

INVOLUTORY AUTOMORPHISMS OF GROUPS OF ODD ORDER AND THEIR FIXED POINT GROUPS

L. G. KOVÁCS AND G. E. WALL

In Memoriam TADASI NAKAYAMA

1. Introduction. Let G be a finite group of odd order with an automorphism θ of order 2. (We use without further reference the fact, established by W. Feit and J. G. Thompson, that all groups of odd order are soluble.) Let G_θ denote the subgroup of G formed by the elements fixed under θ . It is an elementary result that if $G_\theta = 1$ then G is abelian. But if we merely postulate that G_θ be cyclic, the structure of G may be considerably more complicated—indeed G may have arbitrarily large soluble length. E.g. let p be an odd prime and let t denote the largest odd divisor of $p - 1$. Let G be the group formed by the matrices $A = \begin{pmatrix} u + pa & pb \\ pc & v + pd \end{pmatrix}$ of determinant 1, where a, b, c, d, u, v lie in the ring of residue classes (mod p^{k+1}) and $uv \equiv u^t \equiv 1 \pmod{p}$. Let θ be the contragredient automorphism $A \rightarrow (A^{-1})^t$. Then G_θ is cyclic of order p^k . G itself has order $p^{3k}t$ and soluble length m or $m + 1$, where m is the least integer such that $2^m \geq k + 1$. The Fitting subgroup of G is a p -group of order p^{3k} , exponent p^k , and class k .

The theorem proved in this note deals with the case where G_θ is nilpotent. It belongs to the same circle of ideas as the recent results of J. G. Thompson [5], though it is much more special. Let $F(H)$ denote the Fitting subgroup of a group H and $1 = F_0(H) \leq F_1(H) \leq \dots$ the ascending Fitting series of H , defined inductively by $F_{i+1}(H)/F_i(H) = F(G/F_i(H))$.

Theorem Let G be a group of odd order and θ an automorphism of G of order 2 such that G_θ is nilpotent. Suppose either (i) $G = F_2(G)$ or (ii) the Sylow subgroups of G_θ are regular. Then $G/F(G)$ is contained in the variety V generated by G_θ together with the cyclic subgroups of $G/F(G)$.

The following particular case may serve as an illustration of the Theorem:

Received March, 29, 1965.

if G is a group of odd order having an automorphism of order 2 with abelian fixed point group, then G' is nilpotent. The example cited above shows that, even in this case, $G/F(G)$ need not belong to the variety generated by G_0 .

Conditions (i), (ii) in the Theorem are not superfluous, i.e. the nilpotency of G_0 does not imply that $G = F_2(G)$. To see this, let Γ be the group, of order $2 \cdot 3^7 \cdot 7$, formed by the transformations $x \rightarrow ax^\alpha + b$ of the field $K = GF(3^6)$, where a runs over the 7-th roots of unity in K , α over the automorphisms of K and b over all the elements of K . Let G be the subgroup of Γ of index 2 and θ the automorphism $X \rightarrow T^{-1}XT$ of G , where T is the transformation $x \rightarrow x^{27}$. Then G_0 is a (non-regular) 3-group whereas $F_2(G)$ is a subgroup of G of index 3.

On the other hand, the nilpotency of G_0 implies, by the results of Thompson [5], that $G' \leq F_3(G)$ and thus that $G = F_4(G)$. J. N. Ward has proved [6] that, in fact, the nilpotency of G_0 implies $G = F_3(G)$.

We mention in conclusion another unpublished result of Ward dealing with the derived length of $G/F(G)$: if G_0 is nilpotent and $(G_0)^{(n)} = 1$, then $G^{(n)}$ is nilpotent. The point of Ward's result is that it deals with the case where $G/F(G)$ is not necessarily nilpotent.

2. Notation. Our general notation is standard and only the following points need to be mentioned. $M \leq N$ ($M \trianglelefteq N$) means that M is a subgroup (normal subgroup) of the group N . $M < N$ ($M \triangleleft N$) means that M is a subgroup (normal subgroup) of N distinct from N . If $L \trianglelefteq M \leq N$, M/L is called a factor of N .

Throughout the paper, G denotes a group of odd order and θ an automorphism of G of order ≤ 2 such that G_0 is nilpotent. Γ denotes the splitting extension of G by θ . \mathbf{V} denotes the variety generated by G_0 together with the cyclic subgroups of $G/F(G)$; note that every group in \mathbf{V} is nilpotent. If H is a group, $H/\mathbf{V}(H)$ denotes the largest factor group of H belonging to \mathbf{V} . As is well known, $\mathbf{V}(K) \leq \mathbf{V}(H)$ if $K \leq H$, $\mathbf{V}(H/N) = \mathbf{V}(H)N/N$ if $N \trianglelefteq H$, and $\mathbf{V}(H \times K) = \mathbf{V}(H) \times \mathbf{V}(K)$.

