STONE'S THEOREM AND COMPLETENESS OF ORTHOGONAL SYSTEMS

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1. Introduction

It is well known (e.g. Stone [1]) that the Stone-Weierstrass approximation theorem can be used to prove the completeness of various systems of orthogonal polynomials, e.g. Chebyshev polynomials. In this paper, Stone's theorem is used to prove a more general completeness theorem, which includes as special cases Plancherel's theorem, the corresponding theorem for Hankel transforms, the completeness of various polynomial systems, and certain expansions in Jacobian elliptic functions. The essential feature common to all these systems is a certain algebraic structure – if S is an appropriate vector space spanned by orthogonal functions, then the algebra A generated by S is contained in the closure of S in a suitable norm.

Stone's theorem is used in the following form (Naimark [2]). If B is a real algebra of continuous functions on the compact Hausdorff space X, which separates points, then the uniform closure of B either coincides with the algebra of all real continuous functions on X, or with the subalgebra of all real continuous functions which vanish at one particular point of X. (B may be complex if also B contains the conjugate of each of its points.)

Now let X denote a locally compact Hausdorff space, and μ a measure on X; it will always be assumed, without further statement, that $\mu(J) < \infty$ for each compact $J \subset X$, and that the space $L^2(J)$ contains, as a dense subset, $C(J)$, the space of all continuous functions on J; here the inner product on X is $(f, g) = \int_X f\bar{g}d\mu$, and the $L^2(X)$-norm is $||f|| = (f,f)^{\frac{1}{2}}$; denote also by $||f||_J$ the L^2-norm on the subset $J \subset X$. In section 2, $C(X)$ and $L^2(X)$ are restricted to real functions; in Section 3, they relate to complex functions.

If $E \subset C(X) \cap L^2(X)$, $\bar{E}$ denotes the closure of E in the $L^2(X)$-norm $||\cdot||$; $\hat{E}$ denotes the closure of E in the uniform norm on X, $||\cdot||'$; and E^* denotes the closure of E in the norm $||\cdot||^* = ||\cdot||+||\cdot||'$. (If E is compact, then $E^* = \hat{E}$.)

2. Functions on compact spaces

Let X be a compact Hausdorff space, and μ a measure on X, satisfying the above conventions; let $G = \{\phi_n : n = 1, 2, \cdots\}$ be an orthonormal sequence of

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real functions on X, thus $(\phi_m, \phi_n) = \delta_{mn}$. Denote by S_n the vector space spanned by $\{\phi_1, \cdots, \phi_n\}$; $S = \bigcup_1^\infty S_n$; and A the algebra consisting of all finite linear combinations of finite products of elements of S.

LEMMA 1. *If $\phi_m \phi_n \in \bar{S}$ for each pair of integers m, n, then $A \subset \bar{S}$.*

PROOF. It suffices to prove that if g is a finite product of elements of G, such that $g \in \bar{S}$, and $\psi \in G$, then $\psi g \in \bar{S}$. Since $g \in \bar{S}$, $||g - g_p|| \to 0$, where $g_p \in S$, thus g_p is a finite sum $\sum \alpha_j \phi_j$; by hypothesis, $\psi g_p = \sum \alpha_j(\psi\phi_j) \in \bar{S}$; so $||\psi g_p - h_{pq}|| < \varepsilon$ for $q > q(\varepsilon, p)$, where $h_{pq} \in S$. Then

$$\begin{aligned} ||\psi g - h_{pq}|| &\leqq ||\psi(g-g_p)|| + ||\psi g_p - h_{pq}|| \\ &\leqq \sup_X |\psi| \, ||g - g_p|| + ||\psi g_p - h_{pq}|| \\ &< \sup_X |\psi| \varepsilon + \varepsilon \end{aligned}$$

for $p \geqq p(\varepsilon)$ and $q \geqq q(\varepsilon, p(\varepsilon))$. Hence $g\psi \in \bar{S}$.

THEOREM 1. *Let X be compact Hausdorff, μ a measure on X, and $G = \{\phi_n\}$ an orthonormal sequence of real continuous functions on X, such that*

(i) *$\hat{A}$ separates points in X;*
(ii) *for each integer pair m, n, $\phi_m \phi_n \in \bar{S}$;*
(iii) *for each point $x \in X$, there is a function $f \in \hat{A}$ which does not vanish at X.*

Then G is complete.

PROOF. From Lemma 1, $A \subset \bar{S}$. Let $f \in L^2(X)$; since $C(X)$ is dense in $L^2(X)$, there is $g \in C(X)$ with $||f - g|| < \frac{1}{3}\varepsilon$. By Stone's theorem, $\hat{A} = C(X)$, therefore there is $p \in A$ with $\mathrm{Sup}_X |g - p| < \varepsilon/(3\sqrt{\mu(X)})$. Since $A \subset \bar{S}$, there is $h \in S$ with $||p - h|| < \varepsilon/3$; and $h \in S_N$ for some N. Then, for $n \geqq N$,

$$\begin{aligned} ||f - \sum_1^n (f, \phi_j)\phi_j|| &\leqq ||f - h|| \\ &\leqq ||f - g|| + ||g - p|| + ||p - h|| \\ &< \varepsilon/3 + \varepsilon/(3\sqrt{\mu(X)})\sqrt{\mu(X)} + \varepsilon/3 \\ &= \varepsilon, \end{aligned}$$

since $g - p \leqq \sup_X |g - p| \cdot \sqrt{\mu(X)}$. Hence $\bar{S} = L^2(X)$.

