ANNULETS AND α-IDEALS IN A DISTRIBUTIVE LATTICE

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1. Introduction

In a distributive lattice L with 0 the set of all ideals of the form $(x]^*$ can be made into a lattice $A_0(L)$ called the lattice of annulets of L. $A_0(L)$ is a sublattice of the Boolean algebra of all annihilator ideals in L. While the lattice of annulets is no more than the dual of the so-called lattice of filets (carriers) as studied in the theory of l-groups and abstractly for distributive lattices in [1, section 4] it is a useful notion in its own right. For example, from the basic theorem of [3] it follows that $A_0(L)$ is a sublattice of the lattice of all ideals of L if and only if each prime ideal in L contains a unique minimal prime ideal.

For an ideal J in L

$$\alpha(J) = \{(x]^* : x \in J\}$$

is a filter in $A_0(L)$ and conversely

$$\alpha^{\leftarrow}(F) = \{x \in L : (x]^* \in F\}$$

is an ideal in L when F is any filter in $A_0(L)$. An ideal J in L is called an α-ideal if $\alpha^{\leftarrow}\alpha(J) = J$. Then by using the structure of $A_0(L)$ results can be transferred to give information on the ideal structure of L. The most interesting result of this type is that L is a generalized Stone lattice if and only if each prime ideal contains a unique prime α-ideal.

2. Annulets

Throughout the rest of this note all lattices are distributive. Also the terminology of [3] *will be used freely.*

An ideal J of a lattice L with 0 is called an *annihilator ideal* if $J = J^{**}$. This is equivalent to

$$J = \{y \in L : y \wedge s = 0 \text{ for all } s \in S\}$$

where S is some non-empty subset of L. As is well-known the set of annihilator

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ideals $A(L)$ can be made into a Boolean algebra with smallest element $(0]$, largest element L, set-theoretic intersection as the infimum and the map $J \to J^*$ as complementation. Thus the supremum of J and K in $A(L)$ is given by $J \veebar K = (J^* \cap K^*)^*$. This is no more than De Morgan's law.

Call an ideal of the form $(x]^*$, $x \in L$, an *annulet*. Each annulet is an annihilator ideal and hence for two annulets $(x]^*$ and $(y]^*$ their supremum in $A(L)$ is

$$(x]^* \veebar (y]^* = ((x]^{**} \cap (y]^{**})^* = ((x \wedge y]^{**})^* = (x \wedge y]^*.$$

Also their infimum in $A(L)$ is $(x]^* \cap (y]^* = (x \vee y]^*$. We are thus lead to the following result.

PROPOSITION 2.1 *Let L be a lattice with 0. Then the set of annulets $A_0(L)$ of L is a lattice $(A_0(L), \cap, \veebar)$ and sublattice of the Boolean algebra*

$$(A(L), \cap, \veebar, {}^*, (0], L)$$

of annihilator ideals of L. $A_0(L)$ has the same largest element $L = (0]^*$ as $A(L)$ while $A_0(L)$ has a smallest element if and only if L possesses an element d such that $(d]^* = (0]$.

PROOF. All that remains to be verified is the statement concerning the smallest element in $A_0(L)$. If there is an element $d \in L$ such that $(d]^* = (0]$ then plainly $(0]$ is the smallest element in $A_0(L)$. While if there is an element $d \in L$ such that $(d]^*$ is the smallest element then for any $x \in L$

$$(x]^* = (x]^* \veebar (d]^* = (x \wedge d]^*.$$

Thus $x \wedge d = 0$ implies $(x]^* = (0]^* = L$ so that $x = 0$ and hence $(d]^* = (0]$.

We now characterize normal lattices.

PROPOSITION 2.2 *A lattice L with 0 is normal if and only if $A_0(L)$ is a sublattice of the lattice of ideals of L.*

PROOF. $A_0(L)$ is a sublattice of the lattice of ideals of L if and only if, for any $x, y \in L$, $(x]^* \vee (y]^* = (z]^*$ for some $z \in L$. Since

$$(x]^* \vee (y]^* = (z]^* \text{ implies } (z]^{**} = (x]^{**} \cap (y]^{**} = (x \wedge y]^{**},$$

so that $(z]^* = (x \wedge y]^* = (x]^* \veebar (y]^*$ in $A_0(L)$, we see that $A_0(L)$ is a sublattice if and only if $(x]^* \vee (y]^* = (x \wedge y]^*$ for each $x, y \in L$. By [3, Theorem 2.4] this is equivalent to L being normal.

