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On Non-Strongly Free Automorphisms of Subfactors of Type III_0

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Abstract. We determine when an automorphism of a subfactor of type III_0 with finite index is non-strongly free in the sense of C. Winsløw in terms of the modular endomorphisms introduced by M. Izumi.

1 Introduction

In subfactor theory, initiated by V. F. R. Jones in [13], the study of automorphisms of subfactors plays an important role. One of the important properties of an automorphism is strong outerness for automorphisms introduced by Choda-Kosaki in [1]. (Popa also introduced the same property in [27] independently, and called it proper outerness.)

In [29], C. Winsløw introduced the notion of strong freeness for automorphisms of subfactors of type III as a natural generalization of strong outerness and non-pointwise innerness in the sense of [5]. This notion plays an important role in his theory of automorphisms for subfactors of type III.

In [19], H. Kosaki investigated the structure of automorphisms of type III subfactors by using sectors, and characterized non-strongly free automorphisms for subfactors of type III_λ, $\lambda \neq 0$. Namely an automorphism is non-strongly free if and only if it is the composition of a non-strongly-outer automorphism and a modular automorphism. (See Theorem 3.1 below.) However the type III_0 case was left unsolved. Note that the composition of a non-strongly-outer automorphism and an extended modular automorphism is always non-strongly free as shown by Kosaki.

The purpose of this paper is to give a full answer to this problem. Though one may conjecture that modular automorphisms should be replaced with extended modular automorphisms in Kosaki's theorem, we must use modular endomorphisms as defined by M. Izumi in [10] to obtain the correct characterization. (See [11] for details of modular endomorphisms.) In general, there exist subfactors and their non-strongly free automorphisms which are not the composition of non-strongly-outer automorphisms and extended modular automorphisms.

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2 Preliminaries

For the basic facts of index theory and sector theory for subfactors, we refer to [13], [16], [7], [26], [20], [25], [23], [24], [9], [3].

Let E be the minimal conditional expectation from M onto N, φ a faithful normal semifinite weight of N, $\tilde{N} \subset \tilde{M} := N \rtimes_{\sigma^\varphi} \mathbf{R} \subset M \rtimes_{\sigma^{\varphi\circ E}} \mathbf{R}$ the continuous decomposition of $N \subset M$. Throughout this paper, we always assume that type III subfactors have the common flow of weights, that is, $Z(\tilde{N}) = Z(\tilde{M})$.

First we recall the notion of strong outerness for automorphisms of subfactors.

Definition 2.1 **([1, Definition 1], [27, Definition 1.3.1])** An automorphism $\alpha \in \mathrm{Aut}(M, N)$ is said to be strongly outer if there exist no non-zero $a \in \bigcup_k M_k$ such that $\alpha(x)a = ax$ holds for every $x \in M$.

Let α be an automorphism of $N \subset M$. Then we can define the canonical extension $\tilde{\alpha} \in \mathrm{Aut}(\tilde{M}, \tilde{N})$ of α as follows:

$$\tilde{\alpha}(x) = \alpha(x), \quad x \in M,$$
$$\tilde{\alpha}\big(\lambda(t)\big) = (D\varphi \circ \alpha^{-1} : D\varphi)_t \lambda(t),$$

where $\lambda(t)$ is the implementing unitary of σ_t^φ (see [5]).

Definition 2.2 **([29, Definition 3.2])** In the above notations, α is said to be strongly free if there exist no non-zero $a \in \bigcup_k \tilde{M}_k$ satisfying $\tilde{\alpha}(x)a = ax$ for every $x \in \tilde{M}$.

By the above definition, α is non-strongly free if and only if there exist $k > 0$ and non-zero element $a \in \tilde{M}_k$ which satisfies $\tilde{\alpha}(x)a = ax$ for every $x \in \tilde{M}$.

When $N = M$, α is non-strongly free if and only if $\tilde{\alpha}$ is inner. In this case, by [6, Proposition 5.4], α is of the form $\mathrm{Ad}\, u\sigma_c^\varphi$ for some $u \in U(M)$ and an extended modular automorphism σ_c^φ. (For extended modular automorphisms, see [2].)

Next we explain the canonical extension of endomorphisms introduced by Izumi in [10], [11].

Let $\rho \in \mathrm{End}(M)$ be an endomorphism of M with $d\rho < \infty$. We denote by ϕ_ρ the standard left inverse of ρ. Then the canonical extension $\tilde{\rho} \in \mathrm{End}(\tilde{M})$ of ρ is defined as follows:

$$\tilde{\rho}(x) = \rho(x), \quad x \in M,$$
$$\tilde{\rho}\big(\lambda(t)\big) = d\rho^{it}(D\varphi \circ \phi_\rho : D\varphi)_t \lambda(t).$$

It is shown in [11] that the canonical extension is compatible with sector operations.

