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SUMMATION FORMULAE OF MULTIPLICATIVE FUNCTIONS OVER ARITHMETIC PROGRESSIONS AND APPLICATIONS

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ABSTRACT. In this paper, we investigate the asymptotic distribution of a class of multiplicative functions over arithmetic progressions without the Ramanujan conjecture. We also apply these results to some interesting arithmetic functions in automorphic context, such as coefficients of automorphic L-functions, coefficients of their Rankin–Selberg.

1. INTRODUCTION

Problems concerning the asymptotic distribution of arithmetic functions in arithmetic progressions are very classical in analytic number theory, and appear all over the place. Let q be a positive integer and a be an integer prime to q, and let $\{a_n\}_{n=1}^{\infty}$ be an arithmetic sequence of complex numbers. Define

$$\mathcal{S}(x;a,q) = \sum_{\substack{n\leq x \\ n\equiv a \,(\mathrm{mod}\, q)}} a_n.$$

One expects the sequence to be generally well distributed in residue classes to modulo q, namely

$$\mathcal{S}(x;a,q) = \frac{1}{\varphi(q)} \sum_{\substack{n\leq x \\ (a,q)=1}} a_n + \text{small error}, \tag{1.1}$$

where φ is Euler's function. For example, if $a_n = \Lambda(n)$, the von Mangoldt function, the Siegel–Walfisz theorem says that for any $q \leq \log^A x$

$$\sum_{\substack{n\leq x \\ n\equiv a \,(\mathrm{mod}\, q)}} \Lambda(n) = \frac{x}{\varphi(q)} + O\big(x\exp(-c_A\sqrt{\log x})\big),$$

where A is any real number and c_A is some constant depending only on A. If $a_n = \tau_k(n)$, the number of representations of n as the product of k factors, (1.1) holds for $q \leq x^{\theta_k - \varepsilon}$ with

$$\theta_2 = \frac{2}{3}, \quad \theta_3 = \frac{1}{2} + \frac{1}{82}, \quad \theta_4 = \frac{1}{2}, \ldots$$

See the details in [10]. Another example is $a_n = \lambda_f(n)$, the normalized Fourier coefficients of a holomorphic cusp form f, Smith [23] showed that (1.1) holds uniformly for $q \leq x^{\frac{2}{3}}$. Moreover, Murty gave some interesting remarks on Smith's work and said "It is likely that the methods of [23] are applicable for coefficients of Dirichlet series attached to automorphic representation of higher $\mathrm{GL}(n, \mathbb{A}_{\mathbb{Q}})$" at the end of this paper.

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Let $d \geq 2$ be an integer, and let $\mathcal{F}(d)$ be the set of all cuspidal automorphic representations π of $\mathrm{GL}(d)$ over $\mathbb{Q}$ with trivial central character. Let q_π denote the arithmetic conductor of π. For each $\pi \in \mathcal{F}(d)$, the corresponding L-function is defined by absolutely convergent Dirichlet series as

$$L(s,\pi) = \sum_{n=1}^{\infty} \lambda_\pi(n) n^{-s}$$

for $\mathrm{Re}\, s > 1$. Motivated by the remarks of Murty as above, it is interesting to study the distribution of Dirichlet coefficients $\lambda_\pi(n)$ in arithmetic progressions

$$\sum_{\substack{n \leq x \\ n \equiv a \, (\mathrm{mod}\, q)}} \lambda_\pi(n). \tag{1.2}$$

In general, one needs to replace the congruence $n \equiv a \,(\mathrm{mod}\, q)$ in (1.2) by a character sum of additive or multiplicative characters modulo q. Smith [23] chose to use the additive characters and then investigated the properties of generating series of $\lambda_f(n)\mathrm{e}(an/q)$ including the analytical continuation and functional equation, where $\mathrm{e}(x) := \exp(2\pi i x)$ for any $x \in \mathbb{R}$. However, for the higher rank case on $\mathrm{GL}(d)$, the functional equation of Dirichlet series $\sum_{n=1}^{\infty} \lambda_\pi(n)\mathrm{e}(an/q)n^{-s}$ is complicated and lacks a little symmetry structure (see [17] for details). Hence, in contrast to the work of Smith, we shall replace the congruence in (1.2) by a character sum of multiplicative characters, and can prove the following result.

Theorem 1.1. *If $\pi \in \mathcal{F}(d)$ with $(q, aq_\pi) = 1$, then we have*

$$\sum_{\substack{n \leq x \\ n \equiv a \, (\mathrm{mod}\, q)}} \lambda_\pi(n) \ll \begin{cases} \tau_d(q) x^{1-\frac{1}{d}} & \text{if } q \leq x^{\frac{1}{d}}, \\ \tau_{d^2}(q) x^{1-\frac{d+1}{d^2+1}} \log x & \text{if } q \leq x^{\frac{2}{d^2+1}}. \end{cases}$$

Assume the generalized Ramanujan conjecture holds for π, then we have

$$\sum_{\substack{n \leq x \\ n \equiv a \, (\mathrm{mod}\, q)}} \lambda_\pi(n) \ll \tau_d(q) \big(q^{\frac{d-1}{2}} \log q + x^{1-\frac{2}{d+1}} \big)$$

for $q \leq x^{2/(d+1)}$. The implied constants all depend on π only.

Another important arithmetic function is the coefficient $\lambda_{\pi\times\widetilde{\pi}}(n)$ of the Rankin–Selberg L-function $L(s, \pi \times \widetilde{\pi})$, where $\widetilde{\pi}$ denotes the contragredient of $\pi \in \mathcal{F}(d)$. This example is also our motivation for using the multiplicative characters to detect the congruence.

Theorem 1.2. *If $\pi \in \mathcal{F}(d)$ with $(q, aq_\pi) = 1$, then we have*

$$\sum_{\substack{n \leq x \\ n \equiv a \, (\mathrm{mod}\, q)}} \lambda_{\pi\times\widetilde{\pi}}(n) = \frac{c_{\pi,q}}{\varphi(q)} x + O\Big(\tau_{d^2}(q) q^{\frac{d^2-1}{2}} \log q \Big) + O\Big(\tau_{d^2}(q) x^{\frac{d^2-1}{d^2+1}} \Big)$$

for $q \leq x^{\frac{2}{d^2+1}}$, where $c_{\pi,q}$ is defined by $c_{\pi,q} = \operatorname*{Res}_{s=1} \big(L(s, \pi\times\widetilde{\pi}) \big) \prod_{p \mid q} L(1, \pi_p \times \widetilde{\pi}_p)^{-1}$, and the implied constant depends on π only.

As in the argument of Theorem 1.1, if the coefficients $\lambda_\pi(n)$ of L-functions are not all non-negative, we can produce a formula for $\sum_{n \leq x} \lambda_\pi(n)$ in terms of a sum of $\lambda_\pi(n)$ over a

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short interval. Our next goal is to strengthen Theorem 1.1 for special cases by improving some related estimates over short intervals.

Let k and N be positive integers with k even and N square-free, and $\Gamma_0(N)$ be the group of matrices $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})$ with the condition $c \equiv 0 (\bmod N)$. Let $H_k^*(N)$ denote the set of arithmetically normalized primitive cusp forms of weight k for $\Gamma_0(N)$ which are eigenfunctions of all the Hecke operators. Any $f \in H_k^*(N)$ has a Fourier expansion at infinity given by

$$f(z) = \sum_{n=1}^{\infty} \lambda_f(n) n^{\frac{k-1}{2}} \mathrm{e}(nz),$$

where $\lambda_f(1) = 1$ and the eigenvalues $\lambda_f(n) \in \mathbb{R}$. Deligne's bound gives

$$|\lambda_f(n)| \le \tau(n) \tag{1.3}$$

for all $n \ge 1$, where we put as usual $\tau_2(n) = \tau(n)$. The eigenvalues $\lambda_f(n)$ enjoy the multiplicative property

$$\lambda_f(m)\lambda_f(n) = \sum_{\substack{d|(m,n) \\ (d,N)=1}} \lambda_f\left(\frac{mn}{d^2}\right)$$

for all integers $m, n \ge 1$. In particular, $\lambda_f(n)$ are multiplicative. The Hecke L-function $L(s,f)$ associated to f has the Euler product representation

$$L(s,f) = \sum_{n\ge 1} \frac{\lambda_f(n)}{n^s} = \prod_p \left(1 - \frac{\lambda_f(p)}{p^s} + \frac{\psi_0(p)}{p^{2s}}\right)^{-1},$$

where ψ_0 denotes the principal character modulo N. We rewrite the Euler product as

$$L(s,f) = \prod_p \left(1 - \frac{\alpha_f(p)}{p^s}\right)^{-1} \left(1 - \frac{\beta_f(p)}{p^s}\right)^{-1},$$

where $\alpha_f(p), \beta_f(p)$ are complex numbers with

$$\begin{cases} \alpha_f(p) = \varepsilon_p p^{-\frac{1}{2}},\ \beta_f(p) = 0 & \text{if } p|N, \\ \alpha_f(p) = \overline{\beta_f(p)},\ |\alpha_f(p)| = |\beta_f(p)| = 1 & \text{if } p \nmid N \end{cases}$$

and $\varepsilon_p \in \{\pm 1\}$. For each $d \ge 1$, we define the twisted d-th symmetric power L-function by the degree $d+1$ Euler product

$$L(s, \mathrm{sym}^d f) = \prod_p \prod_{0\le j\le d} \left(1 - \frac{\alpha_f(p)^{d-j}\beta_f(p)^j}{p^s}\right)^{-1} := \sum_{n\ge 1} \frac{\lambda_{\mathrm{sym}^d f}(n)}{n^s}. \tag{1.4}$$

Note that $L(s, \mathrm{sym}^1 f) = L(s,f)$.

Recently, Newton and Thorne [19, Theorem B] proved that if $d \ge 1$, then the d-th symmetric power lift $\mathrm{sym}^d f$ corresponds to a cuspidal automorphic representation of $\mathrm{GL}(d+1, \mathbb{A}_{\mathbb{Q}})$ with trivial central character. Moreover, for each prime p, let $\theta_p \in [0,\pi]$ be the unique angel such that $\lambda_f(p) = 2\cos\theta_p$. The Sato–Tate conjecture states that the sequence $\{\theta_p\}$

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is equidistributed in the interval $[0,\pi]$ with respect to the measure $\mathrm{d}\mu_{ST} := (2/\pi)\sin^2\theta \mathrm{d}\theta$. Equivalently, for any continuous function $g \in C([0,\pi])$, one has

$$\sum_{\substack{p\le x\\ p\nmid N}} g(\theta_p) \sim \Big(\int_0^\pi g(\theta)\mathrm{d}\mu_{ST}\Big)\frac{x}{\log x} \qquad \text{as} \quad x \longrightarrow \infty. \tag{1.5}$$

This is now a theorem of Barnet-Lamb, Geraghty, Harris and Taylor [1].

For this special arithmetic function $\lambda_{\mathrm{sym}^d f}(n)$ on GL_{d+1}, we get the following result.

Theorem 1.3. *Let $f \in H_k^*(N)$ and $\lambda_{\mathrm{sym}^d f}(n)$ be the coefficients of $L(s, \mathrm{sym}^d f)$. For $(q, aN) = 1$, we have*

$$\sum_{\substack{n\le x\\ n\equiv a \,(\mathrm{mod}\, q)}} \lambda_{\mathrm{sym}^d f}(n) \ll \tau_{d+1}(q)\Big(q^{\frac{d}{2}}(\log q)^{1-\gamma_d} + x^{\frac{d}{d+2}}(\log x)^{-\gamma_d}\Big)$$

for $q \le x^{\frac{2}{d+2}}$, where $\gamma_d = 1 - \frac{4(d+1)}{d(d+2)\pi}\cot\big(\frac{\pi}{2(d+1)}\big) > 0.15$, and the implied constant depends on f and d.