COROLLARY. *Theorem* 1 *remains true if $\{\phi_n\}$ is not orthogonal.*

PROOF. By Gram-Schmidt orthogonalization, $\{\phi_n\}$ may be replaced by an orthonormal sequence $\{\psi_n\}$; since the algebra A generated by $\{\psi_n\}$ is the same as that generated by $\{\phi_n\}$, the hypotheses of Theorem 1 apply also to $\{\psi_n\}$.

Theorem 1 remains true if the $\{\phi_n\}$ are complex valued, if A (or S) contains the conjugate of each of its elements.

If $X = [a, b]$ is a compact real interval, and $G = \{\phi_n : n = 0, 1, 2, \cdots\}$ is a sequence of real polynomials, such that ϕ_n has degree n, for $n = 0, 1, 2, \cdots$, then $A \subset S$, since $\phi_m \phi_n$ is a polynomial, and the other hypotheses of Theorem 1 are also fulfilled. Thus, for example, the Legendre polynomials, and the Chebyshev polynomials, form complete systems. The same conclusion applies to trigonometric polynomials, with period $b-a$, except that here the two end points of X must be identified. If, as in these instances, $A \subset S$ holds, as well as the hypotheses of Theorem 1, then $S \subset C(X) = \hat{A}$ implies $\hat{S} = C(X)$; thus each continuous function on X is the uniform limit of a sequence of functions in S. For trigonometric polynomials, this result is part of Fejér's theorem.

If G is complete, X compact Hausdorff, and each ϕ_n real continuous, then $A \subset \bar{S}$, since each element of A is continuous; hypothesis (iii) of Theorem 1 holds; and $\bar{S}$ separates points of X; but it is not obvious that $\hat{A}$ separates points of X. So the direct converse to Theorem 1 may not hold, without extra hypothesis.

Theorem 1 can also be applied to certain expansions in Jacobian elliptic functions, analogous to trigonometric Fourier series.

LEMMA 2. *Let the function f be defined on* $[0, \pi]$ *by*

$$f(\theta) = \sum_{n=1}^{\infty} a_n \cos n\theta, \tag{1}$$

where $a_1 = 1$, $\sum_2^\infty |a_n| < 1$, and $\sum a_n \lambda^n$ converges for some $\lambda > 1$. Let ρ and σ denote any positive integers. Then $g(\theta) = f(\rho\theta)f(\sigma\theta)$ is expansible in a uniformly convergent series

$$g(\theta) = \sum_{s=1}^{\infty} b_s f(s\theta). \tag{2}$$

PROOF. If the Laurent series

$$\psi(v) = \sum_{n=-\infty}^{\infty} \alpha_n v^n$$

converges for $\lambda^{-1} < |v| < \lambda$, where $\lambda > 1$, then $\lambda_0^{-1} < |v|^\rho < \lambda_0$, and $\lambda_0^{-1} < |v|^\sigma < \lambda_0$, for suitable $\lambda_0 > 1$. Therefore the series

$$\psi(v^\rho)\psi(v^\sigma) = \sum_{-\infty}^{\infty} \delta_r v^r, \text{ where } \delta_r = \sum_{m\rho + n\sigma = r} \alpha_m \alpha_n, \tag{3}$$

converges for $\lambda_0^{-1} < |v| < \lambda_0$. Hence

$$\sum_{-\infty}^{\infty} |\delta_r| < \infty.$$

Set $\alpha_0 = 0$, and $\alpha_n = \frac{1}{2}a_{|n|}$ for $n \neq 0$; then $f(\theta) = \psi(e^{i\theta})$. Therefore, from (3),

$$g(\theta) = f(\rho\theta)f(\sigma\theta) = \sum_{n=0}^{\infty} d_n \cos n\theta,$$

where $\sum |d_n| < \infty$.

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Now consider

$$\sum_{s=1}^{p} b_s f(s\theta) = \sum_{s=1}^{p} \sum_{n=1}^{\infty} b_s a_n \cos ns\theta = \sum_{r=1}^{\infty} d'_r \cos r\theta$$

where, for $r < p$,

$$d'_r = \sum_{\substack{sn=r\\ s\leqq p}} a_n b_s = b_r + \sum_{\substack{sn=r\\ 1<n\leqq r\\ s\leqq p}} a_n b_s = d_r.$$

Since no pair of subscripts n, s occurs more than once in the array $\{a_n b_s\}$,

$$\sum_{s=1}^{\infty} |b_r| \leqq \sum_{s=1}^{\infty} |d_r| + \sum_{n=2}^{\infty} |a_n| \sum_{s=1}^{\infty} |b_s|.$$

Since $\sum_2^\infty |a_n| < 1$ and $\sum |d_r| < \infty, \sum_1^\infty |b_r| < \infty$.

Therefore

$$\sum_{s=1}^{\infty} |b_s|\,|f(s\theta)| \leqq \sum_{s=1}^{\infty} |b_s| \sum_{n=1}^{\infty} |a_n| < \infty,$$

so (2) converges uniformly.

