GENERATING FUNCTIONS FOR ULTRASPHERICAL FUNCTIONS

B. VISWANATHAN

1. Introduction. The ultraspherical function

$$P_n^{(\lambda)}(x) = \frac{\Gamma(n+2\lambda)}{\Gamma(2\lambda)\Gamma(n+1)} F[-n, n+2\lambda; \lambda+\tfrac{1}{2}; \tfrac{1}{2}(1-x)] \tag{1.1}$$

for $|1-x| < 2$ is a solution of the differential equation

$$(1-x^2)\frac{d^2v}{dx^2} - (2\lambda+1)x\frac{dv}{dx} + n(n+2\lambda)v = 0. \tag{1.2}$$

This equation has two independent solutions; of the two, only $P_n^{(\lambda)}(x)$ is analytic at $x = 1$, aside for some special values of λ, which we shall not consider. The expression (1.1) vanishes identically when n is a negative integer. Hence we choose, when n is a positive integer, the ultraspherical polynomial as

$$P_n^{(\lambda)}(x) = \frac{(2\lambda)_n}{n!} F[-n, n+2\lambda; \lambda+\tfrac{1}{2}; \tfrac{1}{2}(1-x)];$$

otherwise we choose the ultraspherical function as

$$F[-n, n+2\lambda; \lambda+\tfrac{1}{2}; \tfrac{1}{2}(1-x)].$$

Replacing the parameter n in (1.2) by $y\partial/\partial y$, we construct the partial differential equation $Lv = 0$ where

$$L = (1-x^2)\frac{\partial^2}{\partial x^2} - (2\lambda+1)x\frac{\partial}{\partial x} + y^2\frac{\partial^2}{\partial y^2} + (2\lambda+1)y\frac{\partial}{\partial y}. \tag{1.3}$$

This operator L annuls $u(x, y) = v(x)y^n$ if and only if $v(x)$ satisfies (1.2).

We show in § 2 that the partial differential equation $Lu = 0$ admits a three-parameter Lie group. Following the methods of Weisner **(11)**, we use this group to obtain generating functions for ultraspherical functions.

2. Operators. We define the following operators:

$$A = y\partial/\partial y, \qquad B = y^{-1}\left\{(1-x^2)\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}\right\}, \tag{2.1}$$

$$C = y\left\{(1-x^2)\frac{\partial}{\partial x} - xy\frac{\partial}{\partial y} - 2\lambda x\right\},$$

Received July 8, 1966. This paper is a condensation of a doctoral dissertation submitted at the University of New Brunswick in May 1965.

and a linear operator T which satisfies $Tf(x, y) = y^{-2\lambda}f(x, y^{-1})$, where f is an arbitrary function.

The operators A, B, and C satisfy the commutation relations

$$(2.2)\qquad [A, B] = -B, \qquad [A, C] = C, \qquad \text{and} \quad [B, C] = -2A - 2\lambda,$$

where $[A, B] = AB - BA$, and therefore generate a three-parameter Lie group G.

From the relations (2.1) we obtain the relation

$$(2.3)\qquad CB + A^2 + (2\lambda - 1)A = (1 - x^2)L.$$

Hence it follows that A, B, and C each commute with $(1 - x^2)L$ and therefore convert each solution of $Lu = 0$ into another solution. Also we have that the operator T converts every solution of $Lu = 0$ into a solution. In particular,

$$(2.4)\qquad \begin{aligned} &AF[-n, n + 2\lambda; \lambda + \tfrac{1}{2}; \tfrac{1}{2}(1 - x)]y^n \\ &\qquad = nF[-n, n + 2\lambda; \lambda + \tfrac{1}{2}; \tfrac{1}{2}(1 - x)]y^n, \\ &BF[-n, n + 2\lambda; \lambda + \tfrac{1}{2}; \tfrac{1}{2}(1 - x)]y^n \\ &\qquad = nF[-n + 1, n + 2\lambda - 1; \lambda + \tfrac{1}{2}; \tfrac{1}{2}(1 - x)]y^{n-1}, \\ &CF[-n, n + 2\lambda; \lambda + \tfrac{1}{2}; \tfrac{1}{2}(1 - x)]y^n \\ &\qquad = -(n + 2\lambda)F[-n - 1, n + 2\lambda + 1; \lambda + \tfrac{1}{2}; \tfrac{1}{2}(1 - x)]y^{n+1}, \end{aligned}$$

where n is an arbitrary complex number.

The operator A generates a trivial group; $x' = x$ and $y' = ty$ $(t \neq 0)$. The extended form of the group generated by A, B, and C is described by

$$(2.5)\qquad e^{cC}e^{bB}f(x, y) = (1 + 2cxy + c^2y^2)^{-\lambda}f(X, Y),$$

where

$$X = \frac{b + (1 + 2bc)xy + c(1 + bc)y^2}{[(1 + 2cxy + c^2y^2)\{b^2 + 2b(1 + bc)xy + (1 + bc)^2y^2\}]^{\frac{1}{2}}},$$

$$Y = \left[\frac{b^2 + 2b(1 + bc)xy + (1 + bc)^2y^2}{1 + 2cxy + c^2y^2}\right]^{\frac{1}{2}}.$$

b and c are arbitrary constants and $f(x, y)$ is an arbitrary function. The signs of the surds being so chosen that X and Y reduce to x and y, respectively, when $b = 0$ and $c = 0$.

3. Conjugate sets. First we want to examine the functions annulled by L and $R = r_1A + r_2B + r_3C + r_4$, where the r's are arbitrary constants, other than $r_1 = r_2 = r_3 = r_4 = 0$. It is sufficient to consider one operator from each of the conjugate sets into which the operators R fall with respect to the group G.