Remark 1.1. For any fixed $f \in H_k^*(N)$ and $(q, aN) = 1$, Smith [23] obtained a uniform estimate

$$\sum_{\substack{n\le x\\ n\equiv a \,(\mathrm{mod}\, q)}} \lambda_f(n) \ll \tau(q)x^{\frac{1}{3}}\log x$$

for $q \le x^{2/3}$. Compared this with the case $d = 1$ in Theorem 1.3, it is obvious that our result is of a smaller size.

2. The main result

All these results in the theorems above are some specific applications of our technical formulae in Theorem 2.1 below. To state this core result, we need to describe the situation that we consider. Inspired by the series of works [5–8] of Duke and Iwaniec who have developed several techniques for estimating the coefficients of L-functions that satisfy standard functional equations, this paper here is to investigate the average order of a class of multiplicative functions over arithmetic progressions under some similar conditions.

(A1) **Euler product and Dirichlet series**. Let $\mathcal{A} = \{\mathcal{A}_p\}$ be a sequence of square complex matrices of order d indexed by primes, with monic characteristic polynomial $P_p(x) = P_p^{\mathcal{A}}(x) \in \mathbb{C}[x]$ and roots $\alpha_j(p)$. Then our general L-function $L(s,\mathcal{A})$ will be given by

$$L(s,\mathcal{A}) = \prod_p \prod_{j=1}^d \Big(1 - \frac{\alpha_j(p)}{p^s}\Big)^{-1} = \sum_{n=1}^\infty \frac{a_n}{n^s}, \tag{2.1}$$

where we assume that the product and the series are absolutely convergent for $\mathrm{Re}(s) > 1$. Note that $|\alpha_j(p)| \le p$ for all p, which is implied by the convergence of the Euler product for $\mathrm{Re}\, s > 1$.

(A2) **Analytic continuation.** There is some $m = m(\mathcal{A})$ such that $L(s,\mathcal{A})$ can be continued analytically over all of $\mathbb{C}$ except possibly for a pole of order m at $s = 1$.

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(A3) **Functional equation.** Let a Gamma factor $\Delta(s)$ be defined by

$$\Delta(s)=\prod_{j=1}^{d}\Gamma_{\mathbb{R}}(s+\mu_j),$$

where $\Gamma_{\mathbb{R}}(s)=\pi^{-s/2}\Gamma(s/2)$, and μ_j is an arbitrary complex number with $\mathrm{Re}\,\mu_j>-1$ for each $1\le j\le d$. The complete L-function

$$\Lambda(s,\mathcal{A}):=q_{\mathcal{A}}^{\frac{s}{2}}\Delta(s)L(s,\mathcal{A})$$

has finite order, and satisfies the functional equation

$$\Lambda(1-s,\mathcal{A})=\omega_{\mathcal{A}}\overline{\Lambda(1-\bar{s},\mathcal{A})}.$$

where $q_{\mathcal{A}}$ is a positive integer and $\omega_{\mathcal{A}}$ is a complex number with $|\omega_{\mathcal{A}}|=1$, which are called the arithmetic conductor and root number of $\mathcal{A}$, respectively.

(A4) GL(1) **twists.** Let $\chi(\mathrm{mod}\,q)$ be a primitive Dirichlet character with $q>1$ and $(q,q_{\mathcal{A}})=1$. The twisted L-function

$$L(s,\mathcal{A}\otimes\chi)=\prod_{p}\prod_{j=1}^{d}\Big(1-\frac{\alpha_j(p)\chi(p)}{p^s}\Big)^{-1}=\sum_{n=1}^{\infty}\frac{a_n\chi(n)}{n^s}$$

can be analytically continued to be an entire function. Moreover, the complete L-function

$$\Lambda(s,\mathcal{A}\otimes\chi):=q_{\mathcal{A}\otimes\chi}^{\frac{s}{2}}\Delta\big(s+\kappa_{\mathrm{sgn}(\chi)}\big)L(s,\mathcal{A}\otimes\chi)$$

has finite order, and satisfies the functional equation

$$\Lambda(s,\mathcal{A}\otimes\chi)=\omega_{\mathcal{A}\otimes\chi}\overline{\Lambda(1-\bar{s},\mathcal{A}\otimes\chi)},\tag{2.2}$$

where $q_{\mathcal{A}\otimes\chi}>0$ and $\omega_{\mathcal{A}\otimes\chi}$ is a complex number with $|\omega_{\mathcal{A}\otimes\chi}|=1$. We emphasize that the Gamma factor of $\mathcal{A}\otimes\chi$ depends on the parity of χ, but not on the characters χ. For $(q,q_{\mathcal{A}})=1$, we also assume that $q_{\mathcal{A}\otimes\chi}=q_{\mathcal{A}}\,q^d$ and the root number $\omega_{\mathcal{A}\otimes\chi}$ is given by

$$\omega_{\mathcal{A}\otimes\chi}=\eta_{\mathcal{A},\mathrm{sgn}(\chi)}\chi(q_{\mathcal{A}})\Big(\frac{\tau(\chi)}{\sqrt{q}}\Big)^{d},$$

where $\eta_{\mathcal{A},\mathrm{sgn}(\chi)}$ with $|\eta_{\mathcal{A},\mathrm{sgn}(\chi)}|=1$ depends on $\mathcal{A}$ and the parity of χ only, $\tau(\chi)$ is the Gauss sum

$$\tau(\chi)=\sum_{b\,(\mathrm{mod}\,q)}\chi(b)\mathrm{e}\Big(\frac{b}{q}\Big).$$

Some hypotheses about the size of the coefficients have to be assumed in order to prove our result. The Ramanujan conjecture (RC for short) states that for any $\varepsilon>0$, $a_n\ll n^{\varepsilon}$ for all $n\ge1$. As is well known, RC has been proved only for a limited class of functions (the Hecke L-functions, and the L-functions coming from the cuspidal holomorphic forms for congruence groups, see Deligne [4]), although it is generally believed that all the L-functions appearing in number theory should satisfy RC. For example, it is conjectured to hold for the L-functions associated with cuspidal automorphic representations on $\mathrm{GL}(d)$. In general, only some rather weak estimates for the coefficients are at our disposal. Hence, it is interesting to consider the possibility of obtaining some results under some weaker assumptions instead

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of RC. We introduce the following notation: $s_{j,\mathcal{A}}(p)$ denotes the j-th elementary symmetric function of the roots $\alpha_1(p),\ldots,\alpha_d(p)$, that is

$$s_{j,\mathcal{A}}(p)=\sum_{1\le i_1<\cdots<i_j\le d}\alpha_{i_1}(p)\cdots\alpha_{i_j}(p). \tag{2.3}$$

Hypothesis H($\theta_{\mathbf{d}}$): For all primes p with $(p,q_{\mathcal{A}})=1$, one has

$$|\alpha_j(p)|\le p^{\theta_d}\quad\text{and}\quad s_{j,\mathcal{A}}(p)\ll p^{\min\{j,d-j\}\theta_d}\quad\text{for any } 1\le j\le d.$$

Hypothesis S: There exists some $b_{\mathcal{A}}>0$ such that the first moment of absolute values of the coefficients satisfies the bound

$$\sum_{n\le x}|a_n|\ll x(\log x)^{b_{\mathcal{A}}-1}.$$

Our main result states as follows.

Theorem 2.1. *Let $L(s,\mathcal{A})$ be an L-function satisfying the conditions* (A1)–(A4) *with $d\ge 2$, and let $(q,aq_{\mathcal{A}})=1$. Then under Hypothesis* $\mathrm{H}(\theta_{\mathrm{d}})$ *with $\theta_d<1-\frac{1}{d}$ and Hypothesis* S*, we have*

$$\begin{aligned}\sum_{\substack{n\le x\\ n\equiv a\,(\mathrm{mod}\,q)}}a_n=&M_0(x;q)+O\Big(\frac{\tau(q)}{q}y(\log x)^{m-1}\Big)+O\Big(\tau_d(q)q^{\frac{d-1}{2}}(\log q)^{b_{\mathcal{A}}}\Big)\\ &+O\bigg(\tau_d(q)\Big(\frac{qx}{y}\Big)^{\frac{d-1}{2}}(\log x)^{b_{\mathcal{A}}-1}\bigg)+O\bigg(\sum_{\substack{x<n\le x+O(y)\\ n\equiv a\,(\mathrm{mod}\,q)}}|a_n|\bigg),\end{aligned}$$

where y is an arbitrary real number with $0<y<x$, $M_0(x;q)$ is defined by

$$M_0(x;q)=\frac{1}{\varphi(q)}\operatorname*{Res}_{s=1}\Big(\frac{1}{s}L(s,\mathcal{A}\otimes\chi_0)x^s\Big).$$

In addition, if $a_n\ge 0$, we have

$$\sum_{\substack{n\le x\\ n\equiv a\,(\mathrm{mod}\,q)}}a_n=M_0(x;q)+O\Big(\tau_d(q)q^{\frac{d-1}{2}}(\log q)^{b_{\mathcal{A}}}\Big)+O\Big(\tau_d(q)x^{\frac{d-1}{d+1}}(\log x)^{\max\{b_{\mathcal{A}},m\}-1}\Big).$$

We note that the implied constants above depend on $\mathcal{A}$, including the degree d, the parameters μ_j and the arithmetic conductor $q_{\mathcal{A}}$ of $\mathcal{A}$.

Under Hypothesis $\mathrm{H}(\theta_{\mathrm{d}})$ *with $1-\frac{1}{d}\le\theta_d<1$ and Hypothesis* S*, the above two assertions hold provided that $\tau_d(q)$ is replaced by $\tau_{d+1}(q)$ in the error terms.*

Remark 2.1. Chandrasekharan and Narasimhan [3] established these results for $q=1$. Under some additional assumptions on functional equations for additive twists of L-functions, Smith [21] investigated the analogous problem as in Theorem 2.1 for some positive integers q. However, the lack of a good symmetry structure for these functional equations could increase the difficulty of applications, such as in [21,22]. We here take full advantage of multiplicative twists of L-function in this aspect.

In the modern sense, one may apply the Voronoï formula of a_n to study its distribution over arithmetic progressions. However, the corresponding formulae are intricate and constrained for most of our interest objects a_n, such as general divisor functions, coefficients of automorphic L-functions and their Rankin–Selberg convolutions.

The paper is organized as follows. In Section 3, we state a few background results we shall need, including a fact in multiplicative number theory, and some properties about general L-functions. In Sections 4, we prove Theorem 2.1. In order to apply this theorem to the automorphic context, we introduce some related knowledge on automorphic L-functions and their Rankin–Selberg in Section 5. Finally, in Section 6, we explore all various of applications and give the proofs of Theorems 1.1–1.3.

3. Preliminaries

In this section, we present the results and tools needed in our proofs.

The common tool in complex analysis is the method of contour integration, which could give a direct link between the summation associated to an arithmetic function and the corresponding Dirichlet series. The following lemma is a standard contour integration (see [14, Lemma 1], for example).

Lemma 3.1. *If k is any positive integer and $c > 0$, then*

$$\frac{1}{2\pi i}\int_{(c)} \frac{x^s}{s(s+1)\cdots(s+k)}\mathrm{d}s = \begin{cases} \frac{1}{k!}(1-\frac{1}{x})^k & \text{if } x \geq 1, \\ 0 & \text{if } 0 \leq x \leq 1. \end{cases}$$

Now we start to recall and show some uniform estimates for various analytic quantities related to an individual L-function. It turns out that most results for the L-function are expressed conveniently in terms of the analytic conductor. Put

$$q_\infty(s) = \prod_{j=1}^{d}(|s+\mu_j|+3).$$

Then the analytic conductor $q_{\mathcal{A}\otimes\chi}(s)$ is defined by (see, for example, [11, equation (5.6)])

$$q_{\mathcal{A}\otimes\chi}(s) = q_{\mathcal{A}\otimes\chi}q_\infty(s) = q_{\mathcal{A}\otimes\chi}\prod_{j=1}^{d}(|s+\mu_j|+3).$$

We first state the approximate functional equation, which expresses $L(s,\mathcal{A}\otimes\chi)$ in the critical strip as a sum of two Dirichlet series.