THEOREM 2. *The sequence of Jacobian elliptic functions*

(4) $$1, cn(x, k), cn(2x, k), \cdots, cn(nx, k), \cdots$$

is complete on $(0, 2K)$, *where* $4K$ *is the real period of the elliptic functions of modulus* k, *provided that* $0 < k < k_c$, *where* $k_c = 0.99$ *approximately.*

PROOF. Let $X = [0, 2K]$, $d\mu(x) = dx$. Now

$$\mathrm{cn}(x, k) = \frac{2\pi}{k^{\frac{1}{2}}K} \sum_{s=0}^{\infty} \frac{q^{s+\frac{1}{2}}}{1+q^{2s+1}} \cos \frac{(2s+1)\pi x}{2K},$$

where q is the nome. Set $\theta = \frac{1}{2}\pi x/K$, and

$$f(\theta) = \left[\frac{2\pi}{k^{\frac{1}{2}}K} \cdot \frac{q^{\frac{1}{2}}}{1+q}\right]^{-1} \mathrm{cn}(x, k).$$

Since $0 < q < 1, f(\theta)$ satisfies the hypotheses of Lemma 2 if k is such that

$$\sum_{1}^{\infty} \left| \frac{q^{s+\frac{1}{2}}}{1+q^{2s+1}} \right| < \frac{q^{\frac{1}{2}}}{1+q}.$$

This is so if $q^{\frac{3}{2}}/(1-q) < q^{\frac{1}{2}}/(1+q)$, thus if $q < \sqrt{2}-1$, or, from tables [2], if $K \leqq 0.99$. By Lemma 2, if S denotes the space spanned by the functions (4), then the product of any two of the functions (4) is contained in the uniform closure of S, and therefore in $\bar{S}$, since X is compact. The other hypotheses of the Corollary to Theorem 1 are readily verified (orthogonality is not required).

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REMARK. A similar proof, and conclusion, applies to the sequence

$$\mathrm{sn}(x,k),\ \mathrm{sn}(2x,k),\cdots,\ \mathrm{sn}(nx,k),\cdots \tag{5}$$

on $[0, 2K]$, with a similar restriction on k.

Hence, the sequences (4) and (5) combined form a complete system on $[-2K, 2K]$.

3. Generalized Plancherel theorem

Let X be a locally compact Hausdorff space, μ a measure on X, and $\{J_m\}$ an expanding sequence of compact subsets of X, such that each $\mu(J_m) < \infty$, and $\mu(X - \bigcup_1^\infty J_m) = 0$. Assume that $L^2(J_m)$ contains $C_0(J_m)$, the space of continuous complex functions on X whose support is contained in J_m, as a dense subspace; note that a function in $C_0(J_m)$ vanishes at the boundary of J_m. For each $f \in C(J_m)$, extend its definition to all of X by defining $f(x) = 0$ for $x \in X - J_m$. Let J and J' denote elements of $\{J_m\}$, and, for brevity, write $J \uparrow X$ for $J_m \uparrow \bigcup_1^\infty J_m$; the neglected set $X - \bigcup_1^\infty J_m$ does not affect the $L^2(X)$-norm. Let Y, ν, K_n, K, K' satisfy the same hypotheses as X, μ, J_m, J, J'.

A function ϕ, mapping $Y \times X$ into complex numbers, will be called an *orthogonal kernel* if it satisfies the following four conditions:

(i) (*Continuity*) For each $y \in Y$, the function $\phi(y, \cdot)$ is continuous on each J_m. For each $x \in X$, $\phi(\cdot, x)$ is measurable.

(ii) (*Orthogonality*) For each compact $K \in \{K_n\}$,

$$\lim_{J \uparrow X} \int_Y \sigma(y')d\nu(y') \int_Y \overline{\sigma(y)}d\nu(y) \int_J \overline{\phi(y,x)}\phi(y',x)d\mu(x) = \int_Y |\sigma(y)|^2 \, d\nu(y) \tag{6}$$

whenever $\sigma \in D(K)$, a dense (in $L^2(X)$-norm) subset of $C_0(K)$.

(iii) (*Boundedness*) $\phi(y, x)$ is bounded on each compact subset of $Y \times X$.

(iv) (*Conjugacy*) Either ϕ assumes real values only, in which case σ, $C_0(K)$, $C(K)$, $L^2(X)$, $L^2(Y)$ will be restricted to real functions, or, for each $y \in Y$, there exists $y' \in Y$ for which $\overline{\phi(y, x)} = \phi(y', x)$.

If X and Y are vector spaces, then the orthogonality property (6) can be formally expressed, in terms of the Dirac delta function, by

$$\int_X \overline{\phi(y,x)}\phi(y',x)d\mu(x) = \delta(y - y').$$

Denote by S the vector space of functions $T\sigma$, where

$$(T\sigma)(x) = \int_Y \sigma(y)\phi(y,x)d\nu(y), \tag{7}$$

in which $\sigma \in C(K)$, K is a compact subset of Y, and $x \in X$. Denote by A the algebra of all finite linear combinations of finite products of elements of S.