Definition 2.3 **([11, Definition 3.1])** An endomorphism ρ is called a modular endomorphism if $\tilde{\rho}$ is an inner endomorphism, that is, there exist isometries $\{v_i\}_{i=1}^n$ satisfying $\tilde{\rho}(x) = \sum_i v_i x v_i^*$. We denote by $\mathrm{End}(M)_m$ and $\mathrm{Sect}(M)_m$ the set of all modular endomorphisms and the image of $\mathrm{End}(M)_m$ in $\mathrm{Sect}(M)$ by the quotient map respectively.

In the above definition, the number n of isometries $\{v_i\}$ is $d\rho$. Hence the dimension of a modular endomorphism is always an integer.

Take $\rho \in \mathrm{End}(M)_m$. Then we can find isometries $\{v_i\}_{i=1}^n \subset \tilde{M}$ with $\tilde{\rho}(x) = \sum_i v_i x v_i^*$. Define $c_{ij,t} := v_i^* \theta_t(v_j)$. Then $c = (c_{ij,t}) \in Z_\theta^1\Big(\mathbf{R}, U\big(n, Z(\tilde{M})\big)\Big)$. If ρ_1 and ρ_2 are equivalent modular endomorphisms, then the associated cocycles are equivalent. Hence the map $\delta[\rho] := [c] \in H_\theta^1\Big(\mathbf{R}, U\big(d\rho, Z(\tilde{M})\big)\Big)$ is well defined.

Theorem 2.4 **([11, Theorem 3.3])** *The map δ gives the bijective correspondence between* $\mathrm{Sect}(M)_m$ *and* $\bigsqcup_n H_\theta^1\Big(\mathbf{R}, U\big(n, Z(\tilde{M})\big)\Big)$.

We denote by ρ_c a modular endomorphism corresponding to a cocycle c.

3 Non-Strongly Free Automorphisms

In [19], Kosaki obtained the following theorem, which can be considered as a subfactor analogue of [6, Proposition 5.4].

Theorem 3.1 **([19, Theorem 19])** *Let $N \subset M$ be a subfactor of type III_λ, $\lambda \neq 0$. Then every non-strongly free automorphism is of the form $\alpha\sigma_t^\varphi$, where α is a non-strongly-outer automorphism.*

Let γ be Longo's canonical endomorphism of $N \subset M$. Since $\alpha \in \mathrm{Aut}(M, N)$ is non-strongly-outer if and only if α appears in the irreducible decomposition of γ^n, $n \geq 1$ (see [1], [17] and [19]), we can list all non-strongly free automorphisms of subfactors of type III_λ, $0 < \lambda \leq 1$, as long as we know the irreducible decomposition of γ^n. For example see [19, Section 6].

In the case of type III_0 factors, as an analogue of the above theorem, one may consider that every non-strongly free automorphism is of the form $\alpha\sigma_c^\varphi$, where α is a non-strongly-outer automorphism and σ_c^φ is an extended modular automorphism. However this is not true in general, and we will characterize non-strongly free automorphisms by using modular endomorphisms.

Theorem 3.2 *An automorphism α is non-strongly free if and only if there exists a modular endomorphism ρ_c such that $\alpha\rho_c$ appears as an irreducible component of γ^n for some n.*

Proof Let $(X, \nu, \mathcal{F})$ be the flow of weights of M. Hence we have $Z(\tilde{M}) = L^\infty(X, \nu)$ and $\theta_t(f)(w) = f(\mathcal{F}_{-t}w)$. Then we have the common central decomposition $\tilde{N} \subset \tilde{M} = \int_X^\oplus \big(\tilde{N}(w) \subset \tilde{M}(w)\big)\, d\nu(w)$.

Let α be a non-strongly free automorphism. Then there exists $0 \neq a \in \tilde{M}_k$ such that $\tilde{\alpha}(x)a = ax$ holds for every $x \in \tilde{M}$. The argument in [19, p. 436] shows that the Connes-Takesaki module of α is trivial. Hence the above equality can be decomposed as $\int_X^\oplus \alpha_w\big(x(w)\big)\, a(w)\, d\nu(w) = \int_X^\oplus a(w)x(w)\, d\nu(w)$ with $\alpha_w \in \mathrm{Aut}\big(M(w)\big)$. By this decomposition, $\alpha_w\big(x(w)\big)\, a(w) = a(w)x(w)$ holds for a.e. $w \in X$. Since $H_w := \{a \in \tilde{M}_k(w) \mid \alpha_w(x)a = ax \text{ for every } x \in \tilde{M}(w)\}$ is a finite dimensional Hilbert space,

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we can take orthonormal basis $\{a_j(w)\}$ for this Hilbert space. Note that $n := \dim H_w$ is constant for a.e. $w \in X$ because of the ergodicity of θ_t.