As in common practice a lattice L with 0 is called disjunctive if for any a, $b \in L$, $a < b$ implies $a \wedge c = 0$ and $c < b$, for some $o \neq c \in L$. This was the definition of 'disjunctive' used in [3]. However it is easy to see that a lattice L with 0 is disjunctive if and only if $(a]^* = (b]^*$ implies $a = b$ for any a, b in L. We thus have the following corollary to Propositions 2.1 and 2.2.

PROPOSITION 2.3 *A disjuntive normal L is dual isomorphic to its lattice of annulets. Hence L has a largest element if and only if there is an element $d \in L$ such that $(d]^* = (0]$.*

Disjunctive normal lattices are important in compactification theory, see [3, Theorems 7.3 and 7.6]. Actually disjunctive lattices are themselves important in the study of $A_0(L)$; information can be obtained by dualizing Banaschewski's results in [1, section 4].

It is easy to see that a lattice L with 0 is quasi-complemented (for the definition see [3, 5.1]) if and only if for each $x \in L$ there is an x' such that $(x]^{**} = (x']^*$ or equivalently such that

$$x \wedge x' = 0 \text{ and } (x]^* \cap (x']^* = (0].$$

A quasi-complemented lattice has an element d such that $(d]^* = (0]$ so that $A_0(L)$ has smallest element 0. Noting that $x \wedge x' = 0$ is equivalent to

$$(x]^* \underline{\vee} (x']^* = \text{the largest element of } A_0(L)$$

these remarks yields

PROPOSITION 2.4 *A lattice L with 0 is quasi-complemented if and only if $A_0(L)$ is a Boolean subalgebra of $A(L)$.*

We now consider generalized Stone lattices and sectionally quasi-complemented lattices. For the definitions see [3, 5.2, 5.3].

PROPOSITION 2.5 *The lattice of annulets of a generalized Stone lattice is a relatively complemented sublattices of the lattice of ideals of L.*

PROOF. From [3, Proposition 5.5] a generalized Stone lattice L is normal so $A_0(L)$ is a sublattice of the lattice of ideals of L due to Proposition 2.2. We therefore write $\underline{\vee}$ as $\vee$. As $A_0(L)$ is a distributive lattice with largest element L, $A_0(L)$ will be relatively complemented if and only if each interval of the form $[I, L]$, $I \in A_0(L)$, is complemented.

Thus let $J = [(x]^*, L]$ be an interval in $A_0(L)$ and let $(y]^* \in J$. As L is a generalized Stone lattice $(y]^* \vee (y]^{**} = L$. $(y]^* \cap (y]^{**} = (0]$ always holds hence

$$((x] \cap (y]^*) \vee ((x] \cap (y]^{**}) = (x] \text{ and } ((x] \cap (y]^*) \cap ((x] \cap (y]^{**}) = (0].$$

The last two equalities follow from the distributivity of the lattice of ideals of L. We thus have two ideals whose supremum and infimum are principal ideals and by [5, Lemma 2, p. 83] both ideals are themselves principal. Thus $(a] = (x] \cap (y]^*$ for some $a \in L$. As $a \leqq x$, $(x]^* \subseteq (a]^*$ so $(a]^* \in J$. Also $(a] \subseteq (y]^*$ so $(y]^{**} \subseteq (a]^*$, so $(a]^* \vee (y]^* = L$. Now

$$(a]^* \cap (y]^* \cap (x] = (a]^* \cap (a] = (0]$$

so $(a]^* \cap (y]^* \subseteq (x]^*$ but $(x]^* \subseteq (y]^*,(a]^*$ so $(a]^* \cap (y]^* = (x]^*$. Then $(a]^*$ is the required complement of $(y]^*$ in the interval J.

We now improve Proposition 2.5 by showing that the converse is also true. The following lemma is obvious.