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If J is a compact subset of X, denote by $B(J)$ the vector space of functions b, with support in J, such that each b is the limit, in $||\cdot||_X^*$-norm, of a sequence in S. (Then each $b \in B(J)$ vanishes on the boundary of $B(J)$.) The space $B(J)$ will be said to have the *separating property* if

(i) for each two distinct points x_1, x_2 of J, of at which at most one is a boundary point of J, there exists $b \in B(J)$ with $b(x_1) \neq b(x_2)$.

(ii) for each interior point x_1 of J, there exists $b \in B(J)$ with $b(x_1) \neq 0$.

Let $f \in L^2(X)$; for compact $J \subset X$, define f_J by $f_J(x) = f(x)$ for $x \in J, f_J(x) = 0$ for $x \in X - J$. Define the operator Q by

$$(8) \qquad (Qf_J)(y) = \int_X f_J(x)\overline{\phi(y, x)}d\mu(x) \qquad (y \in Y).$$

Define also, where they exist, the following limits:

$$(9) \qquad Qf = \lim_{J \uparrow X} Qf_J \qquad \text{(in } L^2(Y)\text{-norm)}.$$

$$(10) \qquad T(Qf) = \lim_{K \uparrow Y} T(Q_K f) \qquad \text{(in } L^2(X)\text{-norm)},$$

where

$$(Q_K f)(y) = \begin{cases} (Qf)(y) & \text{for } y \in K \\ 0 & \text{for } y \in Y - K. \end{cases}$$

LEMMA 3. *Let ϕ be an orthogonal kernel; let $f \in L^2(X)$; let $\sigma \in D(K)$, where $K \in \{K_n\}$. Then Qf and TQf exist, and*

$$(11) \qquad ||f - T\sigma||^2 \geqq ||f - TQf||^2 = ||f||^2 - ||Qf||^2.$$

PROOF. From (7) and the boundedness property of ϕ, $T\sigma$ is bounded and continuous, for $\sigma \in C(K)$, but not necessarily of bounded support. If $\sigma \in D(K)$ then, from (6) and (7),

$$(12) \qquad ||T\sigma|| = ||\sigma||.$$

By Fubini's theorem, for $\sigma \in D(K)$,

$$(f_J, T\sigma) = (Qf_J, \sigma).$$

From (8) and the boundedness property of ϕ, $Q_K f_J \in C(K)$.

Let $\rho \in L^2(K)$, and let ρ vanish on $Y - K$. Since $D(K)$ is dense in $L^2(K)$, there is $\sigma \in D(K)$ with $||\rho - \sigma||_K < \varepsilon$. By the boundedness property of ϕ, $|\phi(y, x)|$ is bounded by a constant, $\lambda = \lambda(J, K)$ say, on $K \times J$. Define $T\rho$ by (7). Then

$$\begin{aligned} ||T\rho||_J &\leqq ||T(\rho - \sigma)||_J + ||T\sigma|| \\ &\leqq ||\rho - \sigma||_K[\lambda^2\mu(J)\nu(K)]^{\frac{1}{2}} + ||\sigma|| \\ &\quad \text{by (7), applied to } \rho - \sigma, \text{ and (12)} \\ &\leqq ||\rho - \sigma||[\lambda^2\mu(J)\nu(K)]^{\frac{1}{2}} + ||\rho - \sigma|| + ||\rho|| \\ &< \varepsilon[\lambda^2\mu(J)\nu(K)]^{\frac{1}{2}} + \varepsilon + ||\rho||. \end{aligned}$$

Hence $||T\rho||_J < ||\rho||$. Let $J \uparrow Y$, then

$$||T\rho|| \leqq ||\rho||. \tag{14}$$

Now, if $\sigma \in D(K)$, then

$$\begin{aligned} ||f_J - T\sigma||^2 &= (f_J, f_J) - (T\sigma, f_J) - (f_J, T\sigma) + (T\sigma, T\sigma) \\ &= ||f_J||^2 - (Qf_J, \sigma) - (\sigma, Qf_J) + ||\sigma||^2 \\ &\quad \text{by (11) and (12)} \\ &= ||f_J||^2 - ||Qf_J||_K^2 + ||\sigma - Qf_J||_K^2 \end{aligned}$$

where the $L^2(K)$-norms are used, since σ has support in K. Since $Q_K f_J \in C(K)$, and $D(K)$ is dense in $C(K)$, there is $\sigma \in D(K)$ with $||\sigma - Qf_J||_K < \varepsilon$. Hence $||Qf_J||_K < ||f_J||$, so

$$||Qf_J|| = \lim_{K \uparrow Y} ||Qf_J||_K \leqq ||f_J|| \leqq ||f|| < \infty. \tag{15}$$

So, from (14), if $\sigma \in D(K)$, then

$$||f_J - T\sigma||^2 \geqq ||f_J - T(Q_K f_J)||^2 = ||f_J||^2 - ||Q_K f_J||^2. \tag{16}$$

Let $K' \supset K$ and $J' \supset J$. Then

$$\begin{aligned} ||Qf_{J'} - Qf_J|| = ||Qf_{J'-J}|| &\leqq ||f_{J'-J}|| \qquad \text{by (15)} \\ &\to 0 \qquad \text{as } J \uparrow X. \end{aligned}$$

So $\{Qf_J\}$ is a Cauchy sequence in $L^2(Y)$, defining Qf. Similarly

$$\begin{aligned} ||T(Q_{K'}f) - T(Q_K f)|| &= ||TQ_{K'-K} f|| \\ &\leqq ||Q_{K'-K} f|| \qquad \text{by (13)} \\ &\to 0 \qquad\qquad \text{as } K \uparrow Y. \end{aligned}$$

Hence $\{T(Q_K f)\}$ is a Cauchy sequence in $L^2(X)$, defining $T(Qf)$.

