}$$

(same endpoint convention). Applying (2.4) again gives $u = Gf(u) \in C^s$, $\forall s \in (2,6) \cap (2, p+2]$ (same endpoint convention), and repeating this sufficiently often gives the conclusion of Theorem 1.1, for $n = 2$.

We now prove Theorem 1.1 when Ω has dimension $n \geq 3$. Write the hypothesis $p < (n+2)/(n-2)$ as

$$p = \frac{1}{\gamma}\,\frac{n+2}{n-2}, \quad \gamma > 1.$$

This time, we have $u \in H_0^1(\Omega) \subset L^{2n/(n-2)}(\Omega)$, hence

$$f(u) \in L^{2n\gamma/(n+2)}(\Omega). \tag{2.10}$$

An application of (2.3) yields

$$u = Gf(u) \in H^{2,2n\gamma/(n+2)}(\Omega). \tag{2.11}$$

Note that

$$\frac{2n\gamma}{n+2} > \frac{n}{2} \Longleftrightarrow \gamma > \frac{n+2}{4}. \tag{2.12}$$

If (2.12) holds, we have

$$u \in C^s(\overline{\Omega}), \ \text{ for some } \ s \in (0,1). \tag{2.13}$$

In the endpoint case, $\gamma = (n+2)/4$, we have

$$u \in L^q(\Omega), \quad \forall q < \infty, \ \text{ hence } f(u) \in L^q(\Omega), \ \forall q < \infty, \tag{2.14}$$

hence, via (2.3),

$$u \in H^{2,q}(\Omega), \quad \forall q < \infty, \ \text{ hence } \ u \in C^s(\overline{\Omega}), \ \text{ for some } \ s > 0. \tag{2.15}$$

If $\gamma < (n+2)/4$, we have, via (2.1),

$$u \in H^{2,2n\gamma/(n+1)}(\Omega) \subset L^{2n\gamma/(n+2-4\gamma)}(\Omega), \tag{2.16}$$

hence

$$f(u) \in L^{2n\gamma_2/(n+2)}(\Omega), \tag{2.17}$$

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with

$$\gamma_2 = \gamma^2 \frac{n-2}{n+2-4\gamma} > \gamma^2. \tag{2.18}$$

We replace (2.10) with this updated information on $f(u)$, and again apply (2.3), to get

$$u = Gf(u) \in H^{2,2n\gamma_2/(n+2)}(\Omega),$$

in place of (2.11).

We proceed as in (2.12)–(2.18). A finite number of iterations of this procedure yields the property (2.13). This in turn gives $f(u) \in C^s(\overline{\Omega})$, hence $u = Gf(u) \in C^{s+2}(\overline{\Omega})$. From here, an argument parallel to that involving (2.6)–(2.9) and its iteration finishes the proof of Theorem 1.1.

We turn to an examination of local regularity results. In this case, Ω can be any relatively compact open subset of M; no regularity of $\partial\Omega$ is required. Arguments strongly parallel to those used above apply, with (2.3)–(2.4) replaced by associated local regularity results in L^p-Sobolev spaces and Hölder spaces, results that can also be found in [4]. We state the conclusion.

Proposition 2.1 *Let $\Omega \subset M$ be open and relatively compact, and let $u \in H_0^1(\Omega)$ satisfy (1.2), with (1.3) holding. Then $u \in C^{p+2}(\Omega)$, if $p \notin \mathbb{N}$. If p is an odd integer, $u \in C^\infty(\Omega)$. If p is an even integer, $u \in C^s(\Omega)$ for all $s < p+2$, and if p is an even integer and $u \geq 0$ on Ω, $u \in C^\infty(\Omega)$. Finally, if $\mathcal{O} \subset \Omega$ is open and u is nowhere vanishing on $\mathcal{O}$, then $u \in C^\infty(\mathcal{O})$.*

Proof We have addressed all the assertions of Proposition 2.1 except the last. For that, if $|u| \geq \delta > 0$ on $\mathcal{O}$, we can replace $f(u)$ by $g(u)$ with g smooth, and implications like (2.6) $\Rightarrow$ (2.7) get replaced by $u \in C^s(\mathcal{O}) \Rightarrow f(u) \in C^s(\mathcal{O})$, which lead to the final conclusion of Proposition 2.1. ∎

3 Regularity on Lipschitz Domains I

We now tackle Theorem 1.2. In this setting, $\overline{\Omega} \subset M$ is a compact Lipschitz domain. To start things off, we can pick $U \supset \overline{\Omega}$ to be a smoothly bounded, relatively compact, open subset of M, such that if L denotes the Laplace-Beltrami operator on U, with the Dirichlet boundary condition, (1.3) is complemented by

$$\operatorname{Spec}(-L) \subset [\beta, \infty), \quad -\alpha < -\beta < \lambda. \tag{3.1}$$

