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GROWTH THEOREMS FOR HOMOGENEOUS SECOND-ORDER DIFFERENCE EQUATIONS

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Abstract

In this paper we investigate the boundedness and asymptotic behaviour of the solutions of a class of homogeneous second-order difference equations with a single non-constant coefficient. These equations model, for example, the amplitude of oscillation of the weights on a discretely weighted vibrating string. We present several growth theorems. Two examples are also given.

1. Introduction

In this paper we shall study the second-order linear difference equation of the form

$$x_{n+1} + b_n x_n + x_{n-1} = 0, \quad n \in \mathbf{N}, \tag{1}$$

where x_n is the desired solution and b_n is a given real sequence. We shall investigate the boundedness and asymptotic behaviour of the solution of (1).

This equation models, for example, the amplitude of oscillation of the weights on a discretely weighted vibrating string [1, pp. 15–17].

Results for similar problems for second-order differential equations can be found in [3].

If $b_n = -2$, $n \in \mathbf{N}$, we have $x_{n+1} - 2x_n + x_{n-1} = 0$. This equation has a general solution in the form $an + b$, where a, b are arbitrary real numbers and thus has unbounded solutions. It is well-known that if $b_n = d \in \mathbf{R}$, $n \in \mathbf{N}$, where $d \geq 2$ or $d \leq -2$, then the equation also has unbounded solutions. This motivates us to investigate the cases when $-2 < b_n < 2$, $n \in \mathbf{N}$, especially when $b_n \to 0$ as $n \to \infty$.

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The following "symmetry principle" is very useful in the consideration of (1). If we make the change $y_n = (-1)^n x_n$, (1) becomes

$$(-1)^{n+1}(y_{n+1} - b_n y_n + y_{n-1}) = 0, \quad n \in \mathbf{N},$$

that is,

$$y_{n+1} - b_n y_n + y_{n-1} = 0, \quad n \in \mathbf{N}.$$

Thus it is enough to investigate the cases when $-2 < b_n < 0$, $n \in \mathbf{N}$. For example, if we show that, under some conditions, when $-2 < b_n < 0$, $n \in \mathbf{N}$, (1) has either bounded or unbounded solutions, then this also holds for (1) when $0 < b_n < 2$, $n \in \mathbf{N}$.

Using the substitution $b_n = -2/(1 + c_n)$ we may transform the relation of interest into

$$x_{n+1} - 2x_n + x_{n-1} + c_n(x_{n+1} + x_{n-1}) = 0, \quad n \in \mathbf{N}, \tag{2}$$

which is in a more suitable form for the calculations which follow (see also [8]).

2. Preliminaries

For investigation of the boundedness and asymptotic behaviour of the solution x_n, we will need a few auxiliary lemmas. The first of these is a discrete variant of the Bellman-Gronwall lemma. The continuous case of this lemma can be found in [2] and [6].

LEMMA 1. *If $x_n, c_n \ge 0$, c is a positive constant and $x_n \le c + \sum_{i=1}^{n-1} c_i x_i$, $n \in \mathbf{N}$, then $x_n \le c \exp\left(\sum_{i=1}^{n-1} c_i\right)$, $n \in \mathbf{N}$.*

Proof of this lemma and further generalizations can be found in [4] (see also [8]).

LEMMA 2. *Let c_n, $n \in \mathbf{N} \cup \{0\}$, be a positive sequence and x_n be a solution of the difference equation* (2). *Then*

$$(x_{n+1} - x_n)^2 + c_n x_{n+1}^2 + c_{n-1} x_n^2 = (x_1 - x_0)^2 + c_1 x_0^2 + c_0 x_1^2 + \sum_{i=1}^{n-1} (c_{i+1} - c_{i-1}) x_i^2, \quad n \in \mathbf{N}. \tag{3}$$