Since θ_t commutes with $\tilde{\alpha}$, we have $\alpha_{\mathcal{F}_t w}\theta_{(t,w)} = \theta_{(t,w)}\alpha_w$. Hence $\theta_{(t,x)}$ is a unitary operator from H_w to $H_{\mathcal{F}_t w}$. Then we get $\theta_{(t,w)}\big(a_j(w)\big) = \sum_i c_{ij}(t,w)a_i(\mathcal{F}_t w)$ for some unitary matrix $\big(c_{ij}(t,w)\big)_{i,j}$. This $\big(c_{ij}(t,w)\big)_{ij}$ satisfies the following cocycle equations:

$$c_{ij}(t+s,w) = \sum_k c_{ik}(t,\mathcal{F}_s w)c_{kj}(s,w).$$

Define $c_t = (c_{ij,t}) \in M\big(n; Z(\tilde{M})\big)$ by $c_{ij,t}(w) := c_{ij}(t,\mathcal{F}_{-t}w)$. Then $c_t\theta_t(c_s) = c_{t+s}$ holds. Indeed we have the following:

$$\begin{aligned}\sum_k c_{ik,t}(\mathcal{F}_{t+s}w)\theta_t(c_{kj,s})(\mathcal{F}_{t+s}w) &= \sum_k c_{ik,t}(\mathcal{F}_{t+s}w)c_{kj,s}(\mathcal{F}_s w)\\ &= \sum_k c_{ik}(t,\mathcal{F}_s w)c_{kj}(s,w)\\ &= c_{ij}(t+s,w)\\ &= c_{ij,t+s}(\mathcal{F}_{t+s}w).\end{aligned}$$

Thus we have $c \in Z^1_\theta\Big(\mathbf{R}, U\big(n, Z(\tilde{N})\big)\Big)$. By [10, Theorem 3.3], we have the modular endomorphism $\rho_{\bar{c}} \in \operatorname{End}(M)$ associate with $\bar{c}_t$, where $\bar{c}_{ij,t} := c^*_{ij,t}$. Let $\{w_i\}_{i=1}^n \subset \tilde{N}$ be an implementing system of $\rho_{\bar{c}}$. Then we have $\tilde{\rho}_{\bar{c}}(x) = \sum_i w_i x w_i^*$ and $w_i^*\theta_t(w_j) = c^*_{ij,t}$. Set $a_j := \int_X^\oplus a_j(w)\,d\nu(w)$. Then $\theta_t(a_j) = \sum_i c_{ij,t}a_i$ holds. It is easy to show that $\theta_t(\sum_j w_j a_j) = \sum_i w_i a_i$, and this means $\sum_i w_i a_i \in (\tilde{M}_k)^\theta = M_k$. Also we have $\rho_{\bar{c}}\alpha(x)\sum_i w_i a_i = \sum_i w_i a_i x$ for every $x \in M$. In a similar way as in the proof of [1], [17] or [19], we can prove that $\rho_{\bar{c}}\alpha$ appears in the irreducible decomposition of γ^k. ■

When $N \subset M$ is of type III$_\lambda$, $\lambda \neq 0$, every irreducible modular endomorphism is an automorphism. Hence Theorem 3.2 covers Kosaki's theorem.

Corollary 3.3 *If either we have $[M : N] < 4$ or the type II graph and type III graph of $N \subset M$ coincide, then every non-strongly free automorphism is the composition of a non-strongly-outer automorphism and an extended modular automorphism.*

Proof Let α be a non-strongly free automorphism, Δ the set of irreducible sectors appearing in the irreducible decomposition of γ^n, $n > 0$, and $\Delta_m := \Delta \cap \operatorname{Sect}(M)_m$. First assume that $N \subset M$ has the same type II graph and type III graph. In this case, Δ_m is $[\mathrm{id}_M]$ by [11, Theorem 3.5]. By Theorem 3.2, $[\alpha\rho_c]$ is in Δ for some modular endomorphism ρ_c. Since Δ is closed under sector operations, $[\rho_c\rho_{\bar{c}}] = [\alpha\rho_c]\overline{[\alpha\rho_c]}$ can be decomposed in Δ. If $d\rho_c \geq 2$, then nontrivial modular endomorphisms appear in Δ, and this is the contradiction. Hence ρ_c must be an extended modular automorphism, and α can be expressed in the form $\beta\sigma_c^\varphi$ for some non-strongly-outer automorphism β.