Now let $J \uparrow X$, then $K \uparrow Y$, in (16). Since, by (14),

$$||TQ_K f - TQ_K f_J|| = ||TQ_K f_{X-J}|| \leqq ||Q_K f_{X-J}|| \leqq ||f_{X-J}||$$

which $\to 0$ as $J \uparrow X$, the limit of (16) yields (11).

LEMMA 4. *If $A \subset S^*$, then $B(J)$ is an algebra, closed in the uniform norm on J.*

PROOF. Convergence in $||\cdot||^*$ implies both uniform convergence on X and convergence in the $L^2(X)$-norm $||\cdot||$. Let $b, b' \in B(J)$; then $b = \lim s_n$, $b' = \lim s_n'$, where $s_n, s_n' \in S$. Since $A \subset S^*$, $s_n s_n' = \lim a_{n,r}$, where $a_{n,r} \in S$, and the limit is in $||\cdot||^*$ norm. Hence bb' is the uniform limit of a sequence $\{d_n\} \subset S$, where $d_n = a_{n,r(n)}$. Now

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$$
\begin{aligned}
||bb'-d_n|| &\leqq ||b(b'-s_n')||+||b'(b-s_n)||+||(b-s_n)(b'-s_n')||+||s_n s_n'-d_n|| \\
&\leqq \sup_J |b| \cdot ||b'-s_n'||+\sup_J |b'| \cdot ||b-s_n||+\sup_Y |b-s_n| \cdot ||b'-s_n'|| \\
&\quad +||s_n s_n'-d_n|| \\
&< \varepsilon \qquad \text{for } n > N(\varepsilon).
\end{aligned}
$$

So $bb' \in B(J)$; and $B(J)$ is an algebra.

If $\{b_n\} \subset B(J)$, and $\{b_n\} \to b$ uniformly on J, then, since J is compact, $||b_n-b|| \to 0$ also; hence $b \in B(J)$. Thus $B(J)$ is closed in the uniform norm.

LEMMA 5. *If $A \subset S^*$, and $B(J)$ has the separating property, then $B(J) = C_0(J)$.*

PROOF. The algebra $B(J)$ is unaltered by identifying all boundary points of J with a single point x_0; this identification maps J onto a compact set J'. Since $B(J)$ has the separating property, $B(J)$, considered as an algebra on J', separates points of J', and for each point x_1 of J', $x_1 \neq x_0$, there is $f \in B(J)$ with $f(x_1) \neq 0$. By Stone's theorem, $B(J)$ coincides with the algebra of all continuous functions on J' which vanish at x_0; thus $B(J) = C_0(J)$.

THEOREM 3. *Let X and Y have the properties listed above; let ϕ be an orthogonal kernel; let $A \subset S^*$; let $B(J)$ have the separating property, for each $J \in \{J_m\}$; let $f \in L^2(X)$. Then*

$$f = TQf \tag{17}$$

PROOF. Since $f \in L^2(X)$, there exists a compact $J \in \{J_m\}$ and $g \in L^2(X)$, with support of g in J, such that $||f-g|| < \varepsilon/3$. There exists $h \in C_0(J)$ such that $||g-h|| < \varepsilon/3$. By Lemma 5, $B(J) = C_0(J)$, so there exists $s \in S$ such that $||h-s|| < \varepsilon/3$. Therefore $||f-s|| < \varepsilon$.

Now, from (11), since $s = T\sigma$ for some σ, $||f-s|| \geqq ||f-TQf||$. Therefore $||f-TQf|| = 0$, and $f = TQf$.

REMARKS. In the definition of $B(J)$, the uniform convergence on X may be replaced by uniform convergence on J, with bounded convergence almost everywhere on $X-J$; the same results follow, with ess sup replacing sup in the proof of Lemma 4.

If, in particular, $J = \bigcup_1^\infty J_m$ is compact, then $B(J)$ may be replaced by $\bar{A}$, and the hypothesis $A \subset S^*$ by $A \subset \bar{S}$. Then $(\bar{A})^\wedge = \bar{A} \subset \bar{S}$, so that $\bar{A} = C_0(J)$; the rest of the proof is unchanged.

4. Applications

(a). *Plancherel's Theorem.* Let $X = Y = (-\infty, \infty)$, $\mu = \nu =$ Lebesgue measure, $J_m = [-m, m]$, $K_n = [-n, n]$, $\phi(y, x) = \pi^{-\frac{1}{2}}e^{ixy}$, and let $D(K)$ be the space of continuous functions of bounded variation in the compact interval K. It is

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readily verified that ϕ is an orthogonal kernel; the orthogonality follows from

$$(18)\qquad \begin{aligned} &\iint \sigma(y')\overline{\sigma(y)}dy\,dy' \int_{-n}^{n} \overline{e^{ixy}}e^{ixy'}dx \\ &= \iint \sigma(y')\overline{\sigma(y)}dy\,dy' \cdot \frac{2\sin n(y'-y)}{y'-y} \\ &\to \pi \int |\sigma(y)|^2\,dy \qquad \text{as } n\to\infty, \quad \text{if } \sigma \in D(K). \end{aligned}$$

If $T\sigma_1$ and $T\sigma_2$ are elements of S, then their product is $T\sigma$, where σ is the convolution of σ_1 and σ_2, and therefore is continuous and of compact support. Thus $A \subset S$, and a fortiori $A \subset S^*$.