PROOF. Multiplying (2) by $x_{n+1} - x_{n-1} = x_{n+1} - x_n + x_n - x_{n-1}$ we get

$$(x_{n+1} - x_n)^2 - (x_n - x_{n-1})^2 + c_n(x_{n+1}^2 - x_{n-1}^2) = 0, \quad n \in \mathbf{N}. \tag{4}$$

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It follows from (4) that

$$\sum_{i=1}^{n}[(x_{i+1}-x_i)^2-(x_i-x_{i-1})^2]+\sum_{i=1}^{n}c_i(x_{i+1}^2-x_{i-1}^2)=0.$$

Hence for all $n \in \mathbf{N}$ we obtain (3).

3. Main results

We are now in a position to formulate and to prove the main results of this paper. From hereafter we shall exclude the trivial solution from our considerations.

THEOREM 1. *Let c_n, $n \in \mathbf{N} \cup \{0\}$, be a positive nonincreasing sequence and x_n be a solution of* (2). *Then the sequences $x_{n+1}-x_n$ and $c_{n-1}x_n^2$ are bounded. Further, if* $\lim_{n\to\infty} c_n > 0$, *then x_n is bounded.*

PROOF. From (3) we have

$$(x_{n+1}-x_n)^2+c_nx_{n+1}^2+c_{n-1}x_n^2 \le (x_1-x_0)^2+c_1x_0^2+c_0x_1^2$$

since $\sum_{i=1}^{n-1}(c_{i+1}-c_{i-1})x_i^2 \le 0$. From that we have the first part of our theorem. In particular, $c_{n-1}x_n^2 \le (x_1-x_0)^2+c_1x_0^2+c_0x_1^2 = M$, for all $n \in \mathbf{N}$. Therefore if $\lim_{n\to\infty} c_n > 0$, we have

$$x_n^2 \le \frac{M}{c_{n-1}} \le \frac{M}{\lim_{n\to\infty} c_n} < +\infty.$$

Thus the second part of our theorem follows.

THEOREM 2. *Let c_n, $n \in \mathbf{N} \cup \{0\}$, be a positive nondecreasing sequence such that $c_n \ge \delta$ for $n \ge n_0$ and x_n be a solution of* (2). *Then* $\limsup_{n\to\infty} c_nx_n^2 > 0$.

PROOF. Without loss of generality we may suppose that $n_0 = 1$. From (3) we get

$$(x_{n+1}-x_n)^2+c_nx_{n+1}^2+c_{n-1}x_n^2 \ge (x_1-x_0)^2+c_1x_0^2+c_0x_1^2.$$

On the other hand $(x_1-x_0)^2+c_1x_0^2+c_0x_1^2 > 0$, since we may suppose that x_0 and x_1 are not both equal to zero at the same time.

By the inequality between the arithmetic and geometric means and since $c_n \ge \delta$, we have

$$(x_{n+1}-x_n)^2 \le 2(x_{n+1}^2+x_n^2) \le \frac{2}{\delta}(c_nx_{n+1}^2+c_{n-1}x_n^2), \quad n \in \mathbf{N}.$$

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From all of the above we obtain

$$\left(1+\frac{2}{\delta}\right)(c_n x_{n+1}^2 + c_{n-1}x_n^2) \geq (x_1 - x_0)^2 + c_1 x_0^2 + c_0 x_1^2 > 0, \quad n \in \mathbf{N}.$$

Letting $n \to \infty$ in the last inequality we obtain that

$$\left(2+\frac{4}{\delta}\right)\limsup_{n\to\infty} c_n x_n^2 \geq (x_1 - x_0)^2 + c_1 x_0^2 + c_0 x_1^2 > 0$$

Thus the theorem follows.

The following theorem was essentially proved in [8]. In order to make this paper more complete we shall present its proof here.