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Next we assume $[M : N] < 4$. We only have to verify the case that the type III graph of $N \subset M$ is A_{4n-3} and type II graph is D_{2n}. (See [18] or [21].) In this case, we have $\Delta_m = \{[\mathrm{id}_M], [\sigma]\}$, where σ is an extended modular automorphism with period two. Even in this case, it is impossible that $[\alpha\rho_c]$ with $d\rho_c \geq 2$ appears in Δ. ∎

By using Theorem 3.2, we can construct an example of a subfactor and a non-strongly free automorphism which does not have the form $\alpha\sigma_c^\varphi$ with a non-strongly-outer automorphism α. Let M be a type III_0 factor, $(X, \mathcal{F}_t)$ the flow of weights of M, c a minimal cocycle from $(X, \mathcal{F}_t)$ to a noncommutative group G of order 8 in the sense of [31]. Let π be a 2-dimensional irreducible representation of G. Let ρ be a modular endomorphism with $d\rho$ corresponding to a cocycle $\pi \circ c$. (See [11, Section 5].) The fusion rules of sectors generated by $[\rho]$ is the same as that of $\hat{G}$. Hence we have $[\rho^2] = [\mathrm{id}] \oplus [\sigma_1] \oplus [\sigma_2] \oplus [\sigma_1\sigma_2]$, $[\sigma_1\rho] = [\sigma_2\rho] = [\rho]$, $[\sigma_1^2] = [\sigma_2^2] = [\mathrm{id}]$ and $[\sigma_1\sigma_2] = [\sigma_2\sigma_1]$, where σ_i are extended modular automorphisms. Take $\alpha \in \mathrm{Ker}\,\mathrm{mod}$ which is not an extended modular automorphism, and define $\rho_1 \in \mathrm{End}(M)$ as $[\rho_1] = [\alpha\rho] \oplus [\mathrm{id}]$. Then $\rho_1(M) \subset M$ is a subfactor of type III_0 with the common flow of weights. In this case, α is indeed an automorphism of $\rho_1(M) \subset M$ up to inner perturbation because of $[\alpha\rho_1] = [\rho_1\alpha]$. Easy computation shows that the set of irreducible sectors appearing in $(\rho_1\bar{\rho}_1)^n$ is $\{[\mathrm{id}], [\alpha\rho], [\sigma_1], [\sigma_2], [\sigma_1\sigma_2]\}$. By Theorem 3.2, α is non-strongly free, but never has the form $\beta\sigma_c$ for some non-strongly-outer automorphism β and extended modular automorphism σ_c because the only non-strongly-outer automorphisms of $\rho_1(M) \subset M$ are $\{\mathrm{Ad}\,u, \mathrm{Ad}\,u\sigma_1, \mathrm{Ad}\,u\sigma_2, \mathrm{Ad}\,u\sigma_1\sigma_2 \mid u \in U(M)\}$ by the characterization of [1] and [17].

4 The Case of Subfactors with the Principal Graph $D_{2n}^{(1)}$

In the end of the previous section, we construct an example of a subfactor and its non-strongly free automorphism which does not have the form $\alpha\sigma_c^\varphi$ with a non-strongly-outer automorphism α. However this subfactor is reducible. Hence it is natural to ask if there exist examples of irreducible subfactors. In this section, we construct subfactors with the principal graph $D_{2n}^{(1)}$ which have the above property.

First we construct subfactors. In [4, Section 4.7], subfactors with the principal graph $D_n^{(1)}$ has been obtained as $A^{\mathbf{D}_{n-2}} \subset (A \otimes M_2)^{\mathbf{D}_{n-2}}$, where A is a factor with an outer action of a dihedral group $\mathbf{D}_{n-2}$. (Also see [12].)

Let a, b be generators of $\mathbf{D}_{2n}$ with the relations $a^{2n} = b^2 = e$ and $bab = a^{-1}$.

Fix a type III_0 factor A, and let $\sigma^{(0)}$ be an outer action of $\mathbf{D}_{2n}$ on A with the trivial modular invariant.