Starting with

$$u \in H_0^1(\Omega) \subset L^{2n/(n-2)}(\Omega) \Longrightarrow f(u) \in L^{2n/(n-2)p}(\Omega),$$

with the standard convention for $n = 2$, we define $\tilde{f} \in L^{2n/(n-2)p}(U)$ by extending $f(u)$ by 0 on $U \setminus \overline{\Omega}$, and then have

$$u = w - \operatorname{PI}(w\big|_{\partial\Omega}), \tag{3.2}$$

with

$$w = G\tilde{f} = (-L+\lambda)^{-1}\tilde{f}, \tag{3.3}$$

and where $\mathrm{PI}(g) = v$ solves

$$(-\Delta+\lambda)v = 0 \text{ on } \Omega, \quad v\big|_{\partial\Omega} = g. \tag{3.4}$$

Results parallel to (2.3)–(2.4) apply to (3.3).

The first goal will be to show that (3.2)–(3.4) yield a regularity result on u better than $u \in H^1(\Omega)$. Then we plan to iterate the argument, obtaining progressively stronger results on u, until the conclusions of Theorem 1.2 are established. As it turns out, the iterative argument will be much more intricate than that used for smoothly bounded domains in §2, particularly in dimension $n \geq 4$.

To implement this plan, we collect some needed results on $\mathrm{Tr}(u) = u|_{\partial\Omega}$ and on PI that hold when Ω is a Lipschitz domain. These results can be found in [2], [12], and [5] for Lipschitz domains in $\mathbb{R}^n$, and in [6]–[8] for Lipschitz domains in a Riemannian manifold. For the trace map, [5] showed

$$\mathrm{Tr}\colon H^{s+1/p}(\Omega) \longrightarrow B^s_{p,p}(\partial\Omega), \quad 1 < p < \infty,\ 0 < s < 1. \tag{3.5}$$

The Besov space $B^s_{p,p}(\partial\Omega)$ has a characterization parallel to (1.17). More generally,

$$B^s_{p,q}(\partial\Omega) = (L^p(\partial\Omega), H^{1,p}(\partial\Omega))_{s,q},$$

for $0 < s < 1$. Note that if $\partial\Omega$ is Lipschitz, then $L^p(\partial\Omega)$ and $H^{1,p}(\partial\Omega)$ are well defined and invariant under bi-Lipschitz maps. Besov spaces on $\partial\Omega$ are well defined for all real s if $\partial\Omega$ is smooth, and in such a case (3.5) holds for all $s > 0$, but for Lipschitz $\partial\Omega$, (3.5) can fail at $s = 1$.

For PI there are the following results. There exists $\varepsilon = \varepsilon(\Omega) > 0$ such that

$$\mathrm{PI}\colon H^{s,q}(\partial\Omega) \longrightarrow B^{s+1/q}_{q,q\vee 2}(\Omega), \quad 0 \leq s \leq 1,\ q \in (2-\varepsilon, 2+\varepsilon), \tag{3.6}$$

where $q \vee 2 = \max(q,2)$. Furthermore,

$$\mathrm{PI}\colon B^s_{q,q}(\partial\Omega) \longrightarrow H^{s+1/q,q}(\Omega), \text{ for } (s, 1/q) \in \mathcal{R}_\varepsilon, \tag{3.7}$$

where $\mathcal{R}_\varepsilon$ is the interior of the hexagon with vertices at

$$(0,0),\ (\varepsilon,0),\ (1,\tfrac{1}{2}-\varepsilon),\ (1,1),\ (1-\varepsilon,1),\ (0,\tfrac{1}{2}+\varepsilon).$$

The result (3.7) was proved in [5] for Lipschitz domains in $\mathbb{R}^n$, and in [7] for Lipschitz domains in Riemannian manifolds. If $\partial\Omega$ is smooth, it is classical that (3.7) holds for all $s > 0$, $q \in (1,\infty)$; this result together with the extension of (3.5) to all $s > 0$ are key ingredients in the proof of (2.3) in the smooth setting. The case $s = 0$, $q = 2$ of (3.6) is

$$\mathrm{PI}\colon L^2(\partial\Omega) \longrightarrow H^{1/2}(\Omega)$$

(*cf.* [5] for another approach when $\Omega \subset \mathbb{R}^n$), which can be interpolated with

$$\mathrm{PI}\colon L^\infty(\partial\Omega) \longrightarrow L^\infty(\Omega).$$

One further result on PI we will need is that there exists $s_0 = s_0(\Omega) > 0$ such that

$$\mathrm{PI}\colon C^s(\partial\Omega) \longrightarrow C^s(\overline{\Omega}), \quad \forall\, s \in (0, s_0), \tag{3.8}$$

proven in [6] for $\overline{\Omega}$ in a Riemannian manifold.