THEOREM 3. *Let c_n, $n \in \mathbf{N} \cup \{0\}$, be a sequence such that $c_n \geq \delta > 0$, $n \in \mathbf{N}$ and $\sum_{i=1}^{+\infty} |c_{i+1} - c_{i-1}| < \infty$. Then all the solutions of* (2) *are bounded.*

PROOF. From (3) we have

$$c_{n-1}x_n^2 \leq (x_1 - x_0)^2 + c_1 x_0^2 + c_0 x_1^2 + \sum_{i=1}^{n-1} |c_{i+1} - c_{i-1}| x_i^2, \quad n \in \mathbf{N}$$

since $c_i > 0$.

Since $c_n \geq \delta > 0$, $n \in \mathbf{N}$, by the Bellman-Gronwall lemma we get

$$x_n^2 \leq \frac{(x_1 - x_0)^2 + c_1 x_0^2 + c_0 x_1^2}{\delta} \exp\left(\frac{1}{\delta}\sum_{i=1}^{n-1} |c_{i+1} - c_{i-1}|\right).$$

Hence

$$x_n^2 \leq \frac{(x_1 - x_0)^2 + c_1 x_0^2 + c_0 x_1^2}{\delta} \exp\left(\frac{1}{\delta}\sum_{i=1}^{+\infty} |c_{i+1} - c_{i-1}|\right) < +\infty.$$

Therefore all solutions of (2) are bounded.

THEOREM 4. *Let c_n, $n \in \mathbf{N} \cup \{0\}$, be a positive nondecreasing sequence such that $1 \leq m \leq c_{n+1}/c_n \leq M < \infty$, $n \in \mathbf{N} \cup \{0\}$ and x_n be a solution of* (2). *Then*

$$x_n^2 = \mathcal{O}\left(c_n^{M(M+1)-1}\right) \quad \textit{and} \quad x_n^2 = \mathcal{O}\left(c_n^{M^p/m^{p-2}+M^p/m^{p-1}-1}\right)$$

for each $p \geq 2$, $p \in \mathbf{N}$.

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PROOF. From (3) we have

$$c_{n-1}x_n^2 \le (x_1 - x_0)^2 + c_1 x_0^2 + c_0 x_1^2 + \sum_{i=1}^{n-1}(c_{i+1} - c_{i-1})x_i^2, \quad n \in \mathbf{N}$$

since $c_i > 0$.

The last inequality can be writen in the form

$$c_{n-1}x_n^2 \le C + \sum_{i=1}^{n-1} \frac{(c_{i+1} - c_{i-1})}{c_{i+p-1}} c_{i+p-1} x_i^2, \tag{5}$$

where $C = (x_1 - x_0)^2 + c_1 x_0^2 + c_0 x_1^2$ and $p \in \mathbf{N}$ is fixed.

Since $c_n > 0$, (5) is equivalent to

$$c_{n+p-1}x_n^2 \prod_{i=n}^{n+p-1} \frac{c_{i-1}}{c_i} \le C + \sum_{i=1}^{n-1} \frac{(c_{i+1} - c_{i-1})}{c_{i+p-1}} c_{i+p-1} x_i^2,$$

that is,

$$\begin{aligned} c_{n+p-1}x_n^2 &\le \prod_{i=n}^{n+p-1} \frac{c_i}{c_{i-1}} \left(C + \sum_{i=1}^{n-1} \frac{(c_{i+1} - c_{i-1})}{c_{i+p-1}} c_{i+p-1} x_i^2 \right) \\ &\le M^p \left(C + \sum_{i=1}^{n-1} \frac{(c_{i+1} - c_{i-1})}{c_{i+p-1}} c_{i+p-1} x_i^2 \right). \end{aligned}$$

By the condition of the theorem and the discrete Bellman-Gronwall lemma we obtain

$$c_{n+p-1}x_n^2 \le C \exp\left(M^p \sum_{i=1}^{n-1} \frac{c_{i+1} - c_{i-1}}{c_{i+p-1}} \right)$$

since $c_{i+1} - c_{i-1} \ge 0$.