Let $\mathrm{Rep}(\mathbf{D}_{2n})$ be the category of finite dimensional representations of $\mathbf{D}_{2n}$. Since $\sigma^{(0)}$ is a dual action for some Roberts action of $\mathrm{Rep}(\mathbf{D}_{2n})$ on $A^{\sigma^{(0)}}$, we can take a system of isometries $\{v_i^\pi\}_{i=1}^{d\pi} \subset A$ with $v_i^{\pi *} v_j^\pi = \delta_{i,j}$ and $\sum_i v_i^\pi v_i^{\pi *} = 1$, and restriction of $\sigma^{(0)}$ on $\sum_i \mathbf{C} v_i^\pi$ is equivalent to π for each $\pi \in \mathrm{Rep}(\mathbf{D}_{2n})$. Then the dual action of $\mathrm{Rep}(\mathbf{D}_{2n})$ on M is given as follows:

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$$\rho_\pi(x) = \sum_i v_i^\pi x v_i^{\pi*}, \quad x \in A \otimes M_2(\mathbf{C}),$$

$$\rho_\pi(\lambda_g) = \lambda_g.$$

Let χ be a one-dimensional representation of $\mathbf{D}_{2n}$ given by $\chi(a) = 1$ and $\chi(b) = -1$ and χ_1 a one dimensional representation given by $\chi_1(a) = -1$, $\chi_1(b) = 1$, and set $\chi_2 = \chi\chi_1$. Let π_ω be a two-dimensional representation of $\mathbf{D}_{2n}$ given as follows.

$$\pi_{\omega^m}(a) = \begin{pmatrix} \omega^m & 0 \\ 0 & \bar{\omega}^m \end{pmatrix}, \quad \pi_{\omega^m}(b) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

where ω is a $2n$-th primitive root of 1. If $m \neq 0, n$, then $\omega^m \neq \pm 1$ and π_{ω^m} is irreducible, and we have $\pi_1 \cong \mathrm{id} \oplus \chi$ and $\pi_{-1} \cong \chi_1 \oplus \chi_2$. Then $\{\pi_\omega^m\}_{m=1}^{n-1} \cup \{\mathrm{id}, \chi, \chi_1, \chi_2\}$ is the set of representatives of $\widehat{\mathbf{D}_{2n}}$.

Let φ^0 be a dominant weight of A^{σ^0}, and set $\varphi := \varphi^0 \circ E$, where E is the natural conditional expectation from A onto A^{σ^0}. We may assume that (φ, ρ_π) is an invariant pair in the sense of [11, Definition 2.2] for every $\pi \in \widehat{\mathbf{D}_{2n}}$. Then it follows that φ is invariant under σ^0, and v_i^π is in A_φ. The former is trivial, and the latter follows from [8, Theorem 4.19].

Define an action σ of $\mathbf{D}_{2n}$ on $A \otimes M_2(\mathbf{C})$ as in [12, Theorem 5.5]. Then a subfactor $N \subset M := A \rtimes_\sigma \mathbf{D}_{2n} \subset \big(A \otimes M_2(\mathbf{C})\big) \rtimes_\sigma \mathbf{D}_{2n}$ has the principal graph $D_{2n+2}^{(1)}$. Moreover this subfactor has the common flow of weights and its flow is given by $\big(Z(\tilde{A})^\sigma, \theta_t|_{Z(\tilde{A})^\sigma}\big)$. See [15], [28]. The continuous decomposition $\tilde{N} \subset \tilde{M}$ can be identified with $\tilde{A} \rtimes_{\tilde{\sigma}} \mathbf{D}_{2n} \subset \big(\tilde{A} \otimes M_2(\mathbf{C})\big) \rtimes_{\tilde{\sigma}} \mathbf{D}_{2n}$. Here the action $\tilde{\sigma}$ of $\mathbf{D}_{2n}$ on $\tilde{A} \otimes M_2$ is given by the canonical extension of σ. Set $H := \mathrm{Ker\,mod}\,\sigma^{(0)}$, and assume that $H = \{e, a^n\}$. Hence $\mathbf{D}_{2n}/H$ acts on $Z(\tilde{A})$ faithfully. The flow $\{Z(\tilde{A}), \theta_t|_{Z(\tilde{A})}\}$ can be expressed as the skew product of $Z(\tilde{N})$ and its minimal cocycle to $\mathbf{D}_{2n}/H$. (See [31], or [11, Appendix A].) Hence we may assume $Z(\tilde{A}) = Z(\tilde{N}) \otimes l^\infty(\mathbf{D}_{2n}/H)$. We denote by $\dot{x}$ an equivalence class of $x \in \mathbf{D}_{2n}$ in $\mathbf{D}_{2n}/H$, and by $\delta_{\dot{x}}$ the characteristic function of $\{\dot{x}\}$ in $l^\infty(\mathbf{D}_{2n}/H)$.