Remark Though for simplicity we are assuming M carries a smooth Riemannian metric tensor (g_{jk}) here, we mention that [8] proved (3.6), [7] proved (3.7), and [6] proved (3.8), in the setting $g_{jk} \in C^{1+\varepsilon}$.

A couple of other results for which we have occasional use are

$$B^s_{q,q\wedge 2} \subset H^{s,q} \subset B^s_{q,q\vee 2}, \tag{3.9}$$

where $q \wedge 2 = \min(q, 2)$ and $q \vee 2 = \max(q, 2)$, and

$$B^s_{q,p} \subset H^{s-\delta,q}, \quad H^{s,q} \subset B^{s-\delta}_{q,p}, \quad \forall\, \delta > 0,\ p \in [1, \infty] \tag{3.10}$$

(*cf.* [1]).

As mentioned in the introduction, we tackle Theorem 1.2 in stages, starting with $n = 2$ and moving on to $n = 3$, $p \le 4$. These cases are of intrinsic interest and are easier than the general case, while they set the stage for the first round of arguments needed in the general case. Recall that we are dealing with $u \in H^1_0(\Omega)$, solving

$$(-\Delta + \lambda)u = f(u), \tag{3.11}$$

with

$$f(u) = K|u|^{p-1}u, \quad 1 < p < \frac{n+2}{n-2}.$$

In case $n = 2$,

$$\begin{aligned} u \in H^1_0(\Omega) &\Longrightarrow u \in L^q(\Omega), \quad \forall\, q < \infty \\ &\Longrightarrow f(u) \in L^q(\Omega), \quad \forall\, q < \infty. \end{aligned}$$

Invoking (3.2)–(3.3), we have

$$u = w - \mathrm{PI}(w\big|_{\partial\Omega}), \quad w \in H^{2,q}(\Omega), \quad \forall q < \infty.$$

Consequently $w \in C^r(\overline{\Omega})$ for all $r < 2$, so the fact that $u \in C^s(\overline{\Omega})$ for some $s > 0$ follows from (3.8). Sharper Hölder information will be established below. We now pursue the proof of (1.8), *i.e.*,

$$u \in B^{1+1/q}_{q,q}(\Omega), \quad \forall\, q \in [2, 2 + \varepsilon(\Omega)). \tag{3.12}$$

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In fact, the Hölder result on w just established implies

$$w\big|_{\partial\Omega} \in H^{1,q}(\partial\Omega), \quad \forall\, q < \infty. \tag{3.13}$$

We can now apply (3.6), with $s = 1$ (but only for $|q-2| < \varepsilon$), and we have the desired conclusion (3.12). Note that the case $q = 2$ of (3.12) gives

$$u \in H^{3/2}(\Omega) \subset C^{1/2}(\overline{\Omega}), \tag{3.14}$$

when $n = 2$. Furthermore, when $n = 2$,

$$q > 2 \Longrightarrow B^{1+1/q}_{q,q}(\Omega) \subset C^s(\overline{\Omega}), \ \text{ for some } \ s > 1/2. \tag{3.15}$$

This proves Theorem 1.2, in case $n = 2$.

We now take up the case $n = 3,\ 1 < p \le 4$. In such a case,

$$u \in H^1_0(\Omega) \subset L^6(\Omega) \Longrightarrow f(u) \in L^{3/2}(\Omega),$$

so

$$u = w - \mathrm{PI}(w\big|_{\partial\Omega}), \quad w \in H^{2,3/2}(\Omega). \tag{3.16}$$