To show that $B(J)$ has the separating property, it suffices to verify that $\psi_h(x-a) \in B(J)$ for each a interior to J and all h sufficiently small, where $\psi_h(t) = 1-|t|/h$ for $|t| < h$, $\psi_h(t) = 0$ for $|t| > h$. Now $\psi_h(t) = \lim_{n\to\infty} \psi_{h,n}(t)$, where, for $a_n = (2n+\frac{1}{2})\pi/h$,

$$(19)\qquad \psi_{h,n}(t) = \frac{1}{\pi}\int_{-a_n}^{a_n} \frac{1-\cos hy}{y^2}e^{ity}dy \in S.$$

The convergence in (19) is uniform in t, and also in $L^2(-\infty, \infty)$-norm, since for $n' > n$, by (18),

$$(20)\qquad \frac{1}{\pi}\int_{-\infty}^{\infty}\left|\int_{a_n}^{a_{n'}} \frac{1-\cos hy}{y^2}e^{ity}dy\right|^2 dX = \int_{a_n}^{a_{n'}}\left|\frac{1-\cos hy}{y^2}\right|^2 dy$$

which $\to 0$ as $n\to\infty$.

Then Theorem 3 applies, yielding Plancherel's theorem, for $f \in L^2(-\infty, \infty)$:

$$(21)\qquad f(x) = \frac{1}{\pi}\ \underset{n\to\infty}{\mathrm{l.i.m.}} \int_{-n}^{n} e^{ixy}dy\left(\underset{m\to\infty}{\mathrm{l.i.m.}}\int_{-m}^{m} f(t)e^{-ity}dt\right)$$

(b). *Orthonormal sequences of functions.* Let Y be the set of non-negative integers, with the discrete topology; define the measure ν by $\nu(y) = 1$ for each $y \in Y$. Then the compact subsets of Y are just the finite subsets; let $K_n = \{0, 1, 2, \cdots, n\}$. Write $\phi_n(x)$ for $\phi(n, x)(n \in Y)$; then, from (6),

$$(22)\qquad \int_X \phi_m(x)\overline{\phi_n(x)}d\mu(x) = \delta_{mn}.$$

So $\{\phi_n\}$ is a sequence of orthonormal functions, continuous except on the null set $W = X - \bigcup_1^\infty J_m$, and S is the space of finite linear combinations of the ϕ_n. As before, let A denote the algebra generated by S. Then Theorem 3 has the following corollary.

THEOREM 4. *Let X be locally compact; let the measure μ on X satisfy the requirements of Theorem 3. Let $\{\phi_n\}$ be a sequence of orthonormal locally bounded*

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functions on X, continuous except on the null set W. Let each ϕ_n be real, or let S contain the conjugate of each of its elements. Let $A \subset S^*$. For each $x \in X$, which is an interior point of some J_m, and each sufficiently small neighbourhood N of x, let a function f exist, the $||\cdot||^*$-limit of a sequence in S, such that f is positive on N and zero elsewhere. Then the sequence $\{\phi_n\}$ is complete.

REMARKS. The existence of these functions f ensures that each space $B(J_m)$ has the separating property. The requirement of $||\cdot||^*$-convergence can be weakened, as indicated in the remarks following Theorem 3. Theorem 4 is applicable, e.g., to the Hermite polynomials.

(c). *Hankel transform.* Let $X = Y = [0, \infty)$, $d\mu(x) = x^{2\rho+1}dx$, $dv(y) = dy$ where $\rho \geqq 0$ is a constant, $J_m = K_m = [m^{-1}, m]$. Let $\phi(y, x) = y^{\frac{1}{2}}x^{-\rho}J_\rho(yx)$' where $x \in X, y \in Y$. The following results are quoted from the theory of Besse' functions (see, e.g., [4], and the appendix):

$$\phi(y, x) \quad \text{is bounded and continuous on} \quad Y \times X. \tag{23}$$

$$\lim_{r\to\infty} \int_K \sigma(\alpha)\alpha^{\frac{1}{2}}\, d\alpha \int_K \overline{\sigma(\beta)}\beta^{\frac{1}{2}}\, d\beta \int_0^r J_\rho(\alpha x)J_\rho(\beta x)x\, dx = \int_K |\sigma(\alpha)|^2\, d\alpha \tag{24}$$

if $\sigma \in D(K)$, the space of continuous functions of bounded variation on $K \in \{K_m\}$.

$$J_\rho(\alpha x)J_\rho(\beta x) = c_\rho(\alpha\beta x)^\rho \int_{-1}^1 J_\rho(\gamma x)(1-t^2)^{\rho-\frac{1}{2}}\gamma^{-\rho}dt, \tag{25}$$

in which c_ρ is constant, and $\gamma = [\alpha^2+\beta^2-2\alpha\beta t]^{\frac{1}{2}}$. (Then $|\alpha-\beta| \leqq \gamma \leqq |\alpha+\beta|$.)

$$\left| \int_a^b h(\xi)\xi^{\rho+1}J_\rho(\xi y)d\xi \right| \leqq B(1+y^2)^{-1} \tag{26}$$

where B is constant, independent of y, if $0 < a < b < \infty$, and $h \in C^2[0, \infty)$, and has support in $[a, b]$.