A subgroup H of G is called a θ -subgroup if $H = H^\theta$. A factor H/K of G is called a θ -factor if H, K are θ -subgroups. The automorphism θ acts in a natural way as an automorphism of order ≤ 2 on each θ -factor H/K . The subgroup formed by those elements of H/K which are fixed under θ is denoted by $(H/K)_0$.

LEMMA 1. *Let H/K be a θ -factor of G and xK an element of H/K such*

that $(xK)^0 = (xK)^\varepsilon$ ($\varepsilon = \pm 1$). Then xK contains an element y such that $y^0 = y^\varepsilon$.

Proof. The mapping $u \rightarrow (u^\varepsilon)^0$ is a permutation of order ≤ 2 of the set xK of odd order; take y as any of the elements fixed under this permutation.

COROLLARY. $(H/K)_0 = H_0K/K$; thus $(H/K)_0$ is isomorphic to a factor of G_0 .

3. The present section leads up to the proof of the Theorem in Section 4. We derive some properties of the Fitting series of G under the following assumptions:

(3.1) $G/F(G) \in \mathbf{V}$;

(3.2) if K is a θ -factor of G different from G , then $K/F(K) \in \mathbf{V}$.

Set $F = F(G)$, $F_i = F_i(G)$, and $\Phi = \Phi(G)$, the Frattini subgroup of G .

LEMMA 2. F is the unique minimal normal subgroup of Γ . F is its own centralizer $C_\Gamma(F)$ in Γ .

Proof. Suppose $F(\Gamma) \neq F$. Then $|F(\Gamma) : F| = 2$ and the Sylow 2-subgroup of $F(\Gamma)$ (of order 2) is normal in Γ . Thus $\Gamma = \mathbf{gp}\langle \theta \rangle \times G$ and so $G = G_0$, contrary to (3.1). Hence $F(\Gamma) = F$. As Γ is soluble, a result of H. Fitting [2] gives that

(3.3) $C_\Gamma(F) = C_\Gamma(F(\Gamma)) \leq F$.

Let M_1, M_2 be minimal normal subgroups of Γ . Then $M_i \leq F(\Gamma) \leq G$. By (3.2), $\mathbf{V}(G/M_i)$ is nilpotent. Since $G/M_1 \cap M_2$ is isomorphic to a subgroup of $G/M_1 \times G/M_2$, $\mathbf{V}(G/M_1 \cap M_2)$ is also nilpotent; hence, by (3.1), $M_1 \cap M_2 > 1$ and so $M_1 = M_2$. It follows that $M_1 = M_2 = M$ (say) is the unique minimal subgroup of Γ . As Γ is soluble, M is an elementary abelian p -group for some prime p .

By a result of W. Gaschütz [3], $F(G/\Phi) = F/\Phi$. Since $\mathbf{V}(G) \not\leq F$ (by (3.1)), also $\mathbf{V}(G/\Phi) \not\leq F/\Phi$ and therefore $\Phi = 1$ (by (3.2)). As $\Phi(F) \leq \Phi$, F is a direct product of elementary abelian groups. Since M is the unique minimal normal subgroup of Γ , F is in fact an elementary abelian p -group.

Consider $H/M = F(G/M)$. If P/M is the Sylow p -subgroup of H/M , then P is a normal p -subgroup of G : hence $P \leq F$. But clearly $F \leq P$, so that $F = P$. Since $\mathbf{V}(G) \leq H$ (by (3.2)) and $\mathbf{V}(G) \not\leq F$ (by (3.1)), we have

(3.4) $F < H$.

Now F is a normal, elementary abelian Sylow subgroup of H . Therefore

F is a completely reducible (H/F) -module. It follows that $F = M \times N$ with $N \triangleleft H$. Since H/M is nilpotent, H/F acts trivially on N . As F is abelian, we deduce that $N \leq Z(H)$. If $Z(H) > 1$, then $M \leq Z(H)$ and so $F = M \times N \leq Z(H)$, contrary to (3.3) and (3.4). Thus $Z(H) = 1$, $N = 1$, $F = M$, which proves the lemma.

Restating the lemma in module terminology, we have

COROLLARY 1. *F is an elementary abelian p -group for some prime p . F is a faithful, irreducible (Γ/F) -module.*

COROLLARY 2. $F_0 > 1$.