Lemma 3.2. *Let $\chi(\mathrm{mod}\, q)$ be a primitive Dirichlet character with $q > 1$ and $(q, q_{\mathcal{A}}) = 1$. For $0 \leq \mathrm{Re}\, s \leq 1$, there exists a smooth function V_s such that*

$$L(s,\mathcal{A}\otimes\chi) = \sum_{n=1}^{\infty}\frac{a_n\chi(n)}{n^s}V_s\Big(\frac{n}{X\sqrt{q_{\mathcal{A}\otimes\chi}}}\Big) + \omega_{\mathcal{A}\otimes\chi}(s)\sum_{n=1}^{\infty}\frac{\overline{a_n}\,\overline{\chi}(n)}{n^{1-s}}V_{1-s}\Big(\frac{nX}{\sqrt{q_{\mathcal{A}\otimes\chi}}}\Big),$$

where X is an arbitrary positive real number, and

$$\omega_{\mathcal{A}\otimes\chi}(s) = \omega_{\mathcal{A}\otimes\chi}q_{\mathcal{A}\otimes\chi}^{\frac{1}{2}-s}\frac{\Delta(1-s+\kappa_{\mathrm{sgn}(\chi)})}{\Delta(s+\kappa_{\mathrm{sgn}(\chi)})}.$$

The function V_s and its partial derivatives $V_s^{(k)}$ $(k = 1, 2, \ldots)$ satisfy, for any $C > 0$, the following uniform growth estimates at 0 and ∞:

$$V_s(x) = \begin{cases} 1 + O\Big(\big(\frac{x}{q_\infty(s)}\big)^C\Big) \\ O\Big(\big(1+\frac{x}{q_\infty(s)}\big)^{-C}\Big), \end{cases} \qquad V_s^{(k)}(x) = O\Big(\big(1+\frac{x}{q_\infty(s)}\big)^{-C}\Big),$$

where the implied constants depend only on C, k and d.

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Proof. This follows from [11, Theorem 5.3, Proposition 5.4] in the same manner. □

Lemma 3.3. *Let χ is any Dirichlet character $(\bmod\, q)$ with $(q, q_{\mathcal{A}}) = 1$, and let $s = \sigma + it$. Then we have, for $-\varepsilon \le \sigma \le 1 + \varepsilon$ and $|t| \ge 1$,*

$$L(s, \mathcal{A} \otimes \chi) \ll_{\mathcal{A}} (q|t|)^{d(1-\sigma)+\varepsilon}.$$

Proof. Assume $\chi(\bmod\, q)$ is induced by a primitive character $\chi_1(\bmod\, r)$, then

$$L(s, \mathcal{A} \otimes \chi) = L(s, \mathcal{A} \otimes \chi_1) \prod_{p|\frac{q}{r}} \prod_{j=1}^{d} \Big(1 - \frac{\alpha_j(p)\chi_1(p)}{p^s}\Big).$$

Recall that $|\alpha_j(p)| \le p$ in Condition (A1). Thus, we have, for $-\varepsilon \le \sigma \le 1 + \varepsilon$ and $|t| \ge 1$,

$$\prod_{p|\frac{q}{r}} \prod_{j=1}^{d} \Big(1 - \frac{\alpha_j(p)\chi_1(p)}{p^s}\Big) \ll \prod_{p|\frac{q}{r}} \Big(1 + p^{1-\sigma}\Big)^d \le \big(\frac{q}{r}\big)^{d(1-\sigma)+\varepsilon}.$$

Moreover, the convexity bound of $L(s, \mathcal{A} \otimes \chi_1)$ states

$$L(s, \mathcal{A} \otimes \chi_1) \ll q_{\mathcal{A}\otimes\chi_1}(s)^{\frac{1-\sigma}{2}+\varepsilon} \ll_{\mathcal{A}} (r|t|)^{\frac{d(1-\sigma)}{2}+\varepsilon}$$

for $-\varepsilon \le \sigma \le 1 + \varepsilon$ and $|t| \ge 1$ (see [11, equation (5.20)]). Finally, combining these results above, we conclude Lemma 3.3. □

4. Proof of Theorem 2.1

For technical convenience, one usually works with the weighted sum

$$A_\varrho(x; q, a) = \frac{1}{\Gamma(\varrho + 1)} \sideset{}{'}\sum_{\substack{n \le x \\ n \equiv a \,(\bmod\, q)}} a_n (x - n)^\varrho, \tag{4.1}$$

where $(q, a q_{\mathcal{A}}) = 1$, ϱ is a sufficiently large integer, and the symbol $'$ indicates that the last term has to be multiplied by $1/2$ if $\varrho = 0$ and $x = n$. Detecting the congruence condition in (4.1) by the multiplicative characters $\chi(\bmod\, q)$, we obtain the identity

$$\sideset{}{'}\sum_{\substack{n \le x \\ n \equiv a \,(\bmod\, q)}} a_n (x - n)^\varrho = \frac{1}{\varphi(q)} \sum_{\chi (\bmod\, q)} \overline{\chi}(a) \sideset{}{'}\sum_{n \le x} a_n \chi(n) (x - n)^\varrho.$$

Each character $\chi(\bmod\, q)$ can be induced by a primitive character $\chi(\bmod\, r)$ with $r|q$. Note that the character for $\chi(\bmod\, q)$ with the case $r = 1$ is principle. Thus, we get

$$\begin{aligned}
\Gamma(\varrho + 1) A_\varrho(x; q, a) =& \frac{1}{\varphi(q)} \sum_{r|q} \sideset{}{^*}\sum_{\chi (\bmod\, r)} \overline{\chi}(a) \sideset{}{'}\sum_{\substack{n \le x \\ (n, q/r) = 1}} a_n \chi(n) (x - n)^\varrho \\
=& \frac{1}{\varphi(q)} \sum_{r|q} \sideset{}{^*}\sum_{\chi (\bmod\, r)} \overline{\chi}(a) \sum_{h|(q/r)} \mu(h) \chi(h) h^\varrho \sideset{}{'}\sum_{n \le x/h} a_{hn} \chi(n) \Big(\frac{x}{h} - n\Big)^\varrho \qquad (4.2) \\
=& \frac{1}{\varphi(q)} \sum_{hr|q} \mu(h) h^\varrho \sideset{}{^*}\sum_{\chi (\bmod\, r)} \overline{\chi}(a) \chi(h) \sideset{}{'}\sum_{n \le x/h} a_{hn} \chi(n) \Big(\frac{x}{h} - n\Big)^\varrho,
\end{aligned}$$

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where the formula

$$\sum_{d|n} \mu(d) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{otherwise} \end{cases}$$

is used to relax the coprimality condition $(n, q/r) = 1$ above.

The transformation of the innermost sum over n requires factoring the arithmetic function a_{hn}. To this end, we exploit the Euler product for $L(s, \mathcal{A})$. Write

$$L(s, \mathcal{A}) = \prod_p \prod_{j=1}^{d} (1 - \alpha_j(p)/p^s)^{-1} := \prod_p L(s, \mathcal{A}_p).$$

With the notation $s_{j,\mathcal{A}}(p)$ as in (2.3), the reciprocal of the local L-function can be given by

$$L(s, \mathcal{A}_p)^{-1} = 1 - s_{1,\mathcal{A}}(p)p^{-s} + s_{2,\mathcal{A}}(p)p^{-2s} + \cdots + (-1)^d s_{d,\mathcal{A}}(p)p^{-ds}.$$

Thus, we have

$$\Big(1 - s_{1,\mathcal{A}}(p)p^{-s} + s_{2,\mathcal{A}}(p)p^{-2s} + \cdots + (-1)^d s_{d,\mathcal{A}}(p)p^{-ds}\Big)\Big(\sum_{\nu=0}^{\infty} a_{p^\nu} p^{-\nu s}\Big) = 1.$$

Hence for all $\nu \in \mathbb{Z}$, we obtain the recursive relation

$$a_{p^\nu} - s_{1,\mathcal{A}}(p)a_{p^{\nu-1}} + s_{2,\mathcal{A}}(p)a_{p^{\nu-2}} + \cdots + (-1)^d s_{d,\mathcal{A}}(p)a_{p^{\nu-d}} = \delta_{0\nu}$$

subject to the convention that $a_{p^\nu} = 0$ for negative ν. Notice that h is square-free. Now if we suppose $h = \prod p$, we get by the recursion and multiplicativity

$$\begin{aligned}
\sum_{n=1}^{\infty} a_{hn} n^{-s} =& \prod_{p|h} \Big(\sum_{\nu=0}^{\infty} a_{p^{\nu+1}} p^{-\nu s}\Big) \prod_{p \nmid h} L(s, \mathcal{A}_p) \\
=& L(s, \mathcal{A}) \prod_{p|h} \Big(L(s, \mathcal{A}_p)^{-1} \sum_{\nu=0}^{\infty} \frac{a_{p^{\nu+1}}}{p^{\nu s}}\Big) \\
=& L(s, \mathcal{A}) \prod_{p|h} \Big(s_{1,\mathcal{A}}(p) - s_{2,\mathcal{A}}(p)p^{-s} + \cdots + (-1)^{d-1} s_{d,\mathcal{A}}(p)p^{-(d-1)s}\Big).
\end{aligned}$$

Hence it is clear that a_{hn} factors as follows

$$a_{hn} = \sum_{cm=n} a(h,c)a_m, \tag{4.3}$$

where $a(h,c)$ is defined for $c|h^{d-1}$ by

$$a(h,c) = \sum_{h=h_0h_1\cdots h_{d-1}} \prod_{j=0}^{d-1} \prod_{p|h_j} (-1)^j s_{j+1,\mathcal{A}}(p)$$

with $h_0, h_1, \ldots, h_{d-1}$ mutually coprime such that

$$c = \Big(\prod_{p|h_1} p\Big)\Big(\prod_{p|h_2} p\Big)^2 \cdots \Big(\prod_{p|h_{d-1}} p\Big)^{d-1}.$$

Using the above formulas and Hypothesis $\mathrm{H}(\theta_{\mathrm{d}})$, one can show that

$$a(h,c) \ll h^{\frac{d\theta_d}{2}+\varepsilon}. \tag{4.4}$$

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Inserting the identity (4.3) into the innermost sum over n in the last line of (4.2), we get

$$\frac{1}{\Gamma(\varrho+1)}\sideset{}{'}\sum_{n\le x/h} a_{hn}\chi(n)\Big(\frac{x}{h}-n\Big)^{\varrho} = \sum_{c|h^{d-1}} a(h,c)c^{\varrho}\chi(c)B_{\varrho}\Big(\frac{x}{ch},\chi\Big),$$

where

$$B_{\varrho}(y,\chi) = \frac{1}{\Gamma(\varrho+1)}\sideset{}{'}\sum_{m\le y} a_m\chi(m)(y-m)^{\varrho}.$$

Next, we turn to evaluate the summation $B_{\varrho}(y,\chi)$. By Condition (A1), it is known that $L(s,\mathcal{A}\otimes\chi)$ converges absolutely for $\operatorname{Re} s \ge 1+\varepsilon$. Then it follows from Lemma 3.1 that

$$B_{\varrho}(y,\chi) = \frac{1}{2\pi i}\int_{(1+\varepsilon)} \frac{\Gamma(s)}{\Gamma(\varrho+1+s)} L(s,\mathcal{A}\otimes\chi)y^{\varrho+s}\mathrm{d}s,$$

where ϱ is a sufficiently large integer compared to d. Using the analytic properties (A2), (A5) of $L(s,\mathcal{A}\otimes\chi)$ and the bound in Lemma 3.3, we could move the line of integration to $\operatorname{Re} s = -\varepsilon < 0$, change the variable from s to $1-s$ and apply the functional equation (2.2) to get

$$B_{\varrho}(y,\chi) = \delta_{r1}\operatorname*{Res}_{s=1}\Big(\frac{\Gamma(s)}{\Gamma(\varrho+1+s)}L(s,\mathcal{A})y^{\varrho+s}\Big) + \frac{1}{\Gamma(\varrho+1)}L(0,\mathcal{A}\otimes\chi)y^{\varrho} + E_{\varrho}(y,\chi), \quad (4.5)$$

where δ_{r1} denotes the diagonal symbol of Kronecker and

$$E_{\varrho}(y,\chi) = \frac{\omega_{\mathcal{A}\otimes\chi}}{2\pi i}\int_{(1+\varepsilon)} \frac{\Gamma(1-s)\Delta\big(s+\kappa_{\operatorname{sgn}(\chi)}\big)}{\Gamma(\varrho+2-s)\Delta\big(1-s+\kappa_{\operatorname{sgn}(\chi)}\big)} y^{\varrho+1-s} q_{\mathcal{A}\otimes\chi}^{s-\frac{1}{2}} L(s,\overline{\mathcal{A}}\otimes\overline{\chi})\mathrm{d}s.$$