From (23) and (24) it follows that ϕ is an orthogonal kernel. From (25), the product of two elements of S is of the form

$$\begin{aligned} &\int_K \sigma_1(\alpha)\phi(\alpha, x)dv(\alpha) \int_K \sigma_2(\beta)\phi(\beta, x,dv(\beta) \\ &\qquad = \int_K \sigma_1(\alpha)\sigma_2(\beta)d\alpha\, d\beta \int_{-1}^1 (\alpha\beta/\gamma)^{\rho+\frac{1}{2}}\phi(\gamma, x)c_\rho(1-t^2)^{\rho-\frac{1}{2}}\, dt \end{aligned} \tag{27}$$

where $\sigma_1, \sigma_2 \in C(K)$, $K = [a, b]$, $0 < a < b < \infty$; (27) is of the form

$$\int_0^{2b} \phi(\gamma, x)g(\gamma)d\gamma$$

in which $g(\gamma)$ is continuous in $(0, 2b]$. Changing the variables in (27) from t, α, β to γ, α, β, and substituting $u = \alpha+\beta$, $v = \alpha-\beta$, $h(u, v) = \sigma_1(\alpha)\sigma_2(\beta)$, gives, after reduction:

$$g(\gamma) = \tfrac{1}{2}c_\rho \gamma^{-\rho-\frac{1}{2}} \iint h(u, v)(\gamma^2 - v^2)^{\rho-\frac{1}{2}} \left[1 - \frac{\gamma^2 - v^2}{u^2 - v^2}\right]^{\rho-\frac{1}{2}} du\, dv.$$

Now

$$0 < \frac{\gamma^2 - v^2}{u^2 - v^2} = \frac{\gamma^2}{4\alpha\beta} \leqq \frac{\gamma^2}{4a^2},$$

so that if $\gamma < a$, and $M = \sup |h(u, v)|$,

$$\left[1 - \frac{\gamma^2 - v^2}{u^2 - v^2}\right]^{\rho-\frac{1}{2}} \leqq M' = \max\{1, (\tfrac{1}{2})^{\rho-\frac{1}{2}}\}$$

and

$$|g(\gamma)| \leqq \tfrac{1}{2}c_\rho \gamma^{-\rho+\frac{1}{2}} MM' \int_0^{2b} (\gamma^2 - v^2)^{\rho-\frac{1}{2}} dv \leqq \tfrac{1}{2}c_\rho MM' \gamma^{\rho+\frac{1}{2}}.$$

So, if $\rho \geqq 0$, $g \in L^2[0, 2b]$.

Now, from the orthogonality property of ϕ, if $0 < \xi < \eta$, then

$$\lim_{r\to\infty} \int_0^r x^{2\rho+1} dx \left| \int_\xi^\eta \phi(\gamma, x) g(\gamma) d\gamma \right|^2 = \int_\xi^\eta |g(\gamma)|^2 d\gamma$$
$$\to 0 \quad \text{as} \quad \eta \to 0.$$

Therefore (27) equals

$$\lim_{\xi \downarrow 0} \int_\xi^{2b} \phi(\gamma, x) \rho(\gamma) d\gamma,$$

the convergence being in $L^2(X)$-norm, and also uniformly on X, since $g \in L[0, 2b]$ and $\phi(y, x)$ is bounded uniformly in x. Thus (27) is the $\|\cdot\|^*$-limit of elements of S; and $A \subset S^*$.

It remains to show that each $B(J)$ has the separating property. Let $0 < a < b < \infty$; let $h \in C^2[0, \infty)$, with support of h contained in $[a, b]$. Then

$$\Psi(y) = \int_0^\infty h(\xi)\phi(y, \xi) d\mu(\xi)$$

is continuous in $[0, \infty)$, and, from (26), $|\Psi(y)| < By^{\frac{1}{2}}(1 + y^2)^{-1}$, so $\Psi \in L[0, \infty)$. This, with the boundedness of ϕ, implies that

$$H(x) = \lim_{r\to\infty} \int_{r^{-1}}^r \phi(y, x)\Psi(y) dy \tag{28}$$

exists uniformly for $x \in [a, b]$; so H is continuous. Since, for any interval $[a', b'] \subset [a, b]$,

$$\lim_{r\to\infty} \int_{a'}^{b'} dx \int_{r^{-1}}^r \phi(y, x)\Psi(y) dy$$
$$= \lim_{r\to\infty} \int_{a'}^{b'} dx \int_a^b h(\xi)\xi^{\rho+1} x^{-\rho} d\xi \int_0^r J_\rho(\xi y) J_\rho(xy) y\, dy$$
$$= \int_{a'}^{b'} h(x) dx, \qquad \text{using (24)},$$

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it follows that $H(x) = h(x)$ for each x. If $E = K_m - K_n (m > n)$, then

$$\int_0^\infty \left| \int_E \phi(y, x)\Psi(y)dy \right|^2 d\mu(x) = \int_E |\Psi(y)|^2 dy$$
$$\to 0 \quad \text{as} \quad n \to \infty;$$

so (28) converges to h both uniformly and in $L^2(X)$-norm, thus in $||\cdot||^*$-norm.