Proof. Suppose $F_0 = 1$. Then θ must invert all the elements of F . Since F is a faithful (Γ/F) -module, it follows that $\theta F \in Z(\Gamma/F)$ and so $(G/F)_0 = G/F$. By the Corollary to Lemma 1, this implies that $G/F \in \mathbf{V}$, contrary to (3.1).

COROLLARY 3. *If x is a p' -element of G_0 (i.e. an element of G_0 of order prime to p), then $C_F(x) \geq F_0 > 1$.*

Proof. The inclusion $C_F(x) \geq F_0$ follows directly from the nilpotency of G_0 .

LEMMA 3. *Let K/F be a θ -subgroup of G/F distinct from G/F . Suppose there exists a normal θ -subgroup H/F of G/F such that $\mathbf{V}(K) \leq H \leq K$. Then $\mathbf{V}(K/F) = 1$.*

Proof. Since $K = K^0 < G$, $\mathbf{V}(K) \leq F(K)$ by (3.2). Hence $\mathbf{V}(K) \leq F(K) \cap H$. Since $H \leq K$ and $F(K) \cap H \trianglelefteq K$, we have $F(K) \cap H \trianglelefteq H$ and so $F(K) \cap H \leq F(H)$. On the other hand, since $H \trianglelefteq G$, $F(H) \leq F$. It follows that $\mathbf{V}(K) \leq F(K) \cap H \leq F$ and thus that $\mathbf{V}(K/F) = 1$.

COROLLARY. *If H/F is a proper normal θ -subgroup of G/F , then $\mathbf{V}(H/F) = 1$.*

LEMMA 4. F_2/F is the unique minimal normal subgroup of Γ/F . G/F_2 has prime order r and F_2/F is an elementary abelian q -group for some prime $q \neq p$ or r . $Z(G/F) = 1$.

Proof. By the last Corollary, every proper normal θ -subgroup of G/F is nilpotent. Hence either F_2/F is the unique maximal such subgroup or $F_2 = G$. In the former case, F_2/G' is the unique maximal θ -subgroup of G/G' ; it follows readily that G/F_2 has prime order, say r , and that $G/F = \langle \xi, F_2/F \rangle$, where ξ has order $r^k \geq r$ and $\xi^0 = \xi^{\pm 1}$.

Case 1: $Z(G/F) = 1$. Evidently $F_2 < G$. Let R/F be a Sylow r -subgroup of

G/F which contains ξ . If $|R/F| > r$, then $(R/F) \cap (F_2/F) > 1$ and so $1 < Z(R/F) \cap (F_2/F) \leq Z(G/F)$, contrary to assumption. Thus $k = 1$ and F_2/F is an r' -group. Let T/F be a minimal normal subgroup of Γ/F contained in F_2/F . T/F is an elementary abelian q -group for some prime q . Since F_2/F is an r' -group, $q \neq r$. Since $F = F(G)$, $q \neq p$. It remains only to prove that $T = F_2$.

Suppose $T < F_2$. Set $K = \langle \xi, T/F \rangle$. Since ξ has order r , $K/F < G/F$ and since $\xi^0 = \xi^{\pm 1}$, K/F is a θ -subgroup of G/F . By Lemma 3, K/F is nilpotent. In particular, ξ centralizes T/F . Then, since $T/F \leq Z(F_2/F)$, we have $T/F \leq Z(G/F)$, contrary to the assumption that $Z(G/F) = 1$. Thus $T = F_2$ as required.

Case 2: $Z(G/F) > 1$. We have to prove that this assumption leads to a contradiction. Choose $\zeta \in Z(G/F)$ such that $\zeta^0 = \zeta^{\pm 1} \neq 1$. By Lemma 1, $\zeta = zF$, where $z^0 = z^{\pm 1}$. Since $\langle \zeta \rangle \Gamma/F, C_F(z)$ is a normal subgroup of Γ contained in F . By Lemma 2, $C_F(z) = 1$. Then, by Corollary 3 to Lemma 2, $z^0 = z^{-1}$.

Let $F_{-\theta}$ denote the group formed by the elements of F inverted by θ ; then $F = F_0 \times F_{-\theta}$. If $x \in F_{\pm\theta} \cap (F_{\pm\theta})^z$, then $x_{\pm}^{z^{-1}} = (x^{\pm 1})^0 = (x^0)^z = x^{\pm z}$, so that $x \in C_F(z^2) = 1$; hence $F_{\pm\theta} \cap (F_{\pm\theta})^z = 1$. It follows that $|F_{\pm\theta}|^2 = |F_{\pm\theta} \times (F_{\pm\theta})^z| \leq |F|$. Therefore, since $F = F_0 \times F_{-\theta}$, $|F_0| = |F_{-\theta}| = |F|^{1/2}$ and $F = F_0 \times (F_0)^z$.