Let us estimate the sum $\sum_{i=1}^{n-1}(c_{i+1} - c_{i-1})/c_{i+p-1}$. First, we note that

$$\sum_{i=1}^{n-1} \frac{c_{i+1} - c_{i-1}}{c_{i+p-1}} = \sum_{i=1}^{n-1} \frac{c_{i+1} - c_i}{c_{i+1}} \prod_{j=1}^{p-2} \frac{c_{i+j}}{c_{i+j+1}} + \sum_{i=1}^{n-1} \frac{c_i - c_{i-1}}{c_i} \prod_{j=1}^{p-1} \frac{c_{i+j-1}}{c_{i+j}}.$$

By the condition of the theorem we have $c_i/c_{i+1} \le 1/m$ for all $i \in \mathbf{N}$. Thus, for $p \ge 2$, we have

$$\sum_{i=1}^{n-1} \frac{c_{i+1} - c_{i-1}}{c_{i+p-1}} \le \sum_{i=1}^{n-1} \frac{c_{i+1} - c_i}{c_{i+1}} \frac{1}{m^{p-2}} + \sum_{i=1}^{n-1} \frac{c_i - c_{i-1}}{c_i} \frac{1}{m^{p-1}} \tag{6}$$

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and for $p = 1$,

$$\sum_{i=1}^{n-1} \frac{c_{i+1} - c_{i-1}}{c_i} \le \sum_{i=1}^{n-1} \frac{c_{i+1} - c_i}{c_{i+1}} M + \sum_{i=1}^{n-1} \frac{c_i - c_{i-1}}{c_i}.$$

On the other hand, we have

$$\sum_{i=1}^{n-1} \frac{c_{i+1} - c_i}{c_{i+1}} \le \int_{c_1}^{c_n} \frac{dx}{x} \tag{7}$$

and

$$\sum_{i=1}^{n-1} \frac{c_i - c_{i-1}}{c_i} \le \int_{c_0}^{c_{n-1}} \frac{dx}{x}. \tag{8}$$

By (6)–(8), we get

$$\begin{aligned} \sum_{i=1}^{n-1} \frac{c_{i+1} - c_{i-1}}{c_{i+p-1}} &\le \frac{1}{m^{p-2}} \int_{c_1}^{c_n} \frac{dx}{x} + \frac{1}{m^{p-1}} \int_{c_0}^{c_{n-1}} \frac{dx}{x} \\ &= \frac{1}{m^{p-2}}(\ln c_n - \ln c_1) + \frac{1}{m^{p-1}}(\ln c_{n-1} - \ln c_0), \quad \text{for } p \ge 2, \end{aligned}$$

and

$$\sum_{i=1}^{n-1} \frac{c_{i+1} - c_{i-1}}{c_i} \le M(\ln c_n - \ln c_1) + (\ln c_{n-1} - \ln c_0), \quad \text{for } p = 1.$$

Hence, since c_n is nondecreasing, we have

$$\sum_{i=1}^{n-1} \frac{c_{i+1} - c_{i-1}}{c_{i+p-1}} \le \left(\frac{1}{m^{p-2}} + \frac{1}{m^{p-1}}\right) \ln c_n + C, \quad \text{for } p \ge 2$$

and

$$\sum_{i=1}^{n-1} \frac{c_{i+1} - c_{i-1}}{c_i} \le (M+1) \ln c_n + C, \qquad \text{for } p = 1.$$

From all of the above we get

$$c_{n+p-1} x_n^2 \le C \exp\left(M^p \left(\frac{1}{m^{p-2}} + \frac{1}{m^{p-1}}\right) \ln c_n + C\right), \quad \text{for } p \ge 2$$

and $c_n x_n^2 \le C \exp(M(M+1) \ln c_n + C)$, for $p = 1$. Thus we have

$$c_{n+p-1} x_n^2 \le C c_n^{M^p/m^{p-2} + M^p/m^{p-1}} \le C c_{n+p-1}^{M^p/m^{p-2} + M^p/m^{p-1}}, \quad \text{for } p \ge 2,$$

that is,

$$x_n^2 \le C c_{n+p-1}^{M^p/m^{p-2} + M^p/m^{p-1} - 1} \tag{9}$$

and $c_n x_n^2 \le C c_n^{M(M+1)}$, for $p = 1$. From that the theorem follows readily.