If $p < 4$, we have $w \in H^{2,6/p}(\Omega) \subset C(\overline{\Omega})$ for some $r > 0$, but this fails at $p = 4$. We will pick up this Hölder continuity a little further down. Towards the goal of proving (3.12), we start with $\nabla w \in H^{1,3/2}(\Omega)$, and apply (3.5), with $q = 3/2,\ s = 1 - 1/q = 1/3$, to get

$$\begin{aligned} \nabla w\big|_{\partial\Omega} &\in B^{1/3}_{3/2,3/2}(\partial\Omega) \\ &\subset H^{1/3,3/2}(\partial\Omega) \\ &\subset L^2(\partial\Omega), \end{aligned}$$

the latter two inclusions by (3.9) and (3.1) (with Ω replaced by $\partial\Omega$, of dimension 2). Hence $w\big|_{\partial\Omega} \in H^{1,2}(\partial\Omega)$. Applying (3.6), with $s = 1,\ q = 2$, we have

$$u \in H^{3/2}(\Omega), \tag{3.17}$$

which is part of the desired conclusion (3.12). We will need just one more iteration. Note that (3.17) implies $u \in L^q(\Omega)$ for all $q < \infty$, hence $f(u) \in L^q(\Omega)$ for all $q < \infty$, so (3.16) is improved to

$$w \in H^{2,q}(\Omega), \quad \forall\, q < \infty.$$

This implies $w \in C^r(\overline{\Omega})$ for all $r < 2$, and we can now apply (3.8) to get $u \in C^s(\overline{\Omega})$, for some $s > 0$. This observation also gives

$$w\big|_{\partial\Omega} \in H^{1,q}(\partial\Omega), \quad \forall\, q < \infty,$$

and we can now apply (3.6), with $s = 1$, but again only for $|q-2| < \varepsilon(\Omega)$, to obtain the desired conclusion (3.12).

Remark The result (3.12) implies

$$u \in H^{r+1/q,q}(\Omega), \quad \forall\, r < 1,\ q \in (2, 2 + \varepsilon(\Omega)),$$

and since $(r + 1/q)q = qr + 1 > 3$ for some $r < 1,\ q \in (2, 2 + \varepsilon)$, we again get Hölder continuity of u, this time without needing to invoke (3.8). This sort of argument fails to establish Hölder continuity when $n \geq 4$.

We move on to the remaining cases for $n = 3$, namely $p \in (4, 5)$. Let us set $p = 5/\gamma,\ \gamma > 1$, and start with the fact that $u \in H_0^1(\Omega) \subset L^6(\Omega)$ implies

$$f(u) \in L^{6/p}(\Omega) = L^{6\gamma/5}(\Omega). \tag{3.18}$$

Note that $p \in (4, 5) \Leftrightarrow \gamma \in (1, 5/4) \Leftrightarrow 6\gamma/5 \in (6/5, 3/2)$. Parallel to (3.16), we have

$$u = w - \mathrm{PI}(w\big|_{\partial\Omega}), \quad w \in H^{2,6\gamma/5}(\Omega), \tag{3.19}$$

hence

$$\nabla w \in H^{1,6\gamma/5}(\Omega), \tag{3.20}$$

and applying (3.5), with $q = 6\gamma/5,\ s = 1 - 1/q$, we get

$$\begin{aligned} \nabla w\big|_{\partial\Omega} &\in B^{1-5/6\gamma}_{6\gamma/5,6\gamma/5}(\partial\Omega) \\ &\subset H^{1-5/6\gamma,6\gamma/5}(\partial\Omega) \\ &\subset L^{4\gamma/(5-2\gamma)}(\partial\Omega), \end{aligned} \tag{3.21}$$

the second line by (3.9) (and the fact that $6\gamma/5 < 3/2 < 2$), and the third line by (2.1) (with Ω replaced by $\partial\Omega$, of dimension 2). For (2.1) to apply, we need $\gamma < 5/2$, but in fact, in our current setting, $\gamma < 5/4$. Hence

$$w\big|_{\partial\Omega} \in H^{1,r}(\partial\Omega), \quad r = \frac{4\gamma}{5 - 2\gamma}. \tag{3.22}$$

Note that

$$\gamma \in \left(1, \frac{5}{4}\right) \Longrightarrow r \in \left(\frac{4}{3}, 2\right). \tag{3.22}$$

Hence (for the first time, so far) we apply (3.7), with $q = r$ and $s \in (0, 1)$, close to 1. We get

$$\mathrm{PI}(w\big|_{\partial\Omega}) \in H^{s+1/r,r}(\Omega), \quad \forall\, s < 1. \tag{3.23}$$

Together with (3.19), this gives

$$u \in H^{2,6\gamma/5}(\Omega) + H^{s+1/r,r}, \quad \forall\, s < 1, \tag{3.25}$$

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with r as in (3.22). Sobolev's embedding theorem gives

$$H^{1,6\gamma/5}(\Omega) \subset L^{6\gamma/(5-2\gamma)}(\Omega) \tag{3.25}$$

when $\gamma < 5/2$, hence

$$H^{2,6\gamma/5}(\Omega) \subset H^{1,6\gamma/(5-2\gamma)}(\Omega). \tag{3.26}$$