Let t be a p' -element of G_0 . By Corollary 3 to Lemma 2, $F_0 \leq C_F(t)$. Then $(F_0)^z \leq C_F(t)^z = C_F(t^z) = C_F(t)$, since $zF \in Z(G/F)$. Hence $F = F_0 \times (F_0)^z \leq C_F(t)$ and so, by Lemma 2, $t = 1$. Thus G_0 is a p -group. Since F_2/F is a p' -group, $(F_2/F)_0 = 1$. Thus θ inverts all elements of F_2/F and F_2/F is abelian. If G/F were a p' -group, the same argument would prove G/F abelian, contrary to (3.1). Hence $F_2 < G$ and $r = p$. Also, since F_2/F is a p' -group, ξ has order p .

Set $Q = F_2/F, S = Z(G/F), T = \theta(S)$. Clearly, ξ does not centralize Q . On the other hand, the argument used in Case 1 shows that ξ centralizes every proper subgroup of Q which is normal in Γ/F . Thus S is the unique maximal such subgroup. It follows that Q is a q -group for some prime $q (\neq p)$. Also, since $Q/\theta(Q)$ is a completely reducible (Γ/F_2) -module, it is in fact an irreducible module and thus $\theta(Q) = S$. Since an automorphism of Q which leaves the elements of $Q/\theta(Q)$ fixed has q -power order, ξ does not centralize Q/S .

Now, since Q is abelian, the mapping $xS \rightarrow x^q T (x \in Q)$ is a (Γ/F_2) -module epimorphism from Q/S to S/T . Since Q/S is irreducible and $S/T > 1$, Q/S and S/T are isomorphic modules. Since ξ does not centralize Q/S , it does not

centralize S/T . This is a contradiction because $S = Z(G/F)$. The proof is now complete.

4. We are now in a position to prove the Theorem. Let G be a counter-example of least order. Using the Corollary to Lemma 1, we see easily that G satisfies (3.1) and (3.2). Thus the conclusions of all the lemmas in Section 3 hold. In particular, $G/F = gp\langle \xi, F_2/F \rangle$, where $\xi^r = 1$ and $\xi^0 = \xi^{\pm 1}$.

By Corollary 1 to Lemma 2, F provides a faithful, irreducible representation, say f , of Γ/F over the prime field $GF(p)$. Qua representation over the algebraic closure, k , of $GF(p)$, f splits into a direct sum of (absolutely) irreducible representations (cf Curtis and Reiner [1], § 70). Let V be a representation module (over k) corresponding to one of these irreducible parts. We use the customary (additive) module notation in V , writing scalars on the left, and group elements on the right, of vectors. If $x \in \Gamma/F$, V_x denotes the set of vectors fixed by x .

Since the absolutely irreducible parts of f are all algebraically conjugate (Curtis and Reiner l.c.), they all have the same kernel. Thus

$$(4.1) \quad V \text{ is a faithful, irreducible } (\Gamma/F)\text{-module.}$$

Using also Corollaries 2 and 3 of Lemma 2, we get

$$(4.2) \quad V_0 > 0;$$

$$(4.3) \quad \text{if } x \text{ is a } p^l\text{-element of } (G/F)_0, \text{ then } V_0 \leq V_x.$$

Case 1: V is a reducible (G/F) -module. Let W be an irreducible (G/F) -submodule of V , so that $0 < W < V$. Then $W\theta$ is also a (G/F) -submodule of V ; hence $W \cap W\theta$ and $W + W\theta$ are (Γ/F) -submodules and so $V = W \oplus W\theta$. Let x be a p^l -element of $(G/F)_0$. If $w \in W$, then $w(1 + \theta) \in V_0$ and so, by (4.3), $w(1 + \theta) = w(1 + \theta)x$. Hence $w(x - 1) = -w\theta(x - 1) = -w(x - 1)\theta \in W \cap W\theta = 0$, which shows that $w \in V_x$ and $w\theta \in V_x$. Thus $V_x = V$ and so, by (4.1), $x = 1$. This proves that $(G/F)_0$ is a p -group. Hence $(G/F)_0 \cap (F_2/F) = 1$. But $(G/F)_0 > 1$, for otherwise G/F would be abelian, contrary to (3.1). It follows that $r = p$ and $\xi^0 = \xi$.

Now $G > F_2$, so that hypothesis (ii) of the Theorem holds. Hence $V_0(\xi - 1)^{p-1} = 0$. By the argument used above (with $(\xi - 1)^{p-1}$ in place of $(x - 1)$), we deduce that $V(\xi - 1)^{p-1} = 0$. But this is contrary to Theorem *B* of Hall and Higman [4].