Now we will prove $\tilde{\rho}_{\pi_\omega}(x) = w_1\tilde{\alpha}(x)w_1^* + w_2\tilde{\alpha}(x)w_2^*$ for some $\tilde{\alpha} \in \mathrm{Aut}(\tilde{M}, \tilde{N})$ commuting with θ_t. If we can find such $\tilde{\alpha}$, define α as the restriction of $\tilde{\alpha}$ on $M = \tilde{M}^\theta$. Since we have $\tau_{\tilde{M}}\tilde{\rho}_{\pi_\omega} = 2\tau_{\tilde{M}}$ by [11, Proposition 2.5], it is easy to see $\tau\tilde{\alpha} = \tau$. By [14, Lemma 1.1], $\tilde{\alpha}$ is the canonical extension of α. We will show that α is a non-strongly free automorphism. In what follows, we denote ρ_{π_ω} by ρ, and $v_i^{\pi_\omega}$ by v_i for simplicity. We have $\sigma_a(v_0) = \omega v_0$, $\sigma_a(v_1) = \bar{\omega}v_1$, $\sigma_b(v_0) = v_1$ and $\sigma_b(v_1) = v_0$. The canonical extension $\tilde{\rho}$ of ρ is given as follows:

$$\tilde{\rho}(x) = v_0 x v_0^* + v_1 x v_1^*, \quad x \in \tilde{A},$$

$$\tilde{\rho}(\lambda_g) = \lambda_g.$$

Lemma 4.1 *Define w_1 and w_2 as follows.*

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$$w_0 := v_0 \sum_{i=0}^{n-1} \omega^i \delta_{\dot{a}^i} + v_1 \sum_{i=0}^{n-1} \bar{\omega}^i \delta_{\dot{a}^i \dot{b}},$$

$$w_1 := v_1 \sum_{i=0}^{n-1} \bar{\omega}^i \delta_{\dot{a}^i} + v_0 \sum_{i=0}^{n-1} \omega^i \delta_{\dot{a}^i \dot{b}}.$$

Then w_0 and w_1 are mutually orthogonal isometries with support 1.

Proof First we compute $w_0^* w_0$. Then

$$\begin{aligned} w_0^* w_0 &= \Big(\sum_{i=0}^{n-1} v_0^* \bar{\omega}^i \delta_{\dot{a}^i} + v_1^* \omega^i \delta_{\dot{a}^i \dot{b}} \Big) \Big(\sum_{j=0}^{n-1} v_0 \omega^j \delta_{\dot{a}^j} + v_1 \bar{\omega}^j \delta_{\dot{a}^j \dot{b}} \Big) \\ &= \sum_{i=0}^{n-1} \delta_{\dot{a}^i} + \delta_{\dot{a}^i \dot{b}} \\ &= 1. \end{aligned}$$

In a similar way, we can show $w_1^* w_1 = 1$ and $w_0^* w_1 = 0$.

Next we compute $w_0 w_0^*$:

$$\begin{aligned} w_0 w_0^* &= \Big(\sum_{j=0}^{n-1} v_0 \omega^j \delta_{\dot{a}^j} + v_1 \bar{\omega}^j \delta_{\dot{a}^j \dot{b}} \Big) \Big(\sum_{i=0}^{n-1} v_0^* \bar{\omega}^i \delta_{\dot{a}^i} + v_1^* \omega^i \delta_{\dot{a}^i \dot{b}} \Big) \\ &= v_0 v_0^* \sum_{i=0}^{n-1} \delta_{\dot{a}^i} + v_1 v_1^* \sum_{i=0}^{n-1} \delta_{\dot{a}^i \dot{b}}. \end{aligned}$$

In a similar way, we get $w_1 w_1^* = v_1 v_1^* \sum_{i=0}^{n-1} \delta_{\dot{a}^i} + v_0 v_0^* \sum_{i=0}^{n-1} \delta_{\dot{a}^i \dot{b}}$. Now $w_0 w_0^* + w_1 w_1^* = 1$ is clear. ■

Lemma 4.2 *$w_i w_j^* \subset (\tilde{\rho}, \tilde{\rho})$ and $w_1^* \tilde{\rho}(\cdot) w_1 = w_2^* \tilde{\rho}(\cdot) w_2 =: \beta$ is an automorphism.*