Let $x^{\frac{1}{2}}F(x) \in L^2(0, \infty)$; then

$$\int_0^\infty |f(x)|^2 x^{2\rho+1} dx < \infty,$$

where $f(x) = F(x)x^{-\rho}$. By Theorem 3, $f = TQf$. Expressed in terms of F, this gives the Bessel-Plancherel theorem:

$$F(x) = \underset{m\to\infty}{\text{l.i.m.}} \int_0^m J_\rho(yx)y\, dy \left(\underset{n\to\infty}{\text{l.i.m.}} \int_0^n F(\xi)J_\rho(y\xi) \right) \xi\, d\xi. \tag{29}$$

(The replacement of $\int_{n^{-1}}^n$ by $\int_0^n$ is readily validated.)

Appendix

Equation (24) is obtained from Bessel's equation and from the asymptotic expressions [4]:

$$(30) \qquad J_\rho(\alpha n) = \sqrt{\frac{2}{\pi\alpha n}} \cos(\alpha n - q) \cdot \left(1 + O\left(\frac{1}{n}\right)\right)$$
$$J'_\rho(\alpha n) = \sqrt{\frac{2}{\pi\alpha n}} \sin(\alpha n - q) \cdot \left(1 + O\left(\frac{1}{n}\right)\right)$$

(as $n \to \infty$; $\alpha, \beta > 0$; $q = q(\rho)$.). From Bessel's equation,

$$\int_0^n J_\rho(\alpha x)J_\rho(\beta x)x\, dx = \frac{n}{\beta^2 - \alpha^2} [\alpha J_\rho(n\beta)J'_\rho(n\alpha) - \beta J_\rho(n\alpha)J'_\rho(n\beta)]. \tag{31}$$

On substituting the expressions (30), (31) reduces to

$$\frac{\sin n(\beta - \alpha)}{\pi(\beta - \alpha)} \cdot \frac{1}{\alpha^{\frac{1}{2}}\beta^{\frac{1}{2}}} + E + O\left(\frac{1}{n}\right),$$

where E is a finite sum of terms of the form $[\pi(\alpha\beta)^{\frac{1}{2}}(\alpha+\beta)]^{-1}e^{i(np+r)}$, where each r is constant, and each p is a linear combination of α and β. If σ is a continuous function of bounded variation, whose compact support K does not include the origin, then

$$\int_K \sigma(\alpha)\alpha^{\frac{1}{2}}\,d\alpha \int_K \overline{\sigma(\beta)}\beta^{\frac{1}{2}}\,d\beta \int_0^n J_\rho(\alpha x)J_\rho(\beta x)x\,dx$$

$$= \int_K \sigma(\alpha)\alpha^{\frac{1}{2}}\,d\alpha \int_K \overline{\sigma(\beta)}\beta^{\frac{1}{2}}\,d\beta \cdot \frac{\sin n(\beta-\alpha)}{\pi(\beta-\alpha)(\alpha\beta)^{\frac{1}{2}}}$$

$$+ \sum \int_K \sigma(\alpha)\alpha^{\frac{1}{2}}\,d\alpha \int_K \overline{\sigma(\beta)}\beta^{\frac{1}{2}}\,d\beta \cdot \frac{e^{i(np+r)}}{\pi(\alpha+\beta)(\alpha\beta)^{\frac{1}{2}}} + O\left(\frac{1}{n}\right)$$

$$\to \int_K \sigma(\alpha)\overline{\sigma(\alpha)}d\alpha + 0 + 0 \qquad \text{as } n \to \infty.$$

From [4], page 28,

$$\frac{J_\rho(\gamma x)}{\gamma^\rho} = 2^\nu x^{-\rho}\Gamma(\rho)\sum_{m=0}^{\infty}(\rho+m)\frac{J_{\rho+m}(\alpha x)J_{\rho+m}(\beta x)}{(\alpha\beta)^\rho}C_m^\rho(\cos\theta),$$

where $\gamma = (\alpha^2+\beta^2-2\alpha\beta\cos\theta)^{\frac{1}{2}}$, and the Gegenbauer functions $C_m^\rho(\cos\theta)$ satisfy the orthogonality relation ([4], page 77):

$$\int_0^\pi \sin^{2\rho}\theta \cdot C_m^\rho(\cos\theta)\cdot C_n^\rho(\cos\theta)d\theta = \text{const. }\delta_{mn}.$$

Therefore

$$\int_0^\pi \frac{J_\rho(\gamma x)}{\gamma^\rho} C_0^\rho(\cos\theta)\sin^{2\rho}\theta\,d\theta = \text{const. }\frac{J_\rho(\alpha x)J_\rho(\beta x)}{(\alpha\beta x)^\rho}.$$

From this (25) follows, on substituting $\cos\theta = t$, and noting that $C_0^\rho(t) = 1$.

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