Also $H^{1/r,r}(\Omega) \subset L^{3r/2}(\Omega)$ when $\dim \Omega = 3$, so

$$H^{1+1/r,r}(\Omega) \subset H^{1,3r/2}(\Omega) = H^{1,6\gamma/(5-2\gamma)}(\Omega). \tag{3.27}$$

Thus (3.25) gives

$$u \in H^{1,q}(\Omega), \quad \forall\, q < \frac{6\gamma}{5-2\gamma}, \tag{3.29}$$

which improves the original information $u \in H^{1,2}(\Omega)$, since

$$\gamma \in \Big(1, \frac{5}{4}\Big) \Longrightarrow \frac{6\gamma}{5-2\gamma} \in \Big(2\gamma, \frac{12}{5}\gamma\Big).$$

Another application of Sobolev embedding gives, for $s = 6\gamma/(5-2\gamma)$,

$$H^{1,s}(\Omega) \subset L^{3s/(3-s)}(\Omega) = L^{6\gamma/(5-4\gamma)}(\Omega), \tag{3.30}$$

as long as $\gamma < 5/4$, so (3.29) yields

$$u \in L^{q}(\Omega), \quad \forall\, q < \frac{6\gamma}{5-4\gamma}, \tag{3.31}$$

hence

$$f(u) \in L^{q}(\Omega), \quad \forall\, q \in \frac{6\gamma}{5}\,\frac{\gamma}{5-4\gamma}. \tag{3.32}$$

Let us write this as

$$f(u) \in L^{6\gamma_2/5}(\Omega), \quad \forall\, \gamma_2 < \frac{\gamma^2}{5-4\gamma}. \tag{3.33}$$

This improves (3.18), since

$$\gamma \in \Big(1, \frac{5}{4}\Big) \Longrightarrow \gamma_2 > \gamma^2.$$

We hence have (3.19)–(3.20) with γ replaced by γ_2. The analysis from here splits into several cases, depending on the size of γ_2.

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Case I. $\gamma_2 < 5/2$.

Then (3.21)–(3.29) hold with γ replaced by γ_2, with a minor adjustment to the analogue of (3.21) if $6\gamma_2/5 > 2$. From here, there are two sub-cases to consider.

Case IA. $\gamma_2 < 5/4$.

Then (3.30)–(3.32) hold with γ replaced by γ_2, and we improve (3.33) to

$$f(u) \in L^{6\gamma_3/5}(\Omega), \quad \forall\, \gamma_3 < \frac{\gamma_2^2}{5 - 4\gamma_2}.$$

In particular, we have the result for all $\gamma_3 < \gamma^4$.

Case IB. $5/4 \le \gamma_2 < 5/2$.

Then, in place of (3.31)–(3.32), we get

$$u,\ f(u) \in L^q(\Omega), \quad \forall\, q < \infty. \tag{3.34}$$

Case II. $5/2 \le \gamma_2$.

Then the improved information on w in (3.19) (with γ replaced by γ_2) yields

$$w \in H^{2,3}(\Omega) \subset C^r(\overline{\Omega}), \quad \forall\, r < 1.$$

Hence $w|_{\partial\Omega} \in C^r(\partial\Omega)$ and (3.8) gives $u \in C^s(\overline{\Omega})$ for some $s > 0$, which implies

$$f(u) \in L^\infty(\Omega).$$

Having presented the various cases, we proceed. In Case IA, we continue the iteration. We have (3.19)–(3.20) with γ replaced by γ_3. The analysis again splits into three cases, according to whether $\gamma_3 < 5/4$, $5/4 \le \gamma_3 < 5/2$, or $5/2 \le \gamma_3$. We continue this process, eventually reaching either Case IB or Case II. The outcome is that we have established (3.34).

The argument from here proceeds much as for $n = 2$. We have

$$u = w - \mathrm{PI}(w\big|_{\partial\Omega}), \quad w \in H^{2,q}(\Omega),\ \forall\, q < \infty.$$

Hence $w \in C^r(\overline{\Omega})$ for all $r < 2$, so (3.8) implies $u \in C^s(\overline{\Omega})$ for some $s > 0$. Parallel to (3.13)–(3.15), we have

$$w\big|_{\partial\Omega} \in H^{1,q}(\partial\Omega), \quad \forall\, q < \infty,$$

and applying (3.6), with $s = 1$, $|q - 2| < \varepsilon(\Omega)$ gives the conclusion (3.12). This finishes the proof of Theorem 1.2 when $n = 3$.