Case 2: V is an irreducible (G/F) -module. Let U be an irreducible (F_2/F) -submodule of V . Then U is a one-dimensional subspace, spanned by the vector u , say; also, $u\eta = \chi(\eta)u$ for $\eta \in F_2/F$, where χ is a character of F_2/F . Set $u_i = u\xi^{-i}$, $U_i = U\xi^{-i}$. Since $u_i\eta = u\eta^{\xi^i}\xi^{-i} = \chi(\eta^{\xi^i})u_i$, U_i is an (F_2/F) -submodule with corresponding character $\chi_i(\eta) = \chi(\eta^{\xi^i})$. Then $U_0 + \dots + U_{r-1}$ is a (G/F) -submodule, so that $U_0 + \dots + U_{r-1} = V$. Therefore, since V is a faithful module, χ is not the trivial character 1.

We prove next that $\chi_0, \dots, \chi_{r-1}$ are all different. If not, then $\chi_i = \chi_j$ ($i \neq j$) and this gives $\chi(\eta^{\xi^i - \xi^j}) = 1$ for all $\eta \in F_2/F$. Now the kernel of the endomorphism $\eta \rightarrow \eta^{\xi^i - \xi^j}$ of F_2/F is $C_{F_2/F}(\xi^{i-j}) = Z(G/F) = 1$. Therefore the image of this endomorphism is the whole of F_2/F and so our equations give $\chi = 1$, a contradiction. This proves the assertion. It follows that U_0, \dots, U_{r-1} are the only one-dimensional (F_2/F) -submodules of V and that $V = U_0 \oplus U_1 \oplus \dots \oplus U_{r-1}$.

Since $(u_i\theta)\eta = \chi_i(\eta^\theta)(u_i\theta)$, θ permutes the U_i . Since the number, r , of the U_i is odd, θ leaves at least one U_i invariant. We may suppose $U\theta = U$.

Since U affords a 1-dimensional representation of $gp\langle\theta, F_2/F\rangle$ and F_2/F is not contained in the kernel of this representation, the derived group of $gp\langle\theta, F_2/F\rangle$ is a proper subgroup of F_2/F and so $(F_2/F)_\theta > 1$. Let $1 \neq \zeta \in (F_2/F)_\theta$. If $\xi^\theta = \xi$, then ξ, ζ are elements of coprime order in the nilpotent group $(G/F)_\theta$ and so $\xi\zeta = \zeta\xi$. But this implies that $\zeta \in Z(G/F)$, contrary to Lemma 4. Therefore $\xi^\theta = \xi^{-1}$, and consequently $U_i\theta = U_{-i}$ for every i .

The argument used in Case 1 shows that $u_i \in V_\zeta$ whenever $U_i\theta \neq U_i$. Hence

$$(4.4) \quad u_i \in V_\zeta \quad \text{for } 1 \leq i \leq r-1.$$

Since $u_i = u\xi^{-i}$, this gives

$$(4.5) \quad u\xi^{\xi^i} = u \quad \text{for } 1 \leq i \leq r-1.$$

Now, since $\eta \rightarrow \eta^{\xi^{-1}}$ is an automorphism of F_2/F and $1 = \zeta^{\xi^r-1} = \zeta^{(1+\xi+\dots+\xi^{r-1})(\xi-1)}$, we have $\zeta^{1+\xi+\dots+\xi^{r-1}} = 1$. With (4.5), this gives $u\zeta = u$. Hence, by (4.4), $V_\zeta = V$. Since this is contrary to (4.1), the proof is complete.

REFERENCES

[1] C. W. Curtis and I. Reiner, Representation theory of finite groups and associative algebras. Interscience, New York, 1962.
 [2] H. Fitting, Beiträge zur Theorie der Gruppen endlicher Ordnung. Jber. Deutsch.

- Math.-Verein. **48** (1938), 77-141.
- [3] W. Gaschütz, Über die Φ -Untergruppe endlicher Gruppen. Math. Z. **58** (1953), 160-170.
- [4] P. Hall and Graham Higman, On the p -length of p -soluble groups and reduction theorems for Burnside's problem. Proc. London Math. Soc. (3) **6** (1956), 1-42.
- [5] J. G. Thompson, Automorphisms of solvable groups. J. Algebra **1** (1964), 259-267.
- [6] J. N. Ward, Involutory automorphisms of groups of odd order (to appear in J. Australian Math. Soc.).

*Australian National University
and University of Sydney*