Denote the contributions of these three terms on the right hand side of (4.5) to the sum $A_{\varrho}(x;q,a)$ by $M_{\varrho}(x;q)$, $H_{\varrho}(x;q)$ and $S_{\varrho}(x;q)$, respectively. This is to say

$$A_{\varrho}(x;q,a) = M_{\varrho}(x;q) + H_{\varrho}(x;q) + S_{\varrho}(x;q), \quad (4.6)$$

where

$$\begin{aligned} M_{\varrho}(x;q) =& \frac{1}{\varphi(q)}\sum_{h|q}\mu(h)\sum_{c|h^{d-1}} a(h,c)\operatorname*{Res}_{s=1}\Big(\frac{\Gamma(s)}{\Gamma(\varrho+1+s)}L(s,\mathcal{A})\Big(\frac{x}{ch}\Big)^{\varrho+s}\Big),\\ H_{\varrho}(x;q) =& \frac{1}{\Gamma(\varrho+1)\varphi(q)}\sum_{hr|q}\mu(h)\sum_{c|h^{d-1}} a(h,c)\sideset{}{^*}\sum_{\chi(\operatorname{mod} r)}\chi(\overline{a}ch)L(0,\mathcal{A}\otimes\chi)x^{\varrho}, \quad (4.7)\\ S_{\varrho}(x;q) =& \frac{1}{\varphi(q)}\sum_{hr|q}\mu(h)\sum_{c|h^{d-1}} a(h,c)(ch)^{\varrho}\sideset{}{^*}\sum_{\chi(\operatorname{mod} r)}\chi(\overline{a}ch)E_{\varrho}\Big(\frac{x}{ch},\chi\Big). \end{aligned}$$

We introduce the difference operator

$$\Delta_y^{\varrho}F(x) = \sum_{v=0}^{\varrho}(-1)^{\varrho-v}C_{\varrho}^{v}F(x+vy),$$

where y is a positive parameter less than x and C_{ϱ}^{v} denotes the binomial coefficient. If F has ϱ derivatives, then one has

$$\Delta_y^{\varrho}F(x) = \int_x^{x+y}\mathrm{d}t_1\int_{t_1}^{t_1+y}\mathrm{d}t_2\cdots\int_{t_{\varrho-1}}^{t_{\varrho-1}+y}F^{(\varrho)}(t_{\varrho})\mathrm{d}t_{\varrho}, \quad (4.8)$$

where $F^{(\varrho)}$ is the ϱ-th derivative of F.

We first apply the operator Δ_y^{ϱ} to $A_{\varrho}(x;q,a)$ and obtain

$$\Delta_y^{\varrho} A_{\varrho}(x;q,a) = \sideset{}{'}\sum_{\substack{n\le x\\ n\equiv a\,(\mathrm{mod}\,q)}} a_n \frac{\Delta_y^{\varrho}(x-n)^{\varrho}}{\Gamma(\varrho+1)} + \sum_{v=0}^{\varrho}(-1)^{\varrho-v}C_{\varrho}^{v} \sideset{}{'}\sum_{\substack{x<n\le x+vy\\ n\equiv a\,(\mathrm{mod}\,q)}} a_n(x+vy-n)^{\varrho}.$$

Since

$$\frac{1}{\Gamma(\varrho+1)}\Delta_y^{\varrho}(x-n)^{\varrho} = y^{\varrho},$$

we get

$$\Delta_y^{\varrho} A_{\varrho}(x;q,a) = y^{\varrho}A_0(x;q,a) + O_{\varrho}\Bigg(y^{\varrho}\sum_{\substack{x<n\le x+\varrho y\\ n\equiv a\,(\mathrm{mod}\,q)}}|a_n|\Bigg). \tag{4.9}$$

Furthermore, if $a_n \ge 0$, then $A_0(x;q,a)$ is monotone. Thus, it follows from (4.8) that

$$\Delta_y^{\varrho} A_{\varrho}(x-\varrho y;q,a) \le y^{\varrho}A_0(x;q,a) \le \Delta_y^{\varrho} A_{\varrho}(x;q,a). \tag{4.10}$$

Next, we shall apply the operator Δ_y^{ϱ} to $M_{\varrho}(x;q)$, $H_{\varrho}(x;q)$ and $S_{\varrho}(x;q)$, separately. From now on, we assume that the implied constant in the notation $\ll$ or O is allowed to depend on $\mathcal{A}, \varrho$ for convenience.

4.1. **Computation of $\Delta_y^{\varrho}S_{\varrho}(x;q)$.** By the Dirichlet series expression of $L(s,\overline{\mathcal{A}}\otimes\overline{\chi}))$, we can rewrite $E_{\varrho}(y,\chi)$ as

$$E_{\varrho}(y,\chi) = \omega_{\mathcal{A}\otimes\chi}q_{\mathcal{A}\otimes\chi}^{\varrho+\frac12}\sum_{n=1}^{\infty}\frac{\overline{a_n}\,\overline{\chi}(n)}{n^{1+\varrho}}J\left(\frac{ny}{q_{\mathcal{A}\otimes\chi}}\right), \tag{4.11}$$

where

$$J(x) = \frac{1}{2\pi i}\int_{(c)}\frac{\Gamma(1-s)\Delta\big(s+\kappa_{\mathrm{sgn}(\chi)}\big)}{\Gamma(\varrho+2-s)\Delta\big(1-s+\kappa_{\mathrm{sgn}(\chi)}\big)}x^{\varrho+1-s}\mathrm{d}s.$$

We shall deal with the integral $J(x)$ by means of the following result (see [3, equations (4.5) and (4.11)] or [14, Theorem 3]).

Lemma 4.1. *With the notation as before, suppose $d \ge 2$. Let $0 \le \varrho \in \mathbb{Z}$ and $c \in \mathbb{R}$. Then for suitable choices c and ϱ, we have*

$$J(x) = O\Big(x^{\frac12+\left(1-\frac1d\right)\varrho-\frac{1}{2d}}\Big) \quad \textit{and} \quad J^{(\varrho)}(x) = O\Big(x^{\frac12-\frac{1}{2d}}\Big).$$

Combining (4.11) with the expression of $S_{\varrho}(x;q)$ in (4.7), we conclude

$$S_{\varrho}(x;q) = \frac{1}{\varphi(q)}\sum_{hr|q}\mu(h)\sum_{c|h^{d-1}}a(h,c)(ch)^{\varrho}\sum_{n=1}^{\infty}\frac{\overline{a_n}}{n^{1+\varrho}}\sideset{}{^*}\sum_{\chi(\mathrm{mod}\,r)}\chi(\overline{a}nch)\omega_{\mathcal{A}\otimes\chi}q_{\mathcal{A}\otimes\chi}^{\varrho+\frac12}J\left(\frac{nx}{chq_{\mathcal{A}\otimes\chi}}\right).$$

Recall that

$$q_{\mathcal{A}\otimes\chi} = q_{\mathcal{A}}\,r^d \quad \text{and} \quad \omega_{\mathcal{A}\otimes\chi} = \eta_{\mathcal{A},\mathrm{sgn}(\chi)}\chi(q_{\mathcal{A}})\Big(\frac{\tau(\chi)}{\sqrt{r}}\Big)^d$$

for $(r,q_{\mathcal{A}})=1$. Since the $\eta_{\mathcal{A},\mathrm{sgn}(\chi)}$ and $J(x)$ depend on the parity of χ, but not on the character itself, we need to break up the sum over χ separately into even and odd characters. We put

$$K_{\pm}(a,r) = \frac12\sideset{}{^*}\sum_{\chi(\mathrm{mod}\,r)}(1\pm\chi(-1))\overline{\chi}(a)\Big(\frac{\tau(\chi)}{\sqrt{r}}\Big)^d.$$

Moreover, we display the dependence by writing J_+ and J_- respectively in place of J. Thus, we have

$$S_\varrho(x;q) = \frac{1}{\varphi(q)} \sum_{hr|q} \mu(h) \sum_{c|h^{d-1}} a(h,c)\,(chq_{\mathcal{A}}r^d)^\varrho\,(q_{\mathcal{A}}r^d)^{\frac{1}{2}}$$
$$\times \sum_{\pm} \eta_{\mathcal{A},\mathrm{sgn}(\chi)} \sum_{n=1}^{\infty} \frac{\overline{a_n}}{n^{1+\varrho}} K_\pm(an\overline{chq_{\mathcal{A}}}, r) J_\pm\left(\frac{nx}{chq_{\mathcal{A}}r^d}\right).$$

Lemma 4.2. *Let $K_\pm(a,r)$ be as above with $(a,r)=1$. Then we have*

$$|K_\pm(a,r)| \le \varphi(r) r^{-\frac{1}{2}} \tau_d(r).$$

Proof. It is clear that

$$K_\pm(a,r) = \frac{1}{2}\left(K(a,r) \pm K(-a,r)\right), \tag{4.12}$$

where

$$K(a,r) = \sideset{}{^*}\sum_{\chi (\operatorname{mod} r)} \overline{\chi}(a) \Big(\frac{\tau(\chi)}{\sqrt{r}}\Big)^d.$$

In fact, $K(a,r)$ appears in a long list of literature, such as the series works of Duke and Iwaniec about estimating coefficients of L-functions (see [5–9]), the work of Luo, Rudnick and Sarnak on the Selberg conjecture [16] and the work of Luo about nonvanishing of GL(d) L-functions [15]. It plays a key role in making these remarkable achievements.

As in the proof of [9], by the definition of Gauss sum, we infer that

$$r^{\frac{d}{2}} K(a,r) = \sideset{}{^*}\sum_{\chi (\operatorname{mod} r)} \overline{\chi}(a) \left(\sum_{b (\operatorname{mod} r)} \chi(b) \mathrm{e}\left(\frac{b}{r}\right)\right)^d.$$

Changing the order of summation and using the relation [11, equation (3.8)]

$$\sideset{}{^*}\sum_{\chi \bmod r} \chi(m) = \sum_{l|(m-1,r)} \varphi(l)\mu\left(\frac{r}{l}\right)$$

when $(r,m)=1$, we get

$$r^{\frac{d}{2}} K(a,r) = \sum_{lk=r} \varphi(l)\mu(k) \sideset{}{^*}\sum_{\substack{b_1,\dots,b_d (\operatorname{mod} r)\\ b_1\cdots b_d \equiv a (\operatorname{mod} l)}} \mathrm{e}\left(\frac{b_1+\cdots+b_d}{r}\right)$$
$$= \sum_{\substack{lk=r\\ (l,k)=1}} \varphi(l)\mu(k)^{d+1} \sideset{}{^*}\sum_{\substack{b_1,\dots,b_d (\operatorname{mod} l)\\ b_1\cdots b_d \equiv a (\operatorname{mod} l)}} \mathrm{e}\left(\frac{(b_1+\cdots+b_d)\overline{k}}{r}\right).$$