4 Regularity on Lipschitz Domains II: $n \geq 4$

Here we continue the study of solutions $u \in H_0^1(\Omega)$ to (3.11), *i.e.*, to

$$(-\Delta + \lambda)u = f(u),$$

with

$$f(u) = K|u|^{p-1}u, \quad 1 < p < \frac{n+2}{n-2}.$$

This time, $n \geq 4$. Parallel to (3.18), we set

$$p = \frac{1}{\gamma}\,\frac{n+2}{n-2}, \quad 1 < \gamma < \frac{n+2}{n-2},$$

and start with

$$u \in H_0^1(\Omega) \subset L^{2n/(n-2)}(\Omega),$$

which implies

$$f(u) \in L^{2n\gamma/(n+2)}(\Omega). \tag{4.1}$$

As in (3.19), we have

$$u = w - \mathrm{PI}(w\big|_{\partial\Omega}), \quad w \in H^{2,2n\gamma/(n+2)}(\Omega). \tag{4.2}$$

Hence

$$\nabla w \in H^{1,2n\gamma/(n+2)}(\Omega),$$

so

$$\begin{aligned} \nabla w\big|_{\partial\Omega} &\in B^{1-(n+2)/2n\gamma}_{2n\gamma/(n+2),2n\gamma/(n+2)}(\partial\Omega) \\ &\subset H^{s-(n+2)/2n\gamma,2n\gamma/(n+2)}(\partial\Omega), \quad \forall\, s < 1. \end{aligned} \tag{4.3}$$

From here, the analysis splits into several cases, depending on the size of γ.

Case I. $\gamma < (n+2)/2$.

Then the Sobolev embedding theorem yields

$$\nabla w\big|_{\partial\Omega} \in L^r(\partial\Omega), \quad \forall\, r < \frac{2(n-1)\gamma}{n+2-2\gamma},$$

since

$$\frac{2n\gamma(n-1)/(n+2)}{n-1-(2n\gamma-n-2)/(n+2)} = \frac{2(n-1)\gamma}{n+2-2\gamma}. \tag{4.4}$$

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Hence

$$w\big|_{\partial\Omega} \in H^{1,r}(\partial\Omega), \quad \forall\, r < \frac{2(n-1)\gamma}{n+2-2\gamma}. \tag{4.5}$$

What to do next depends on whether (4.5) forces $r < 2$. Note that

$$\begin{aligned}\frac{2(n-1)\gamma}{n+2-2\gamma} < 2 &\iff (n-1)\gamma < n+2-2\gamma \\ &\iff \gamma < \frac{n+2}{n+1}.\end{aligned}$$

Thus we have two sub-cases.

Case IA. $\gamma < (n+2)/(n+1)$.
Then $r < 2$ in (4.4), and we have

$$w\big|_{\partial\Omega} \in B^s_{r,r}(\partial\Omega), \quad \forall\, s < 1, \tag{4.6}$$

and (3.7) implies

$$\mathrm{PI}(w\big|_{\partial\Omega}) \in H^{s+1/r,r}(\Omega), \quad \forall\, s < 1, \tag{4.7}$$

with r as in (4.5). From (4.2), we have

$$u \in H^{2,2n\gamma/(n+2)}(\Omega) + H^{s+1/r,r}(\Omega), \quad \forall\, s < 1,\ r < \frac{2(n-1)\gamma}{n+2-2\gamma}. \tag{4.8}$$

We have

$$H^{1,2n\gamma/(n+2)}(\Omega) \subset L^{2n\gamma/(n+2-2\gamma)}(\Omega), \tag{4.9}$$

provided $\gamma < (n+2)/2$, and hence

$$H^{2,2n\gamma/(n+2)}(\Omega) \subset H^{1,2n\gamma/(n+2-2\gamma)}(\Omega). \tag{4.10}$$

Meanwhile,

$$H^{1/r,r}(\Omega) \subset L^{nr/(n-1)}(\Omega) \Longrightarrow H^{1+1/r,r}(\Omega) \subset H^{1,nr/(n-1)}(\Omega), \tag{4.11}$$

so

$$r = \frac{2(n-1)\gamma}{n+2-2\gamma} \Longrightarrow H^{1+1/r,r}(\Omega) \subset H^{1,2n\gamma/(n+2-2\gamma)}(\Omega). \tag{4.12}$$

Together, (4.8)–(4.12) yield

$$\begin{aligned} u \in H^{1,s}(\Omega), \quad &\forall\, s < 2\gamma \frac{n}{n+2-2\gamma} \\ &\subset H^{1,2\gamma}(\Omega) \\ &\subset L^{2n\gamma/(n-2\gamma)}(\Omega), \end{aligned} \tag{4.13}$$

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provided $\gamma < n/2$, which follows from $\gamma < (n+2)/(n+1)$ if $n \geq 4$. From here, we have the following improvement of (4.1):

$$f(u) \in L^{2n\gamma/p(n-2\gamma)}(\Omega) = L^{2n\gamma_2/(n+2)}(\Omega), \tag{4.14}$$

where

$$\gamma_2 = \gamma^2 \frac{n-2}{n-2\gamma} > \gamma^2. \tag{4.15}$$

Case IB. $(n+2)/(n+1) \leq \gamma < (n+2)/2$.