LEMMA 2.6 Let $I = [0,x]$, $0 < x$, *be an interval in lattice* L *with* 0. *For* $a \in I$, $(a]^+$ *is the annihilator of* a *with respect to* I;

$$(a]^+ = \{y \in I : y \wedge a = 0\} = \{y \in I : y \wedge b = 0 \text{ for } all\ b \in I\ and\ b \leqq a\}.$$

Then,

(1) *if* $a, b \in I$ *and* $(a]^+ \subseteq (b]^+$ *it follows that* $(a]^* \subseteq (b]^*$,

(2) *if* $w \in L$, $(w]^* \cap I = (w \wedge x]^+$.

PROPOSITION 2.7 *The lattice of annulets of a lattice* L *with* 0 *is relatively complemented if and only if* L *is sectionally quasi-complemented.*

PROOF. Suppose $A_0(L)$ is relatively complemented. We must show that $I = [0, x]$ is a quasi-complemented lattice for each $0 < x \in L$. Let a, $b \in I$ and suppose

$$(a]^+ \subseteq (b]^+ \subseteq I = (0]^+.$$

From the lemma, $(a]^* \subseteq (b]^* \subseteq L$. The interval $[(a]^*, L]$ is complemented in $A_0(L)$ so that there is an element $w \in L$ such that

$$(b]^* \cap (w]^* = (a]^* \text{ and } (b]^* \veebar (w]^* = L.$$

Then $(b]^* \veebar (w]^* = (b \wedge w]^*$ gives $b \wedge w = 0$. Then $b \wedge (w \wedge x) = 0$ so

$$(b]^+ \veebar (w \wedge x]^* = (a]^+,$$

due to Lemma 2.6. It follows that $A_0(L)$ is complemented and so by Proposition 2.4 (or rather a variation on it) I is quasi-complemented.

Suppose L is sectionally quasi-complemented. To prove $A_0(L)$ is relatively complemented it suffices to prove that each interval $[(a]^*, L]$ is complemented as $A_0(L)$ is distributive (Proposition 2.1). Let $(b]^* \in [(a]^*, L] \subseteq A_0(L)$ and consider the interval $I = [0, a \vee b]$ in L. Then

$$(a]^+ = (a]^* \cap I \subseteq (b]^* \cap I = (b]^+ \subseteq I$$

so there is an element $w \in I$ such that $(w]^+ \cap (b]^+ = (a]^+$ and $(w]^+ \veebar (b]^+ = I$ as I is quasi-complemented and so $A_0(I)$ is complemented by Proposition 2.5. Then

$$(w \vee b]^+ = (w]^+ \cap (b]^+ = (a]^+$$

so $(b]^* \cap (w]^* = (b \vee w]^* = (a]^*$ by Lemma 2.6. Also $(w \wedge b]^+ = I$ so $w \wedge b = 0$ hence

$$(b]^* \veebar (w]^* = L.$$

As $(w]^* \in [(a]^*, L]$ it follows that $A_0(L)$ is relatively complemented.

From Proposition 2.7 we now obtain the improvement Proposition 2.5. Incidentally, the proof of the following proposition supplies an alternative proof of Proposition 2.5.

PROPOSITION 2.8 *A lattice L with* 0 (*resp. with* 0 *and* 1) *is a generalized Stone lattice* (*resp. Stone lattice*) *if and only if the lattice of annulets is a relatively complemented* (*resp. Boolean algebra and*) *sublattice of the ideals of L.*

PROOF. Suppose L is a lattice with 0 but not necessarily with a largest element 1. Then L is a generalized Stone lattice by Propositions 2.2 and 2.7 together with [3, Theorem 5.7].

When $1 \in L$ the assertion follows from Propositions 2.2 and 2.5 together with [3, Theorem 5.6].

3. α-ideals

We now replace Propositions 2.7 and 2.8 by propositions concerning the prime ideal structure of L. To do this we introduce a special class of ideals and our first job is to elucidate this class of ideals.

The proof of the following proposition is quite routine and will be omitted.