Note that the innermost sum is the generalized Kloosterman sum for which Deligne [4] has established the bound $\tau_d(l) l^{\frac{d-1}{2}}$. Employing Deligne's bound, we directly have

$$|r^{\frac{d}{2}} K(a,r)| \le \sum_{lk=r} \varphi(l)\tau_d(l) l^{\frac{d-1}{2}} \le \varphi(r) r^{\frac{d-1}{2}} \tau_d(r) \sum_{k|r} \frac{1}{\varphi(k) k^{\frac{d-1}{2}}} \ll \varphi(r) r^{\frac{d-1}{2}} \tau_d(r),$$

which implies this lemma from (4.12). $\square$

We continue to compute $\Delta_y^\varrho S_\varrho(x;q)$. Now we apply the operator Δ_y^ϱ to $S_\varrho(x;q)$ and obtain from Lemma 4.2 that

$$\begin{aligned}\Delta_y^\varrho S_\varrho(x;q) \ll& \frac{1}{\varphi(q)}\sum_{hr|q}|\mu(h)|\sum_{c|h^{d-1}}|a(h,c)|(chr^d)^\varrho r^{\frac{d}{2}}\\ &\times\sum_{\pm}\sum_{n=1}^{\infty}\frac{|a_n|}{n^{1+\varrho}}\left|K_\pm(an\overline{chq_{\mathcal{A}}},r)\right|\left|\Delta_y^\varrho J_\pm\left(\frac{nx}{chq_{\mathcal{A}}r^d}\right)\right|\\ \ll&\frac{1}{\varphi(q)}\sum_{hr|q}|\mu(h)|\sum_{c|h^{d-1}}|a(h,c)|(chr^d)^\varrho\varphi(r)r^{\frac{d-1}{2}}\tau_d(r)\\ &\times\sum_{\pm}\sum_{n=1}^{\infty}\frac{|a_n|}{n^{1+\varrho}}\left|\Delta_y^\varrho J_\pm\left(\frac{nx}{chq_{\mathcal{A}}r^d}\right)\right|.\end{aligned}\tag{4.13}$$

By definition of the operator Δ_y^ϱ and Lemma 4.1, one easily has

$$\Delta_y^\varrho J_\pm(x)=\begin{cases}O(|J_\pm(x)|)=O\Big(x^{\frac{1}{2}+\left(1-\frac{1}{d}\right)\varrho-\frac{1}{2d}}\Big),\\ O(y^\varrho|J_\pm^{(\varrho)}(x)|)=O\Big(y^\varrho x^{\frac{1}{2}-\frac{1}{2d}}\Big).\end{cases}$$

Thus, we have

$$\Delta_y^\varrho J_\pm\left(\frac{nx}{chq_{\mathcal{A}}r^d}\right)\ll_{\mathcal{A}}\min\left\{\left(\frac{nx}{chr^d}\right)^{\frac{1}{2}+\left(1-\frac{1}{d}\right)\varrho-\frac{1}{2d}},\ \left(\frac{ny}{chr^d}\right)^\varrho\left(\frac{nx}{chr^d}\right)^{\frac{1}{2}-\frac{1}{2d}}\right\}.$$

We divide the innermost summation in (4.13) into two parts by the parameter $z>0$, which shall be chosen later. For any $\varepsilon>0$, under Hypothesis S, we get

$$\begin{aligned}\sum_{n>z}\frac{|a_n|}{n^{1+\varrho}}\left|\Delta_y^\varrho J_\pm\left(\frac{nx}{chq_{\mathcal{A}}r^d}\right)\right|&\ll\sum_{n>z}\frac{|a_n|}{n^{1+\varrho}}\left(\frac{nx}{chr^d}\right)^{\frac{1}{2}+\left(1-\frac{1}{d}\right)\varrho-\frac{1}{2d}}\\ &\ll\left(\frac{x}{chr^d}\right)^{\frac{1}{2}+\left(1-\frac{1}{d}\right)\varrho-\frac{1}{2d}}z^{\frac{1}{2}-\frac{\varrho}{d}-\frac{1}{2d}}(\log z)^{b_{\mathcal{A}}-1},\end{aligned}$$

and

$$\begin{aligned}\sum_{n\le z}\frac{|a_n|}{n^{1+\varrho}}\left|\Delta_y^\varrho J_\pm\left(\frac{nx}{chq_{\mathcal{A}}r^d}\right)\right|&\ll\sum_{n\le z}\frac{|a_n|}{n^{1+\varrho}}\left(\frac{ny}{chr^d}\right)^\varrho\left(\frac{nx}{chr^d}\right)^{\frac{1}{2}-\frac{1}{2d}}\\ &\ll\left(\frac{y}{chr^d}\right)^\varrho\left(\frac{xz}{chr^d}\right)^{\frac{1}{2}-\frac{1}{2d}}(\log z)^{b_{\mathcal{A}}-1}.\end{aligned}$$

On taking $z=\frac{chr^dx^{d-1}}{y^d}$, we have

$$\sum_{n=1}^{\infty}\frac{|a_n|}{n^{1+\varrho}}\left|\Delta_y^\varrho J_\pm\left(\frac{nx}{chq_{\mathcal{A}}r^d}\right)\right|\ll\left(\frac{y}{chr^d}\right)^\varrho\left(\frac{x}{y}\right)^{\frac{d-1}{2}}(\log x)^{b_{\mathcal{A}}-1}.$$

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Inserting this into (4.13) and applying the estimate (4.4) yield

$$\begin{aligned}\Delta_y^\varrho S_\varrho(x;q) &\ll \left(\frac{x}{y}\right)^{\frac{d-1}{2}} \frac{y^\varrho(\log x)^{b_{\mathcal{A}}-1}}{\varphi(q)} \sum_{hr|q} |\mu(h)| \sum_{c|h^{d-1}} |a(h,c)| \varphi(r) r^{\frac{d-1}{2}} \tau_d(r) \\ &\ll \left(\frac{x}{y}\right)^{\frac{d-1}{2}} \frac{y^\varrho(\log x)^{b_{\mathcal{A}}-1}}{\varphi(q)} \sum_{hr=q} h^{\frac{d\theta_d}{2}+\varepsilon} \varphi(r) r^{\frac{d-1}{2}} \tau_d(r).\end{aligned} \tag{4.14}$$

It is easy to deduce

$$\Delta_y^\varrho S_\varrho(x;q) \ll \left(\frac{qx}{y}\right)^{\frac{d-1}{2}} y^\varrho (\log x)^{b_{\mathcal{A}}-1} \tau_d(q)$$

when $\theta_d < 1 - \frac{1}{d}$, and

$$\Delta_y^\varrho S_\varrho(x;q) \ll \left(\frac{qx}{y}\right)^{\frac{d-1}{2}} y^\varrho (\log x)^{b_{\mathcal{A}}-1} \tau_{d+1}(q)$$

when $1 - \frac{1}{d} \leq \theta_d < 1$.

4.2. **Computation of $\Delta_y^\varrho H_\varrho(x;q)$.**

Lemma 4.3. *Let $(r, aq_{\mathcal{A}}) = 1$. Then we have*

$$\sum_{\chi (\mathrm{mod}\, r)}^{*} \chi(a) L(0, \mathcal{A} \otimes \chi) \ll \varphi(r) r^{\frac{d-1}{2}} \tau_d(r) (\log r)^{b_{\mathcal{A}}}.$$

Proof. By the approximate functional equation in Lemma 3.2 with $X = r^{-d/3}$, we have

$$L(0, \mathcal{A}\otimes\chi) = \sum_{n \leq r^{d/6+\varepsilon}} a_n \chi(n) V_0\Big(\frac{n}{q_{\mathcal{A}}^{1/2} r^{d/6}}\Big) + \omega_{\mathcal{A}\otimes\chi}(0) \sum_{n \leq r^{5d/6+\varepsilon}} \frac{\overline{a_n}\, \overline{\chi}(n)}{n} V_1\Big(\frac{n}{q_{\mathcal{A}}^{1/2} r^{5d/6}}\Big) + O(r^{-2020}).$$

We average the approximate functional equation over all primitive characters $(\mathrm{mod}\, r)$. Thus, the sum

$$\sum_{\chi (\mathrm{mod}\, r)}^{*} \chi(a) L(0, \mathcal{A} \otimes \chi)$$

is decomposed into two parts T_1 and T_2 with negligible error $O(r^{-2019})$. Since $L(s, \mathcal{A})$ is absolutely convergent for $\mathrm{Re}\, s > 1$, we get

$$T_1 = r \sum_{n \leq r^{d/6+\varepsilon}} |a_n| \ll r^{\frac{d}{6}+1+\varepsilon}. \tag{4.15}$$

To treat the contribution of T_2, we first note that $\omega_{\mathcal{A}\otimes\chi}(s)$ and V_s depend on the parity of χ, but not on the characters χ. Similar to the previous argument for $S_\varrho(x;q)$, we break up the sum T_2 over χ separately into even and odd characters, and then get

$$\begin{aligned} T_2 &= \sum_{\substack{n \leq r^{5d/6+\varepsilon} \\ (n,r)=1}} \frac{\overline{a_n}}{n} \sum_{\chi (\mathrm{mod}\, r)}^{*} \chi(a\overline{n}) \omega_{\mathcal{A}\otimes\chi}(0) V_1\Big(\frac{n}{q_{\mathcal{A}}^{1/2} r^{5d/6}}\Big) \\ &\ll r^{\frac{d}{2}} \sum_{\pm} \sum_{\substack{n \leq r^{5d/6+\varepsilon} \\ (n,r)=1}} \frac{|a_n|}{n} |K_{\pm}(n \overline{a q_{\mathcal{A}}}, r)|. \end{aligned}$$

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Using Hypothesis S and Lemma 4.2, we therefore have

$$T_2 \ll \varphi(r) r^{\frac{d-1}{2}} \tau_d(r) (\log r)^{b_{\mathcal{A}}}. \tag{4.16}$$

Collecting (4.15) and (4.16), Lemma 4.3 immediately follows. □

If the operator Δ_y^ϱ acts on $H_\varrho(x;q)$, then we obtain from Lemma 4.3 that

$$\begin{aligned}\Delta_y^\varrho H_\varrho(x;q) =& \frac{1}{\Gamma(\varrho+1)\varphi(q)} \sum_{hr|q} \mu(h) \sum_{c|h^{d-1}} a(h,c) y^\varrho \sideset{}{^*}\sum_{\chi (\bmod r)} \chi(\overline{a}ch) L(0, \mathcal{A}\otimes\chi) \\ \ll& \frac{y^\varrho}{\varphi(q)} \sum_{hr|q} |\mu(h)| \sum_{c|h^{d-1}} |a(h,c)| \varphi(r) r^{\frac{d-1}{2}} \tau_d(r) (\log r)^{b_{\mathcal{A}}}.\end{aligned}$$

Similar to the previous estimate for (4.14), we get

$$\Delta_y^\varrho H_\varrho(x;q) \ll y^\varrho q^{\frac{d-1}{2}} \tau_d(q) (\log q)^{b_{\mathcal{A}}}$$

when $\theta_d < 1 - \frac{1}{d}$, and

$$\Delta_y^\varrho H_\varrho(x;q) \ll y^\varrho q^{\frac{d-1}{2}} \tau_{d+1}(q) (\log q)^{b_{\mathcal{A}}}$$

when $1 - \frac{1}{d} \le \theta_d < 1$.