We consider separately the cases $\gamma = (n+2)/(n+1)$ and $(n+2)/(n+1) < \gamma < (n+2)/2$. In the former case, reasoning as in (4.6)–(4.13) gives

$$u \in H^{1,2\gamma}(\Omega) \subset L^{2n\gamma/(n-2\gamma)}(\Omega),$$

and hence $f(u)$ satisfies (4.14), with γ_2 as in (4.15). In the latter case, we can take $r > 2$ in (4.5), and we can apply (3.6) to get

$$\mathrm{PI}(w\big|_{\partial\Omega}) \in B^{1+1/q}_{q,q}(\Omega), \quad \forall\, q \in [2, 2+\varepsilon),$$

for some $\varepsilon > 0$. Since this arises from no other information on w than in (4.2), we must also have $H^{2,2n\gamma/(n+2)}(\Omega) \subset B^{1+1/q}_{q,q}(\Omega)$, for some such q, hence

$$u \in B^{1+1/q}_{q,q}(\Omega), \quad \forall\, q \in [2, 2+\varepsilon). \tag{4.16}$$

This takes care of Case I.

Case II. $\gamma \geq (n+2)/2$.

Then $2n\gamma/(n+2) \geq n$, so, by (4.2),

$$w \in H^{2,n}(\Omega) \subset C^r(\overline{\Omega}), \quad \forall\, r < 1.$$

It follows that $w|_{\partial\Omega} \in C^r(\overline{\Omega})$, and (3.8) gives $u \in C^{(}\overline{\Omega})$ for some $s > 0$. Also, an analogue of (4.3) gives

$$w\big|_{\partial\Omega} \in H^{1,q}(\partial\Omega), \quad \forall\, q < \infty,$$

and applying (3.6), with $s = 1$ and $|q-2| < \varepsilon(\Omega)$, gives again (4.16).

This completes our discussion of the cases. We now have a machine for progressively improving the provable regularity of the solution u. Here is the algorithm. Start with the knowledge that $f(u)$ satisfies (4.1) for some $\gamma \in (1, (n+2)/(n-2))$. Consider whether γ falls in Case IA, IB, or II. In the latter part of Case IB and in Case II, we obtain the desired result (4.16). In Case IA and the first part of Case IB, we obtain the improvement (4.14) of (4.1), *i.e.*, we get (4.1) with γ replaced by $\gamma_2 > \gamma^2$. We then see which of these cases apply to γ_2 and continue this argument. After a finite number of iterations, we obtain (4.16).

Except when we land in Case II, which is not guaranteed, we do not get the Hölder condition $u \in C^s(\overline{\Omega})$ from this argument, and, as noted in Section 3, this result does not follow from (4.16) when $n \geq 4$. Thus, for $n \geq 4$, a further argument is required to show that $u \in C^s(\overline{\Omega})$ for some $s > 0$ and finish off the proof of Theorem 1.2.

So here is where we stand. We have

$$w\big|_{\partial\Omega} \in H^{1,r}(\partial\Omega), \tag{4.17}$$

for some $r > 2$, and we want to build on this to get $u \in C^s(\overline{\Omega})$ for some $s > 0$. If (4.17) holds with $r > n-1$, then $w|_{\partial\Omega} \in C^s(\partial\Omega)$ for some $s > 0$, and (3.8) gives PI $w|_{\partial\Omega} \in C^s(\overline{\Omega})$ (perhaps with s decreased). Also, since the sole source of (4.17) is the application of the trace theorem to known Sobolev space regularity of u on Ω, that regularity must be sufficient to imply $w \in C^s(\overline{\Omega})$, hence $u \in C^s(\overline{\Omega})$.

If (4.17) holds with $r = n-1$, we will simply record that

$$w\big|_{\partial\Omega} \in H^{1,r}(\partial\Omega), \quad \forall\, r < n-1,$$

and proceed to the next step.