Proof In the proof of the above lemma, we already know $w_0 w_0^* = v_0 v_0^* e_0 + v_1 v_1^* e_1$, where $e_0 = \sum_{i=0}^{n-1} \delta_{\dot{a}^i}$ and $e_1 = \sum_{i=0}^{n-1} \delta_{\dot{a}^i \dot{b}}$. Since e_0 and e_1 are in $Z(\tilde{A})$ and $\tilde{\rho}(x) = v_0 x v_0^* + v_1 x v_1^*$ for $x \in \tilde{A}$, it is shown that $w_0 w_0^* \tilde{\rho}(x) = \tilde{\rho}(x) w_0 w_0^*$ holds for $x \in \tilde{A}$. We have $\mathrm{Ad}\, \lambda_a v_i v_i^* = v_i v_i^*$ and $\mathrm{Ad}\, \lambda_a e_i = e_i$. Hence we get $w_0 w_0^* \tilde{\rho}(\lambda_a) = \tilde{\rho}(\lambda_a) w_0 w_0^*$. Similarly $w_0 w_0^* \tilde{\rho}(\lambda_b) = \tilde{\rho}(\lambda_b) w_0 w_0^*$ holds. Hence $w_0^* \tilde{\rho}(\cdot) w_0$ is an endomorphism of $\tilde{M}$. If $x \in \tilde{A}$, then

$$\begin{aligned} w_0^* \tilde{\rho}(x) w_0 &= \Big(\sum_{i=0}^{n-1} v_0^* \bar{\omega}^i \delta_{\dot{a}^i} + v_1^* \omega^i \delta_{\dot{a}^i \dot{b}} \Big) (v_0 x v_0^* + v_1 x v_1^*) \Big(\sum_{j=0}^{n-1} v_0 \omega^j \delta_{\dot{a}^j} + v_1 \bar{\omega}^j \delta_{\dot{a}^j \dot{b}} \Big) \\ &= e_0 x + e_1 x \\ &= x \end{aligned}$$

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holds.

On the other hand, we have

$$\begin{aligned} w_0^*\tilde{\rho}(\lambda_a)w_0 &= \Big(\sum_{i=0}^{n-1} v_0^*\bar{\omega}^i\delta_{\dot{a}^i} + v_1^*\omega^i\delta_{\dot{a}^i\dot{b}}\Big)\lambda_a\Big(\sum_{j=0}^{n-1} v_0\omega^j\delta_{\dot{a}^j} + v_1\bar{\omega}^j\delta_{\dot{a}^j\dot{b}}\Big) \\ &= \Big(\sum_{i=0}^{n-1} v_0^*\bar{\omega}^i\delta_{\dot{a}^i} + v_1^*\omega^i\delta_{\dot{a}^i\dot{b}}\Big)\Big(\sum_{j=0}^{n-1} v_0\omega^{j+1}\delta_{\dot{a}^{j+1}} + v_1\bar{\omega}^{j+1}\delta_{\dot{a}^{j+1}\dot{b}}\Big)\lambda_a \\ &= \Big(\Big(\sum_{i=0}^{n-1}\bar{\omega}^i\delta_{\dot{a}^i}\Big)\Big(\sum_{j=0}^{n-1}\omega^{j+1}\delta_{\dot{a}^{j+1}}\Big) + \Big(\sum_{i=0}^{n-1}\omega^i\delta_{\dot{a}^i\dot{b}}\Big)\Big(\sum_{j=0}^{n-1}\omega^{j+1}\delta_{\dot{a}^{j+1}\dot{b}}\Big)\Big)\lambda_a \\ &= \Big(-\delta_{\dot{e}} - \delta_{\dot{b}} + \sum_{i=1}^{n-1}\delta_{\dot{a}^i} + \delta_{\dot{a}^i\dot{b}}\Big)\lambda_a. \end{aligned}$$

Put $u = (-\delta_{\dot{e}} - \delta_{\dot{b}} + \sum_{i=1}^{n-1}\delta_{\dot{a}^i} + \delta_{\dot{a}^i\dot{b}})$. By similar computation, we can verify $w_0^*\tilde{\rho}(\lambda_b)w_0 = -u\lambda_b$. Due to $u^2 = 1$, $\beta := w_0^*\tilde{\rho}(\cdot)w_0$ is an automorphism with period 2.

In the same way as above, $w_1^*\tilde{\rho}(x)w_1 = \beta(x)$ can be shown. ■

By the above lemma, $\{w_iw_j^*\}_{0\le i,j\le 1}$ is a matrix unit in $(\tilde{\rho},\tilde{\rho})$.