4.3. **Computation of $\Delta_y^\varrho M_\varrho(x;q)$.** By the relation (4.3), we have

$$M_\varrho(x;q) = \frac{1}{\varphi(q)} \operatorname*{Res}_{s=1} \Big(\frac{\Gamma(s)}{\Gamma(\varrho+1+s)} L(s, \mathcal{A}\otimes\chi_0) x^{\varrho+s} \Big),$$

where χ_0 is the principle character $(\bmod\, q)$. Let $\mathcal{C}_\varepsilon$ be a cycle with a center at $s=1$ and a radius of ε. Then $M_\varrho(x;q)$ can also be written as

$$M_\varrho(x;q) = \frac{1}{\varphi(q)} \frac{1}{2\pi i} \int_{\mathcal{C}_\varepsilon} \frac{\Gamma(s)}{\Gamma(\varrho+1+s)} L(s, \mathcal{A}\otimes\chi_0) x^{\varrho+s} \mathrm{d}s.$$

In dealing with $\Delta_y^\varrho M_\varrho(x;q)$, the identity (4.8) immediately implies

$$\Delta_y^\varrho M_\varrho(x;q) = \int_x^{x+y} \mathrm{d}t_1 \int_{t_1}^{t_1+y} \mathrm{d}t_2 \cdots \int_{t_{\varrho-1}}^{t_{\varrho-1}+y} M_0(t_\varrho;q) \mathrm{d}t_\varrho.$$

By introducing the change of variables $t_j \mapsto y\, v_j + t_{j-1}$ for $1 \le j \le \varrho$ with $t_0 = x$, we have

$$\Delta_y^\varrho M_\varrho(x;q) = y^\varrho \int_0^1 \cdots \int_0^1 M_0(x + y(v_1 + \cdots + v_\varrho); q) \mathrm{d}v_1 \cdots \mathrm{d}v_\varrho.$$

Then the first mean value theorem for integrals implies that

$$\Delta_y^\varrho M_\varrho(x;q) = y^\varrho M_0(x + \xi y; q)$$

for some $0 < \xi < \varrho$. From the differential form of the mean value theorem, we have

$$\Delta_y^\varrho M_\varrho(x;q) = y^\varrho M_0(x;q) + \xi y^{\varrho+1} M_0'(x + \xi_1 y; q)$$

for some $0 < \xi_1 < \xi$, where $M_0'(x;q)$ is the derivative of $M_0(x;q)$ given by

$$M_0'(x;q) = \frac{1}{\varphi(q)} \frac{1}{2\pi i} \int_{\mathcal{C}_\varepsilon} L(s, \mathcal{A}\otimes\chi_0) x^{s-1} \mathrm{d}s.$$

We can rewrite $L(s, \mathcal{A}\otimes\chi_0)$ as

$$L(s, \mathcal{A}\otimes\chi_0) = G_q(s, \mathcal{A}) L(s, \mathcal{A}),$$

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where

$$G_q(s,\mathcal{A})=\prod_{p|q}\prod_{j=1}^{d}\Big(1-\frac{\alpha_j(p)}{p^s}\Big).$$

For any $j\geq 0$, we obtain from general Leibniz rule that

$$\frac{q}{\varphi(q)}G_q^{(j)}(1,\mathcal{A})\ll(\log q)^j\prod_{p|q}\Big(1+\frac{1}{p^{1-\theta_d}}\Big)^d\Big(1-\frac{1}{p}\Big)^{-1}\ll\tau(q)(\log q)^j$$

if $\theta_d<1$. The residue theorem then yields

$$\begin{aligned}M_0'(x;q)=&\frac{1}{\varphi(q)}\operatorname*{Res}_{s=1}\Big(G_q(s,\mathcal{A})L(s,\mathcal{A})x^{s-1}\Big)\\ \ll&\frac{1}{q}(|G_q^{(m-1)}(1,\mathcal{A})|+|G_q(1,\mathcal{A})|(\log qx)^{m-1})\\ \ll&\frac{\tau(q)(\log qx)^{m-1}}{q}.\end{aligned}$$

Thus, we have

$$\Delta_y^{\varrho}M_{\varrho}(x;q)=y^{\varrho}\left(M_0(x;q)+O\left(\frac{\tau(q)}{q}y(\log x)^{m-1}\right)\right).$$

At last, we just note that these terms do not exist when the pole order m of $L(s,\mathcal{A})$ at $s=1$ equals zero, which means that $L(s,\mathcal{A})$ is an entire function.

4.4. **The finishing touches.** We first assume $\theta_d<1-\frac{1}{d}$. Applying the operator Δ_y^{ϱ} to both sides of (4.6), we have

$$\Delta_y^{\varrho}A_{\varrho}(x;q,a)=\Delta_y^{\varrho}M_{\varrho}(x;q)+\Delta_y^{\varrho}H_{\varrho}(x;q)+\Delta_y^{\varrho}S_{\varrho}(x;q).$$

Collecting these estimates of $\Delta_y^{\varrho}M_{\varrho}(x;q)$, $\Delta_y^{\varrho}H_{\varrho}(x;q)$ and $\Delta_y^{\varrho}S_{\varrho}(x;q)$ as in Sections 4.1–4.3, it follows that

$$\begin{aligned}\frac{\Delta_y^{\varrho}A_{\varrho}(x;q,a)}{y^{\varrho}}=&M_0(x;q)+O\left(\frac{\tau(q)}{q}y(\log qx)^{m-1}\right)+O\left(q^{\frac{d-1}{2}}\tau_d(q)(\log q)^{b_{\mathcal{A}}}\right)\\ &+O\left(\tau_d(q)\left(\frac{qx}{y}\right)^{\frac{d-1}{2}}(\log x)^{b_{\mathcal{A}}-1}\right).\end{aligned}\tag{4.17}$$

Thus, we conclude the first assertion of Theorem 2.1 from (4.9).

In addition $a_n\geq 0$, the differential form of the mean value theorem gives

$$\begin{aligned}M_0(x;q)-M_0(x-\varrho y;q)\ll&y\max_{\xi\ll 1}|M_0'(x+\xi y;q)|\\ \ll&\frac{\tau(q)}{q}y(\log qx)^{m-1}.\end{aligned}$$

From the estimates (4.17), it is easy to derive that

$$\Delta_y^{\varrho}A_{\varrho}(x-\varrho y;q,a)\quad\text{and}\quad\Delta_y^{\varrho}A_{\varrho}(x;q,a)$$

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are equal to

$$\begin{aligned}&M_0(x;q)+O\left(\frac{\tau(q)}{q}y(\log qx)^{m-1}\right)+O\left(q^{\frac{d-1}{2}}\tau_d(q)(\log q)^{b_{\mathcal{A}}}\right)\\&+O\left(\tau_d(q)\left(\frac{qx}{y}\right)^{\frac{d-1}{2}}(\log x)^{b_{\mathcal{A}}-1}\right).\end{aligned}\tag{4.18}$$

Using the inequalities (4.10), we then infer $A_0(x;q,a)$ also asymptotically equals (4.18). On taking $y=qx^{\frac{d-1}{d+1}}$, we finally derive

$$A_0(x;q,a)=M_0(x;q)+O\left(q^{\frac{d-1}{2}}\tau_d(q)(\log q)^{b_{\mathcal{A}}}\right)+O\left(\tau_d(q)x^{\frac{d-1}{d+1}}(\log x)^{\max\{b_{\mathcal{A}},m\}-1}\right),$$

which completes the proof of the second assertion in Theorem 2.1.

If $1-\frac{1}{d}\leq\theta_d<1$, we get analogous conclusions, where the only difference is that the divisor function $\tau_d(q)$ in the error terms is replaced by $\tau_{d+1}(q)$.

5. Background on automorphic L-functions and their Rankin–Selberg

We are mainly interested in some arithmetic functions arising from cuspidal automorphic representations. So we recall and show some standard facts about L-functions related to cuspidal automorphic representations in this section. We refer the reader to [24, Section 2] for a more detailed overview.

5.1. **Standard L-functions.** For $\pi=\otimes_p\pi_p\in\mathcal{F}(d)$ with $d\geq 2$, the standard L-function $L(s,\pi)$ associated to π is of the form

$$L(s,\pi)=\prod_{p<\infty}L(s,\pi_p)=\sum_{n=1}^{\infty}\frac{\lambda_\pi(n)}{n^s}.$$

The Euler product and Dirichlet series converge absolutely for $\mathrm{Re}(s)>1$. For each (finite) prime p, the inverse of the local factor $L(s,\pi_p)$ is a polynomial in p^{-s} of degree $\leq d$

$$L(s,\pi_p)^{-1}=\prod_{j=1}^{d}\left(1-\frac{\alpha_{j,\pi}(p)}{p^s}\right)$$

for suitable complex numbers $\alpha_{j,\pi}(p)$. With this convention, we have $\alpha_{j,\pi}(p)\neq 0$ for all j whenever $p\nmid q_\pi$, and it might be the case that $\alpha_{j,\pi}(p)=0$ for some j when $p\mid q_\pi$, where q_π is the arithmetic conductor of π. At the archimedean place of $\mathbb{Q}$, there are d complex Langlands parameters $\mu_{j,\pi}$ from which we define

$$L(s,\pi_\infty)=\prod_{j=1}^{d}\Gamma_{\mathbb{R}}(s+\mu_{j,\pi}).$$

For all primes p, it is known that there exists a constant

$$\theta_d\in\left[0,\,\frac{1}{2}-\frac{1}{d^2+1}\right]\tag{5.1}$$

such that

$$|\alpha_{j,\pi}(p)|\leq p^{\theta_d}\qquad\text{and}\qquad\mathrm{Re}(\mu_{j,\pi})\geq-\theta_d$$

for all j. Furthermore, for any unramified prime p and any $1 \le j \le d$, one has

$$p^{-\theta_d} \le |\alpha_{j,\pi}(p)| \le p^{\theta_d} \qquad \text{and} \qquad |\operatorname{Re}(\mu_{j,\pi})| \le \theta_d. \tag{5.2}$$

The generalized Ramanujan conjectures assert that θ_d may be taken as 0.

With all the local factors defined as above, we can turn to the functional equation. The contragredient $\widetilde{\pi}$ of $\pi \in \mathcal{F}(d)$ is also an irreducible cuspidal automorphic representation in $\mathcal{F}(d)$. Thus, we have

$$\{\alpha_{j,\widetilde{\pi}}(p) :\ 1 \le j \le d\} = \{\overline{\alpha_{j,\pi}(p)} :\ 1 \le j \le d\}$$

for each $p < \infty$, and

$$\{\mu_{j,\widetilde{\pi}} :\ 1 \le j \le d\} = \{\overline{\mu_{j,\pi}} :\ 1 \le j \le d\}.$$

Define the completed L-function

$$\Lambda(s,\pi) = q_\pi^{s/2} L(s,\pi) L(s,\pi_\infty).$$

Thus, $\Lambda(s,\pi)$ extends to an entire function. Moreover, $\Lambda(s,\pi)$ is bounded in vertical strips and satisfies a functional equation of the form

$$\Lambda(s,\pi) = \omega_\pi \Lambda(1-s,\widetilde{\pi}),$$

where ω_π is a complex number of modulus 1.

5.2. **Rankin–Selberg L-functions.** Now we turn to the Rankin–Selberg L-functions. Let $\pi = \otimes_p \pi_p \in \mathcal{F}(d)$ and $\pi' = \otimes_p \pi'_p \in \mathcal{F}(d')$. The Rankin–Selberg L-function $L(s, \pi \times \pi')$ associated to π and π' is of the form

$$L(s,\pi\times\pi') = \prod_p L(s,\pi_p\times\pi'_p) = \sum_{n=1}^{\infty} \frac{\lambda_{\pi\times\pi'}(n)}{n^s}.$$

The Euler product and Dirichlet series converge absolutely for $\operatorname{Re}(s) > 1$. For each (finite) prime p, the inverse of the local factor $L(s,\pi_p \times \pi'_p)$ is a polynomial in p^{-s} of degree $\le dd'$

$$L(s,\pi_p\times\pi'_p)^{-1} = \prod_{j=1}^{d}\prod_{j'=1}^{d'} \Big(1 - \frac{\alpha_{j,j',\pi\times\pi'}(p)}{p^s}\Big) \tag{5.3}$$

for suitable complex numbers $\alpha_{j,j',\pi\times\pi'}(p)$. With θ_d as in (5.1), we have the pointwise bound

$$|\alpha_{j,j',\pi\times\pi'}(p)| \le p^{\theta_d + \theta_{d'}}. \tag{5.4}$$

If $p \nmid q_\pi$ or $p \nmid q_{\pi'}$, then we have the equality of sets

$$\{\alpha_{j,j',\pi\times\pi'}(p) :\ j \le d,\ j' \le d'\} = \{\alpha_{j,\pi}(p)\alpha_{j',\pi'}(p) :\ j \le d,\ j' \le d'\}. \tag{5.5}$$

At the archimedean place of $\mathbb{Q}$, there are dd' complex Langlands parameters $\mu_{j,j',\pi\times\pi'}$ from which we define

$$L(s,\pi_\infty\times\pi'_\infty) = \pi^{-\frac{dd's}{2}} \prod_{j=1}^{d}\prod_{j'=1}^{d'} \Gamma\Big(\frac{s+\mu_{j,j',\pi\times\pi'}}{2}\Big).$$

These parameters satisfy the equality

$$\{\mu_{j,j',\widetilde{\pi}\times\widetilde{\pi}'}\} = \{\overline{\mu_{j,j',\pi\times\pi'}}\}$$

for $1 \le j \le d$, $1 \le j' \le d'$ and the pointwise bound

$$\operatorname{Re}(\mu_{j,j',\pi\times\pi'}) \ge -\theta_d - \theta_{d'}. \tag{5.6}$$

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The complete L-function

$$\Lambda(s,\pi\times\pi')=q_{\pi\times\pi'}^{s/2}L(s,\pi\times\pi')L(s,\pi_\infty\times\pi'_\infty)$$

has a meromorphic continuation and is bounded (away from its poles) in vertical strips. Under our normalization on the central characters, $\Lambda(s,\pi\times\pi')$ is entire if and only if $\widetilde{\pi}\neq\pi'$. Moreover, $\Lambda(s,\pi\times\pi')$ satisfies the functional equation

$$\Lambda(s,\pi\times\pi')=\omega_{\pi\times\pi'}\Lambda(1-s,\widetilde{\pi}\times\widetilde{\pi}'),$$

where $\omega_{\pi\times\pi'}$ is a complex number of modulus 1.