PROPOSITION 3.1 *Let L be a lattice with* 0. *The following holds*:

(*a*) *for an ideal J in L*, $\alpha(J) = \{(x]^* : x \in J\}$ *is a filter in* $A_0(L)$,

(*b*) *for a filter F in* $A_0(L)$, $\alpha^{\leftarrow}(F) = \{x \in L : (x]^* \in F\}$ *is an ideal in L*,

(*c*) *if* J_1, J_2 *are ideals in L then* $J_1 \subseteq J_2$ *implies* $\alpha(J_1) \subseteq \alpha(J_2)$; *and if* F_1, F_2 *are filter in* $A_0(L)$ *then* $F_1 \subseteq F_2$ *implies* $\alpha^{\leftarrow}(F_1) \subseteq \alpha^{\leftarrow}(F_2)$,

(*d*) *the map* $I \to \alpha^{\leftarrow}\alpha(I)\{ = \alpha^{\leftarrow}(\alpha(I))\}$ *is a closure operation on the lattice of ideals of L, i.e.*

(*i*) $\alpha^{\leftarrow}\alpha(\alpha^{\leftarrow}\alpha(I)) = \alpha^{\leftarrow}\alpha(I)$

(*ii*) $I \subseteq \alpha^{\leftarrow}\alpha(I)$,

(*iii*) $I \subseteq J$ *implies* $\alpha^{\leftarrow}\alpha(I) \subseteq \alpha^{\leftarrow}\alpha(J)$,

for any ideals I, J in L.

An ideal I is an called *α-ideal* if $\alpha^{\leftarrow}\alpha(I) = I$.

Thus α-ideals are simply the closed elements with respect to the closure operation of Proposition 3.1. From this propositon, the following is an immediate consequence.

PROPOSITION 3.2 *The α-ideals of a lattice L with* 0 *form a complete distributive* *lattice isomorphic to the lattice of filters, ordered by set-inclusion, of the lattice* $A_0(L)$ *of annulets of* L.

The infimum of a set of α-ideals J_i is $\cap J_i$, their set-theoretic intersection. The supremum is $\alpha^{\leftarrow}\alpha(\vee J)$ where $\vee J$ is their supremum in the lattice of ideals of L.

The following proposition gives equivalent conditions for an ideal to be an α-ideal.

PROPOSITION 3.3 *For an ideal* I *in a lattice* L *with* 0 *the following are equivalent*:

(*a*) I *is an* α*-ideal*,

(*b*) *for* $x, y \in L$, $(x]^* = (y]^*$ *and* $x \in I$ *implies* $y \in I$,

(*c*) $I = \bigcup_{x \in I}(x]^{**}$ (*here* $\bigcup$ = *set-theoretic union*),

(*d*) *for* $x, y \in L$, $h(x) = h(y)$ *and* $x \in I$ *implies* $y \in I$, *where* $h(\cdot)$ *is the hull of* $(\cdot)$ *with respect to the minimal prime ideals in* L.

PROOF. The equivalence of (a) and (b) is trivial.

(b) $\Rightarrow$ (c). If $x \in I$ and $y \in (x]^{**}$ then $(x]^* \subseteq (y]^*$ so

$$(y]^* = (x]^* \veebar (y]^* = (x \wedge y]^*$$

and $x \wedge y \in I$, so $y \in I$. That is, $\bigcup_{x \in I}(x]^{**} \subseteq I$ whence (c) follows.

(c) $\Rightarrow$ (b) is trivial.

(b) $\Leftrightarrow$ (d). Here we freely make use of results on minimal prime ideals and the space of minimal primes (under the hull-kernel topology) implied by [7]. Suppose $h(x) = h(y)$. Then $g(x) = g(y)$ where $g(\cdot)$ is the complement of $h(\cdot)$ in the set of minimal primes. Then

$$(x]^* = \cap\{P : P \in g(x)\} = \cap\{P : P \in g(y)\} = (y]^*$$

because the intersection of all the minimal primes of L is $(0]$. While if $(x]^* = (y]^*$ then

$$\cap\{P : P \in g(x)\} = \cap\{P : P \in g(y)\}$$

so $h(\cap\{P : P \in g(x)\}) = h(\cap\{P : P \in g(y)\})$ so $g(x) = g(y)$. Since $g(x)$, $g(y)$ are closed and $h(\cap\{P : P \in g(x)\})$ is the closure of $g(x)$ etc. Then $h(x) = h(y)$. *That is* $h(x) = h(y)$ *if and only if* $(x]^* = (y]^*$ *and* (b)$\Leftrightarrow$(d) *follows.*

Examples of α-ideals are provided by annihilator ideals, the ideal

$$O(P) = \{x \in L : x \wedge y = 0 \text{ for some } y \notin P\}$$

where P is a prime ideal, and minimal prime ideals (because of [7, Lemma 3.1]).