Lemma 4.3 *We have $\theta_t\beta\theta_{-t} = \operatorname{Ad} u_t^*\beta$ for some θ_t-cocycle u_t.*

Proof Since θ_t commutes with $\tilde{\rho}$, we have $\tilde{\rho}(x)\theta_t(w_0) = \theta_t(w_0)\theta_t\beta\theta_{-t}(x)$. Since $\theta_t(w_0w_0^*)$ is equivalent to $w_0w_0^*$ in $(\tilde{\rho},\tilde{\rho})$, β and $\theta_t\beta\theta_{-t}$ are unitary equivalent. Hence there exists a unitary u_t with $\operatorname{Ad} u_t^*\beta = \theta_t\beta\theta_{-t}$. Since $\theta_{t+s}\beta\theta_{-(t+s)} = \theta_t\theta_s\beta\theta_{-s}\theta_{-t}$, u_{t+s} and $u_t\theta_t(u_s)$ differ by a center valued 2-cocycle. However it is always a coboundary (for example see [2, Appendix]), and hence we can choose u_t as a θ-cocycle. ■

By stability of θ_t, there exists a unitary u with $u^*\theta_t(u) = u_t$. Then $\operatorname{Ad} u\beta$ commutes with θ_t. Let α be the restriction of $\operatorname{Ad} u\beta$ on M. As remarked in the beginning of this section, $\operatorname{Ad} u\beta$ is the canonical extension of α.

Proposition 4.4 *An automorphism α is a non-strongly free automorphism.*

Proof Since ρ is an irreducible component of γ, and $\rho\alpha$ is a modular endomorphism, the conclusion follows by Theorem 3.2. ■

Theorem 4.5 *Let $N \subset M$ be $A \rtimes_\sigma \mathbf{D}_{4n} \subset \big(A \otimes M_2(\mathbf{C})\big) \rtimes_\sigma \mathbf{D}_{4n}$ and α an automorphism constructed above. Then α is not the composition of a non-strongly-outer automorphism and an extended modular automorphism.*

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Proof By examining the fusion rule of Δ, we can show

$$\Delta_m = \{[\mathrm{id}], [\rho_\chi], [\rho_{\chi_1}], [\rho_{\chi_2}]\} \cup \{[\rho_{\pi_{\omega^{2m}}}]\}_{m=1}^{n-1}.$$

For example, we have $[\rho_{\pi_\omega}\rho_{\pi_\omega}] \cong [\mathrm{id}] \oplus [\rho_\chi] \oplus [\rho_{\pi_{\omega^2}}]$. Since an outer period of $\tilde{\alpha}$ is two, $\rho_{\pi_\omega}\rho_{\pi_\omega}$ is a modular endomorphism. Hence its irreducible components are also modular endomorphisms. Non-strongly-outer automorphisms of $N \subset M$ are inner perturbation of id, ρ_χ, ρ_{χ_1} and ρ_{χ_2}, and all of them are extended modular automorphisms. On the other hand, α is not an extended modular automorphism. Hence α is not the composition of a non-strongly-outer automorphism and an extended modular automorphism. ∎

Remark 1. If $N \subset M$ is a subfactor constructed in this section by using $\mathbf{D}_{4n-2}$, then Δ_m associated with this subfactor is $\{[\mathrm{id}], [\sigma_\chi]\} \cup \{\rho_{\pi_{\omega^{2m}}}\}_{m=1}^{n-1}$. Hence ρ_{χ_1} and ρ_{χ_2} are non-strongly-outer automorphism and neither of them is an extended modular automorphism. In this case, all non-strongly free automorphism are the composition of ρ_{χ_1} and modular extended modular automorphisms. In fact, it is easy to see $[\tilde{\rho}_{\chi_1}] = [\tilde{\rho}_{\chi_2}] = [\tilde{\alpha}]$.

2. We have a cocycle θ from $(X, \mathcal{F})$ to the Loi part of $\tilde{N}(w) \subset \tilde{M}(w)$. In a similar way as in [19], we can prove that ρ_c appears in γ^n if and only if there exist $\{v_i\} \in \tilde{M}' \cap \tilde{M}_k$ satisfying $\theta_t(v_j) = \sum_i c_{ij,t} v_i$, and this is a representation of a cocycle θ. There exists a 1 to 1 correspondence between the representation of a cocycle and the representation of the minimal group associated with a cocycle θ. (See [31].) Then the minimal group associated with this cocycle is $\mathbf{D}_{2n}/\{e, a^n\}$. (Note that the Loi part of subfactors with index 4 is completely determined by Loi in [22].) Moreover this minimal cocycle determines the relative flow of weights for $N \subset M$, which plays an essential role in the classification of type III$_0$ subfactors in [30].

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