Finally, we recall some estimates for $\pi'=\widetilde{\pi}$. It is known from [13, Lemma 3.1] that

$$|\lambda_\pi(n)|^2\leq\lambda_{\pi\times\widetilde{\pi}}(n) \tag{5.7}$$

hold for all positive integer n. Moreover, $L(s,\pi\times\widetilde{\pi})$ extends to the complex plane with a simple pole at $s=1$. Hence, Landau's lemma [2, Theorem 3.2] gives

$$\sum_{n\leq x}\lambda_{\pi\times\widetilde{\pi}}(n)=c_\pi x+O_\pi\left(x^{\frac{d^2-1}{d^2+1}}\right) \tag{5.8}$$

for some constant $c_\pi>0$.

5.3. **Twists.** Let $\chi(\operatorname{mod} q)$ be a primitive Dirichlet character with $(q,q_\pi)=1$. As is well known, χ corresponds to a Hecke character of the idele class group $\mathbb{A}^\times/\mathbb{Q}^\times$ trivial on $\mathbb{R}_+^\times$, so χ is of the form $\chi=\otimes_p\chi_p$.

We apply the Rankin–Selberg theory described above to the following situation: Fix π in $\mathcal{F}(d)$ with $m\geq 2$, and let χ be a primitive Dirichlet character modulo q. Take $\pi'=\chi$. The twisted L-function is given by

$$L(s,\pi\otimes\chi)=\sum_{n=1}^\infty\frac{\lambda_\pi(n)\chi(n)}{n^s}.$$

The corresponding complete L-function

$$\Lambda(s,\pi\otimes\chi)=(q_\pi q^d)^{s/2}L(s,\pi_\infty\times\chi_\infty)L(s,\pi\otimes\chi)$$

has an analytic continuation to the whole complex plane and satisfies the following functional equation:

$$\Lambda(s,\pi\otimes\chi)=\omega_{\pi\otimes\chi}\Lambda(1-s,\widetilde{\pi}\otimes\overline{\chi}),$$

where $L(s,\pi_\infty\times\chi_\infty)$ is given by

$$L(s,\pi_\infty\otimes\chi_\infty)=\prod_{j=1}^d\Gamma_{\mathbb{R}}\big(s+\mu_{j,\pi\otimes\chi}\big).$$

Similarly, if we take $\pi'=\widetilde{\pi}(\chi):=\widetilde{\pi}\otimes\chi$, then we have

$$L(s,\pi\times\widetilde{\pi}(\chi))=\sum_{n=1}^\infty\frac{\lambda_{\pi\times\widetilde{\pi}}(n)\chi(n)}{n^s}.$$

The complete L-function

$$\Lambda(s,\pi\times\widetilde{\pi}(\chi))=(q_{\pi\times\widetilde{\pi}}q^{2d})^{s/2}L(s,\pi_\infty(\chi_\infty)\times\widetilde{\pi}_\infty)L(s,\pi\times\widetilde{\pi}(\chi))$$

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has an analytic continuation to the whole complex plane and satisfies the following functional equation:

$$\Lambda(s,\pi\times\widetilde{\pi}(\chi))=\omega_{\pi\times\widetilde{\pi}(\chi)}\Lambda(1-s,\pi\times\widetilde{\pi}(\overline{\chi})),$$

where

$$L(s,\pi_\infty\times\widetilde{\pi}_\infty(\chi_\infty))=\prod_{j=1}^{d}\prod_{j'=1}^{d}\Gamma_{\mathbb{R}}\big(s+\mu_{j,j',\pi\times\widetilde{\pi}(\chi)}\big).$$

Due to the work of Müller and Speh [18, Proof of Lemma 3.1], all local Langlands parameters $\mu_{j,\pi\otimes\chi}$ and $\mu_{j,j',\pi\times\widetilde{\pi}(\chi)}$ depend on π and the parity of χ at most (see also [24, Proof of Lemma 2.1]). Moreover, the relatively explicit expressions of $\omega_{\pi\otimes\chi}$ and $\omega_{\pi\times\widetilde{\pi}(\chi)}$ are required. We adopt the argument of Barthel–Ramakrishnan [2, Proposition 4.1] or Luo–Rudnick–Sarnak [16, Lemma 2.1] and show the following result.

Lemma 5.1. *Let $\pi\in\mathcal{F}(d)$, and let $\chi(\bmod q)$ be a primitive Dirichlet character with $(q,q_\pi)=1$. Then we have*

$$\omega_{\pi\otimes\chi}=\eta_{\pi,\mathrm{sgn}(\chi)}\chi(q_\pi)\tau(\chi)^dq^{-\frac{d}{2}},$$

where $\eta_{\pi,\mathrm{sgn}(\chi)}$ depends on π and the parity of χ only, and $|\eta_{\pi,\mathrm{sgn}(\chi)}|=1$.

Proof. Let the ϵ-factor be defined by

$$L(s,\pi_\infty\otimes\chi_\infty)L(s,\pi\otimes\chi)=\epsilon(s,\pi\otimes\chi)L(1-s,\pi_\infty\otimes\chi_\infty)L(1-s,\pi\otimes\chi).$$

By the functional equation, the relation between the ϵ-factor and the root number is

$$\epsilon(s,\pi\otimes\chi)=(q_\pi q^d)^{\frac{1}{2}-s}\omega_{\pi\otimes\chi}.$$

Moreover, it can be written as a product of local factors by fixing an additive character $\psi=\prod_{p\leq\infty}\psi_p$:

$$\epsilon(s,\pi\otimes\chi)=\prod_{p\leq\infty}\epsilon(s,\pi_p\otimes\chi_p,\psi_p).\tag{5.9}$$

If $p\nmid q_\pi q$, where π_p and χ_p are both unramified, then

$$\epsilon(s,\pi_p\otimes\chi_p,\psi_p)=1.\tag{5.10}$$

Suppose that $p^{r(\chi_p)}\parallel q$, in which case χ_p is ramified with conductor $p^{r(\chi_p)}$. By assumption, π_p is the canonical component of $\pi_q=\mathrm{Ind}\,(\mathrm{GL}_d,B;\mu_1,\dots,\mu_d)$ where B is the Borel subgroup of GL_m and $\mu_j(x)=|x|^{u_j}$ are unramified characters. Then $\pi_q\otimes\chi_p=\mathrm{Ind}\,(\mathrm{GL}_d,B;\chi\mu_1,\dots,\chi\mu_d)$. Thus, we have

$$\begin{aligned}\epsilon(s,\pi_p\otimes\chi_p,\psi_p)&=\prod_{j=1}^{d}\epsilon(s,\mu_j\otimes\chi_p,\psi_p)\\&=\prod_{j}^{d}\epsilon\left(s,\mu_j\chi_p,\psi_p\right)\\&=\prod_{j}^{d}\epsilon\left(s+u_j,\chi_p,\psi_p\right),\end{aligned}$$

where the abelian ϵ-factor (for χ primitive) is given by

$$\epsilon\left(s,\chi,\psi_q\right)=\tau(\chi)p^{-r(\chi_p)s}.$$

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Since $\epsilon(s,\pi_p,\psi_p)=1$ and the central character of π is trivial, which means that $\sum_{j=1}^m u_j=0$, we have

$$\begin{aligned}\epsilon(s,\pi_p\otimes\chi_p,\psi_p) &= \prod_j^d \tau(\chi)p^{-r(\chi_p)(s+u_j)} \\ &= \tau\left(\chi,\psi_p\right)^d p^{-dr(\chi_p)s}\epsilon(s,\pi_p,\psi_p).\end{aligned} \tag{5.11}$$

Suppose that $p^{r(\pi_p)}\parallel q_\pi$, in which case χ_p is unramified given by $\chi_p(x)=|x|^{v_p}$. With this given, we have

$$\begin{aligned}\epsilon(s,\pi_p\otimes\chi_p,\psi_p) &= \epsilon(s+v_p,\pi_p,\psi_p) \\ &= \omega_{\pi_p}p^{r(\pi_p)\left(\frac{1}{2}-s-v_p\right)} \\ &= \chi\left(p^{r(\pi_p)}\right)\epsilon(s,\pi_p,\psi_p),\end{aligned} \tag{5.12}$$

Consider the archimedean place. It is known from [12] that $\epsilon(s,\pi_\infty,\psi_\infty)$ and $\epsilon(s,\pi_\infty\otimes\chi_\infty,\psi_\infty)$ are constants, hence equal to the corresponding values at $s=1/2$. Since $\chi_\infty(x)=\operatorname{sgn}(x)|x|^{v_\infty}$, the constant $\epsilon(s,\pi_p\otimes\chi_p,\psi_p)$ depends only on π and the parity of χ.

Finally, inserting (5.10), (5.11) and (5.12) into (5.9), we get

$$\begin{aligned}\epsilon(s,\pi\otimes\chi) =&\left(\prod_{p|q}\tau\left(\chi,\psi_p\right)^d p^{-dr(\chi_p)s}\epsilon(s,\pi_p,\psi_p)\right)\left(\prod_{p|q_\pi}\chi\left(p^{r(\pi_p)}\right)\epsilon(s,\pi_p,\psi_p)\right) \\ &\times\frac{\epsilon_\infty(\frac{1}{2},\pi_\infty\otimes\chi_\infty,\psi_\infty)}{\epsilon_\infty(\frac{1}{2},\pi_\infty,\psi_\infty)}\epsilon_\infty(s,\pi_\infty,\psi_\infty) \\ =& c_{\pi,\operatorname{sgn}(\chi)}\chi(q_\pi)\tau(\chi)^m q^{-ds}\epsilon(s,\pi),\end{aligned} \tag{5.13}$$

where $c_{\pi,\operatorname{sgn}(\chi)}:=\epsilon_\infty(1/2,\pi_\infty\otimes\chi_\infty,\psi_\infty)/\epsilon_\infty(1/2,\pi_\infty,\psi_\infty)$ is a constant depending on π and the parity of χ only. Thus, the relation (5.13) of ϵ-factors gives

$$\begin{aligned}\omega_{\pi\otimes\chi} =&\left(\prod_{p|q}\tau\left(\chi,\psi_p\right)^d p^{-dr(\chi_p)s}\epsilon(s,\pi_p,\psi_p)\right)\left(\prod_{p|q_\pi}\chi\left(p^{r(\pi_p)}\right)\epsilon(s,\pi_p,\psi_p)\right) \\ &\times\frac{\epsilon_\infty(\frac{1}{2},\pi_\infty\otimes\chi_\infty,\psi_\infty)}{\epsilon_\infty(\frac{1}{2},\pi_\infty,\psi_\infty)}\epsilon_\infty(s,\pi_\infty,\psi_\infty) \\ =& c_{\pi,\operatorname{sgn}(\chi)}\chi(q_\pi)\tau(\chi)^d q^{-\frac{d}{2}}\omega_\pi,\end{aligned}$$

which implies $|c_{\pi,\operatorname{sgn}(\chi)}|=1$ in turn. On putting $\eta_{\pi,\operatorname{sgn}(\chi)}=c_{\pi,\operatorname{sgn}(\chi)}\omega_\pi$, we complete the proof of this lemma. □

Similar to Lemma 5.1, we can also show the following lemma.