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REMARK 1. Throughout the above proof we used C to denote a positive constant, the value of which may vary from line to line.

COROLLARY 1. *Let c_n, $n \in \mathbf{N}$, be a positive bounded nondecreasing sequence such that $1 \le m \le c_{n+1}/c_n \le M < \infty$, $n \in \mathbf{N}$. Then all the solutions of* (2) *are bounded.*

REMARK 2. Note that the condition $1 \le m \le c_{n+1}/c_n \le M < \infty$, $n \in \mathbf{N}$, implies that the sequence (c_n) in Theorem 1 and Corollary 1 is nondecreasing.

THEOREM 5. *Let d_n, $n \in \mathbf{N}$, be a positive, unbounded, strictly concave sequence. Then Equation* (1) *where $b_n = -(d_{n+1} + d_{n-1})/d_n$ has unbounded solution and $b_n \to -2$ as $n \to \infty$.*

PROOF. It is obvious that d_n is an unbounded solution of (1). Since d_n is a strictly concave sequence, that is, $d_{n+1}+d_{n-1}<2d_n$, $n\in\mathbf{N}$, we have $d_{n+1}-d_n<d_n-d_{n-1}$, $n\in\mathbf{N}$. Thus the sequence $d_{n+1} - d_n$ is decreasing. Therefore there exists $\lim_{n\to\infty}(d_{n+1} - d_n)$ of finite or infinite value (that is, $-\infty$). Let $\lim_{n\to\infty}(d_{n+1} - d_n) = d$. If $d < 0$ from $d_n = d_1 + \sum_{i=1}^{n-1}(d_{i+1} - d_i)$ we conclude that d_n is negative for sufficiently large n. Then we arrive at a contradiction with the positivity assumption on d_n. So we have $d \ge 0$. Hence $d_{n+1} \ge d_n$, $n \in \mathbf{N}$. Since

$$\lim_{n\to\infty}(d_{n+1} - d_n) = \lim_{n\to\infty} d_n \left(\frac{d_{n+1}}{d_n} - 1\right) = d < +\infty$$

and $\lim_{n\to\infty} d_n = +\infty$, we get $\lim_{n\to\infty}(d_{n+1}/d_n) = 1$. It follows that

$$\lim_{n\to\infty} b_n = -\lim_{n\to\infty} \frac{d_{n+1} + d_{n-1}}{d_n} = -2.$$

EXAMPLE 1. Consider the difference equation

$$x_{n+1} - \frac{\ln(n+1) + \ln(n-1)}{\ln n} x_n + x_{n-1} = 0, \quad n \ge 2.$$

This equation is of the form of (1) and obviously has solution $x_n = \ln n$ and $d_n = \ln n$, $n \ge 2$, satisfying the conditions of Theorem 5.

EXAMPLE 2. Consider the difference equation

$$x_{n+1} - \frac{(n+1)^\alpha + (n-1)^\alpha}{n^\alpha} x_n + x_{n-1} = 0, \quad n \in \mathbf{N},\ \alpha \in (0, 1).$$

This equation is of the form of (1) and obviously it has solution $x_n = n^\alpha$. It is clear that $d_n = n^\alpha$ is a positive, unbounded and strictly concave sequence, since the function $f(x) = x^\alpha$, $\alpha \in (0, 1)$, is such a function.

By the "symmetry principle" we can obtain analogous theorems in the case $b_n \in (0, 2)$, $n \in \mathbf{N}$.

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