We are left with the task of treating (4.17) when

$$2 < r < n-1. \tag{4.18}$$

In this situation, our strategy will be to apply (3.7) with

$$s < \frac{2}{q}, \quad 2 < q < \infty,$$

in which case $(s, 1/q) \in \mathcal{R}_\varepsilon$. We have

$$H^{1,r}(\partial\Omega) \subset H^{2/q,q}(\partial\Omega) \tag{4.19}$$

when

$$H^{1-2/q,r}(\partial\Omega) \subset L^q(\partial\Omega).$$

Sobolev's embedding theorem gives

$$H^{1-2/q,r}(\partial\Omega) \subset L^{r(n-1)/(n-1-r+2r/q)}(\partial\Omega),$$

so (4.19) holds when

$$\frac{r(n-1)}{n-1-r+2r/q} = q,$$

i.e., when

$$q = \frac{r(n-3)}{n-1-r}.$$

Note that (4.18) gives $r < q < \infty$. From (4.17) and (4.19), plus (3.10), we have

$$w\big|_{\partial\Omega} \in B^{s}_{q,q}(\partial\Omega), \quad \forall\, s < \frac{2}{q},\ q = \frac{r(n-3)}{n-n-r}.$$

Hence, by (3.7),

$$\mathrm{PI}(w\big|_{\partial\Omega}) \in H^{s+1/q,q}(\Omega), \tag{4.20}$$

for such s, q. Now the fact that (4.20) arises from (4.17) implies that $w \in H^{s+1/q,q}(\Omega)$, for such s, q, and hence

$$u \in H^{s+1/q,q}(\Omega), \quad \forall\, s < \frac{2}{q}, \quad q = \frac{r(n-3)}{n-n-r}.$$

Now

$$H^{3/q,q}(\Omega) \subset L^{qn/(n-3)}(\Omega) = L^{rn/(n-1-r)}(\Omega),$$

so

$$u \in L^{q}(\Omega), \quad \forall\, q < \frac{rn}{n-1-r}.$$

Hence

$$f(u) \in L^{q\gamma(n-2)/(n+2)}(\Omega), \quad \forall\, q < \frac{rn}{n-1-r}.$$

This gives

$$w \in H^{2,b}(\Omega), \quad \forall\, b < \frac{rn}{n-n-r} \cdot \frac{\gamma(n-2)}{n+2}.$$

Hence $\nabla w \in H^{1,b}(\Omega)$ for such b, so

$$\begin{aligned} \nabla w\big|_{\partial\Omega} &\in B^{1-1/b}_{b,b}(\partial\Omega) \\ &\subset H^{s-1/b,b}(\partial\Omega), \quad \forall\, s < 1, \end{aligned}$$

for such b. In turn, this gives

$$\nabla w\big|_{\partial\Omega} \in L^{m}(\partial\Omega), \quad \forall\, m < \frac{(n-1)b}{n-b},$$

provided $b < n$. If $b \geq n$, then $w \in C^{r}(\overline{\Omega})$ for all $r < 1$, so $u \in C^{s}(\overline{\Omega})$ for some $s > 0$, and we are done. If $b < n$, we get

$$w\big|_{\partial\Omega} \in H^{1,m}, \quad \forall\, m < r\,\frac{\gamma(n-1)(n-2)}{(n-1-r)(n+2) - r\gamma(n-2)}.$$

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In other words,

$$w\big|_{\partial\Omega} \in H^{1,\sigma r}(\partial\Omega), \quad \forall\, \sigma < \sigma_0 = \frac{\gamma(n-1)(n-2)}{(n-1-r)(n+2) - r\gamma(n-2)}. \tag{4.21}$$

Claim $\sigma_0 > \gamma$.

This is equivalent to

$$(n-1-r)(n+2) - r\gamma(n-2) < (n-1)(n-2). \tag{4.22}$$

The left side of (4.22) is equal to

$$n^2 + n - 2rn - 2 - (\gamma - 1)r(n-2) \le n^2 - 3n - 2,$$

since $r > 2,\ \gamma > 1$, and the right side of (4.22) is equal to $n^2 - 3n + 2$. This establishes the claim.

We now finish the proof of Theorem 1.2, showing that $u \in C^s(\overline{\Omega})$ for some $s > 0$. We have from (4.17), with $2 < r < n-1$, the improvement (4.21), with $\sigma > \gamma$. Iteration eventually produces u satisfying (4.17) with $r > n-1$, which as we have seen leads to the desired conclusion that $u \in C^s(\overline{\Omega})$ for some $s > 0$. The proof of Theorem 1.2 is complete. ■

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Mathematics Department, University of North Carolina, Chapel Hill NC 27599, USA
e-mail: met@math.unc.edu