Lemma 5.2. *Let $\pi\in\mathcal{F}(d)$ be a cuspidal automorphic representation of* $\mathrm{GL}(d)$ *of conductor q_π with trivial central character, and $\chi(\operatorname{mod} q)$ be a primitive Dirichlet character with $(q,q_\pi)=1$. Then we have*

$$\omega_{\pi\times\widetilde{\pi}(\chi)}=\eta_{\pi\times\widetilde{\pi},\operatorname{sgn}(\chi)}\chi(q_{\pi\times\widetilde{\pi}})\tau(\chi)^{d^2}q^{-\frac{d^2}{2}},$$

where $\eta_{\pi\times\widetilde{\pi},\operatorname{sgn}(\chi)}$ depends on π and the parity of χ only, and $|\eta_{\pi\times\widetilde{\pi},\operatorname{sgn}(\chi)}|=1$.

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6. Applications of Theorems 2.1

6.1. **Proof of Theorem 1.2.** From the discussion in Section 5, we see that the Rankin–Selberg L-function $L(s,\pi\times\widetilde{\pi})$ satisfies Conditions (A1)–(A3) with $m=1$, and its twisted L-function $L(s,\pi\otimes\chi)$ satisfies Condition (A4), where the later follows from Lemma 5.2.

Next, we discuss the sizes of various types for the coefficients $\lambda_{\pi\times\widetilde{\pi}}(n)$. The asymptotic formula (5.8) yields Hypothesis S with $b_{\pi\times\widetilde{\pi}}=1$. Since the central character of π is trivial, one has

$$s_{d,\pi}(p)=\alpha_{1,\pi}(p)\alpha_{2,\pi}(p)\cdots\alpha_{d,\pi}(p)=1$$

for all primes p with $(p,q_\pi)=1$. Then it follows from (5.2) and (5.5) that

$$|\alpha_{j,j',\pi\times\pi'}(p)|\le p^{2\theta_d},\qquad s_{j,\pi\times\widetilde{\pi}}(p)\ll p^{2\min\{j,d^2-j\}\theta_d}$$

for any prime p with $(p,q_\pi)=1$ and any $1\le j\le d^2$, which implies Hypothesis $\mathrm{H}(\theta_{\mathrm{d}^2})$ with $\theta_{d^2}=2\theta_d\le 1-\frac{2}{d^2+1}<1-\frac{1}{d^2}$. Therefore, we can apply Theorem 2.1 to the non-negative coefficients $\lambda_{\pi\times\widetilde{\pi}}(n)$, and then obtain

$$\sum_{\substack{n\le x\\ n\equiv a\,(\mathrm{mod}\,q)}}\lambda_{\pi\times\widetilde{\pi}}(n)=M_0(x;q)+O_\pi\Big(\tau_{d^2}(q)q^{\frac{d^2-1}{2}}\log q\Big)+O_\pi\Big(\tau_{d^2}(q)x^{\frac{d^2-1}{d^2+1}}\Big),$$

where the main term is given by

$$M_0(x;q)=\frac{1}{\varphi(q)}\operatorname*{Res}_{s=1}\Big(\frac{1}{s}L\big(s,\pi\times\widetilde{\pi}(\chi_0)\big)x^s\Big).$$

Since

$$L\big(s,\pi\times\widetilde{\pi}(\chi_0)\big)=L(s,\pi\times\widetilde{\pi})\prod_{p|q}L(s,\pi_p\times\widetilde{\pi}_p)^{-1}$$

and $L(s,\pi\times\widetilde{\pi})$ has a simple pole at $s=1$, we have

$$M_0(x;q)=\frac{1}{\varphi(q)}\operatorname*{Res}_{s=1}\big(L(s,\pi\times\widetilde{\pi})\big)\prod_{p|q}L(1,\pi_p\times\widetilde{\pi}_p)^{-1}x.$$

This completes the proof of Theorem 1.2.

6.2. **Proof of Theorem 1.1.** Similar to the argument in Section 6.1, we can apply Theorem 2.1 to the coefficients $\lambda_\pi(n)$. By applying the Cauchy–Schwarz inequality, (5.7) and (5.8), we get

$$\sum_{\substack{x<n\le x+y\\ n\equiv a\,(\mathrm{mod}\,q)}}|\lambda_\pi(n)|\ll_\pi\Big(\frac{xy}{q}\Big)^{1/2}\tag{6.1}$$

for any $q\le y\le x$, which yields Hypothesis S with $b_\pi=1$. Since $L(s,\pi)$ is entire, the main term and the first error term do not exist when applying Theorem 2.1. Thus, we obtain

$$\sum_{\substack{n\le x\\ n\equiv a\,(\mathrm{mod}\,q)}}\lambda_\pi(n)\ll_\pi\tau_d(q)q^{\frac{d-1}{2}}\log q+\tau_d(q)\Big(\frac{qx}{y}\Big)^{\frac{d-1}{2}}+\sum_{\substack{x<n\le x+O(y)\\ n\equiv a\,(\mathrm{mod}\,q)}}|\lambda_\pi(n)|.\tag{6.2}$$

Inserting the bound (6.1) and taking $y=qx^{1-\frac{2}{d}}$, we get the first bound

$$\sum_{\substack{n\le x\\ n\equiv a\,(\mathrm{mod}\,q)}}\lambda_\pi(n)\ll_\pi\tau_d(q)x^{1-\frac{1}{d}}$$

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for $q \le x^{\frac{1}{d}}$.

Moreover, it follows from Theorem 1.2 that

$$\sum_{\substack{x<n\le x+y\\ n\equiv a \,(\mathrm{mod}\, q)}} \lambda_{\pi\times\widetilde{\pi}}(n) \ll_\pi \frac{c_{\pi,q}}{\varphi(q)}y + O\Big(\tau_{d^2}(q)q^{\frac{d^2-1}{2}}\log q\Big) + O\Big(\tau_{d^2}(q)x^{\frac{d^2-1}{d^2+1}}\Big)$$

for $q \le x^{\frac{2}{d^2+1}}$. By (5.3) and (5.4), the constant $c_{\pi,q}$ satisfies

$$c_{\pi,q} \ll_\pi \prod_{p|q}\big(1+p^{-\frac{2}{d^2+1}+\varepsilon}\big)^{d^2} \ll \tau(q).$$

Note that $q/\varphi(q) \ll \log q$. Further, we get from (5.7) that

$$\sum_{\substack{x<n\le x+y\\ n\equiv a \,(\mathrm{mod}\, q)}} |\lambda_\pi(n)| \ll \tau_{d^2}(q)\log x\left(\frac{y}{q} + \sqrt{\frac{y}{q}}\cdot x^{\frac{1}{2}-\frac{1}{d^2+1}}\right)$$

for $q \le x^{\frac{2}{d^2+1}}$. On taking $y = qx^{1-\frac{2d}{d^2+1}}$, the estimate (6.2) gives the second bound

$$\sum_{\substack{n\le x\\ n\equiv a \,(\mathrm{mod}\, q)}} \lambda_\pi(n) \ll \tau_{d^2}(q)x^{1-\frac{d+1}{d^2+1}}\log x$$

for $q \le x^{\frac{2}{d^2+1}}$.

Assume the Ramanujan conjecture holds for π, the Brun–Titchmarsh inequality (see Shiu [20, Theorem 1]) yields

$$\sum_{\substack{x<n\le x+y\\ n\equiv a \,(\mathrm{mod}\, q)}} |\lambda_\pi(n)| \le \frac{y}{\varphi(q)\log x}\exp\Bigg(\sum_{\substack{p\le x\\ p\nmid q}}\frac{|\lambda_\pi(p)|}{p}\Bigg) \tag{6.3}$$

provided that $q \le y^{1-\varepsilon}$ and $x^\varepsilon \le y \le x$. By Mertens' theorem and the prime number theorem for Rankin–Selberg L-function $L(s,\pi\times\widetilde{\pi})$ (see [13, Page 630]), one has

$$\sum_{p\le x}\frac{|\lambda_\pi(p)|}{p} \ll \Big(\sum_{p\le x}\frac{1}{p}\Big)^{\frac{1}{2}}\Big(\sum_{p\le x}\frac{\lambda_{\pi\times\widetilde{\pi}}(p)}{p}\Big)^{\frac{1}{2}} \ll \log\log x.$$

Inserting this estimate into (6.3), we obtain

$$\sum_{\substack{x<n\le x+y\\ n\equiv a \,(\mathrm{mod}\, q)}} |\lambda_\pi(n)| \ll \frac{y}{\varphi(q)}$$

provided that $q \le y^{1-\varepsilon}$ and $x^\varepsilon \le y \le x$. Substitute this into (6.2) and taking $y = qx^{1-\frac{2}{d+1}}$, the last assertion follows.

6.3. **Proof of Theorem 1.3.** We begin with evaluating the summation about $\lambda_{\mathrm{sym}^d f}(n)$ in a short interval.

Lemma 6.1. *Let $f \in H_k^*(N)$ and $\lambda_{\mathrm{sym}^d f}(n)$ be the coefficients of $L(s,\mathrm{sym}^d f)$. For $(q,aN) = 1$, we have*

$$\sum_{\substack{x<n\le x+y\\ n\equiv a \,(\mathrm{mod}\, q)}} |\lambda_{\mathrm{sym}^d f}(n)| \ll \frac{y}{\varphi(q)(\log x)^{\gamma_d}}$$

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provided that $q \le y^{1-\varepsilon}$ *and* $x^{\varepsilon} \le y \le x$*, where* $\gamma_d = 1 - \frac{4(d+1)}{d(d+2)\pi}\cot\left(\frac{\pi}{2(d+1)}\right)$ *and* $0.15 < \gamma_d < 0.19$.

Proof. Let

$$U_d(\cos\theta_p) = \frac{\sin((d+1)\theta_p)}{\sin\theta_p}$$

be the d-th Chebyshev polynomial of the second type. One can easily check via (1.4) that

$$\lambda_{\mathrm{sym}^d f}(p) = U_d(\cos\theta_p), \qquad p \nmid N.$$

By the Sato–Tate conjecture (1.5) and a straightforward calculation of Maple, we get

$$\begin{aligned}\sum_{\substack{p\le x\\ p\nmid q}} |\lambda_{\mathrm{sym}^d f}(p)| \le &\sum_{\substack{p\le x\\ p\nmid N}} |\lambda_{\mathrm{sym}^d f}(p)| + O(1)\\ \sim&\Big(\int_0^{\pi} \frac{|\sin((d+1)\theta)|}{\sin\theta}\mathrm{d}\mu_{ST}\Big)\frac{x}{\log x}\\ \sim&\frac{4(d+1)}{d(d+2)\pi}\cot\left(\frac{\pi}{2(d+1)}\right)\frac{x}{\log x}.\end{aligned}$$

Hence, we derive by partial summation and substituting this into (6.3) that

$$\sum_{\substack{x<n\le x+y\\ n\equiv a\,(\mathrm{mod}\,q)}} |\lambda_{\mathrm{sym}^d f}(n)| \ll \frac{y}{\varphi(q)(\log x)^{\gamma_d}},$$

where $\gamma_d = 1 - \frac{4(d+1)}{d(d+2)\pi}\cot\left(\frac{\pi}{2(d+1)}\right)$. It is clear that γ_d is strictly increasing. Thus, for any $d \ge 1$, we have

$$0.15 < 1 - \frac{8}{3\pi} = \gamma_1 \le \gamma_d \le \lim_{d\to\infty}\gamma_d = 1 - \frac{8}{\pi^2} < 0.19.$$

□

Finally, the proof of Theorem 1.3 is completed if we combine the first assertion of Theorem 2.1 with Lemma 6.1, the choice $y = qx^{\frac{d}{d+2}}$ and the fact $q/\varphi(q) \le \tau(q)$.

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