The following proposition is of intrinsic interest. It is not hard to establish if the criterion for a disjunctive lattice, as mentioned just before Proposition 2.3, is used. The proof is omitted.

PROPOSITION 3.4 *Let L be a lattice with* 0. *The following are equivalent*:

(*a*) *each prime ideal is an α-ideal*,

(*b*) *each ideal is an α-ideal*,

(*b*) *L is disjunctive*

Proposition 3.2 implies that there is an order isomorphism between the prime α-ideals and the prime filters of the lattice of annulets. Also it is not hard to show that each α-ideal is an intersection of prime α-ideals. We now come to our objectives but first we need a standard result. It was proved for bounded lattices in [9] and announced in general in [8]; an explicit proof is given in [6, p. 276].

LEMMA 3.5 *A lattice with* 0 *is relatively complemented if and only if every prime filter is an ultrafilter.*

THEOREM 3.6 *Let L be a lattice with* 0. *The following are equivalent*:

(*a*) *L is sectionally quasi-complemented*,

(*b*) *each prime α-ideal is a minimal prime ideal*,

(*c*) *each α-ideal is an intersection of minimal prime ideals.*

Moreover, the above conditions are equivalent to L being quasi-complemented if and only if there is an element $d \in L$ such that $(d]^* = (0]$.

PROOF. The equivalence of (a), (b) and (c) is an immediate consequence of Lemma 3.5 and the remarks immediately preceding it together with Proposition 2.7.

The remaining assertion follows from Proposition 2.4 and 2.7 or more simply from [3, Proposition 5.5]

THEOREM 3.7 *A lattice with* 0 (*resp. and* 1) *is a generalized Stone lattice* (*resp. Stone lattice*) *if and only if each prime ideal contains a unique prime α-ideal.*

PROOF. Since minimal prime ideals are α-ideals the condition in the theorem implies normality. It also implies L is sectionally quasi-complemented by Theorem 2.8. Observing this, the result follows immediately from [3, Theorems 5.6, 5.7].

Remarks 1. It is possible to prove the following result. For a lattice L with 0, $A_0(L)$ is isomorphic to the lattice $(H, \cap, \cup)$, where $H = \{h(x) : x \in L\}$ and $h(\cdot)$ is the hull with respect to the minimal primes. The isomorphism is the map $(x]^* \to h(x)$. This is easily shown to be a lattice homomorphism, and the slightly more troublesome fact that it is a bijection has already been established in the proof (b)$\Leftrightarrow$(d) of Proposition 3.3.

2. In view of the preceding remark and Proposition 3.3 we see that there is nothing very novel about either the lattice of annulets or the lattice of α-ideals. Generally constructions of ideals like α-ideals are carried out with respect to

maximal ideals. In fact Subramanian [10, section 4.3, p. 201] used this idea to obtain "h-ideals" with respect to the space of maximal l-ideals in an f-ring. Of course our α-ideals and his h-ideals were both suggested by the "z-ideals" of Gillman and Jerison [4, Chapter 2] where one is, implicitly, taking hulls with respect to the space of maximal ideals of $C(X)$.

3. Bigard [2] has also studied α-ideals in the context of lattice-ordered groups. His definition is in terms of (b) of Proposition 3.3 – of course for him $(x]^*$ is replaced by the polar $\{y : |y| \wedge |x = 0\}$. Instead of annulets he uses the dual lattice of carriers. He gives some interesting results, all without proof, including an analogue of Theorem 3.7. Though we arrived at the notion of α-ideal independently of Bigard, Theorem 3.7 was directly suggested by his assertion.

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