As in **(11**, p. 1035**)**, we have

(3.1) $e^{aA}Be^{-aA} = e^{-a}B, \qquad e^{aA}Ce^{-aA} = e^{a}C,$

(3.2) $e^{bB}Ae^{-bB} = A + bB, \qquad e^{bB}Ce^{-bB} = -2bA - b^2B + C - 2\lambda b,$

(3.3) $e^{cC}Ae^{-cC} = A - cC, \qquad e^{cC}Be^{-cC} = 2cA + B - c^2C + 2\lambda c,$

(3.4) $SAS^{-1} = (1 + 2bc)A + bB - c(1 + bc)C + 2\lambda bc,$

where $S = e^{cC}e^{bB}$.

It follows that R is conjugate to $mA + n$ for suitable choices of a, b, c, m, and n, except when $r_1^2 + 4r_2 r_3 = 0$, in which case it may be inferred that R is conjugate to $mB + n$ from (3.3).

4. Generating functions annulled by operators of the first order. We observe that

$$u_1 = F[-\nu, \nu + 2\lambda; \lambda + \tfrac{1}{2}; \tfrac{1}{2}(1 - x)]y^{\nu} \qquad \text{for} \quad |1 - x| < 2$$

and

$$u_2 = F[-\nu, \nu + 2\lambda; \lambda + \tfrac{1}{2}; \tfrac{1}{2}(1 + x)]y^{\nu} \qquad \text{for} \quad |1 + x| < 2,$$

where ν is an arbitrary constant, are both annulled by L and $A - \nu$.

Hence from (2.6) and (3.4) it follows that

(4.1) $G_1(x, y) = M^{\nu}(1 + 2cxy + c^2y^2)^{-\lambda-\nu/2}F[-\nu, \nu + 2\lambda; \lambda + \tfrac{1}{2}; \tfrac{1}{2}(1-X)]$

and $G_2(x, y) = M^{\nu}(1 + 2cxy + c^2y^2)^{-\lambda-\nu/2}F[-\nu, \nu + 2\lambda; \lambda + \tfrac{1}{2}; \tfrac{1}{2}(1 + X)]$

where

$$M = [b^2 + 2b(1 + bc)xy + (1 + bc)^2y^2]^{\frac{1}{2}}$$

and

$$X = \frac{b^2 + (1 + 2bc)xy + c(1 + bc)y^2}{M(1 + 2cxy + c^2y^2)^{\frac{1}{2}}}$$

are both annulled by L and

$$R = (1 + 2bc)A + bB - c(1 + bc)C + 2\lambda bc - \nu.$$

In the following work, we shall be examining G_1 or G_2 depending on which is analytic at $x = 1$.

Case 1. In (4.1) putting $b = -1$ and $c = 0$, we obtain $R = A - B - \nu$ and

$$G_2(x, y) = \rho^{\nu}F[-\nu, \nu + 2\lambda; \lambda + \tfrac{1}{2}; \tfrac{1}{2}(1 + X)]$$

where $\rho = (1 - 2xy + y^2)^{\frac{1}{2}}$ and $X = (-1 + xy)/\rho$.

This function has an expansion of the form

$$\sum_{n=0}^{\infty} c_n P_n^{(\lambda)}(x)y^n.$$

The constant c_n is determined by putting $x = 1$.

Thus

$$(4.2)\qquad \rho^{\nu}F[-\nu, \nu + 2\lambda; \lambda + \tfrac{1}{2}; \tfrac{1}{2}(1 - X)] = \sum_{n=0}^{\infty} \frac{(-\nu)_n}{(2\lambda)_n} P_n^{(\lambda)}(x)y^n$$

where $\rho = (1 - 2xy + y^2)^{\frac{1}{2}}$ and $X = (1 - xy)/\rho$ for $|y| < |x \pm (x^2 - 1)^{\frac{1}{2}}|$. This is equivalent to that of Brafman (**2**, p. 945, eq. 18).

Special cases. When $\nu = -2\lambda$, we obtain

$$(4.3)\qquad (1 - 2xy + y^2)^{-\lambda} = \sum_{n=0}^{\infty} P_n^{(\lambda)}(x)y^n,$$

which is sometimes taken as a definition for ultraspherical polynomials.

When $\nu = -(\lambda + \frac{1}{2})$, we obtain

$$(4.4)\qquad \rho^{-1}\left(\frac{1 + \rho - xy}{2}\right)^{\frac{1}{2}-\lambda} = \sum_{n=0}^{\infty} \frac{(\lambda + \frac{1}{2})_n}{(2\lambda)_n} P_n^{(\lambda)}(x)y^n;$$

cf. (**7**, p. 82, eq. 4.7.16).

When $\nu = -(\lambda - \frac{1}{2})$, we obtain

$$(4.5)\qquad \left(\frac{1 - xy + \rho}{2}\right)^{\frac{1}{2}-\lambda} = \sum_{n=0}^{\infty} \frac{(\lambda - \frac{1}{2})_n}{(2\lambda)_n} P_n^{(\lambda)}(x)y^n;$$

Carlitz (**5**, p. 151, eq. 9) has given an equivalent result for the Jacobi polynomials.

When $\nu = n$, a positive integer, (4.2) reduces to a polynomial identity:

$$(4.6)\qquad \rho^n P_n^{(\lambda)}\left(\frac{1 - xy}{\rho}\right) = \frac{(2\lambda)_n}{n!} \sum_{m=0}^{n} \frac{(-n)_m}{(2\lambda)_m} P_m^{(\lambda)}(x)y^m;$$

cf. (**2**, p. 946, eq. 22).

The above expansion (4.2) is valid only in $|y| < |x \pm (x^2 - 1)^{\frac{1}{2}}|$; $\rho = (1 - 2xy + y^2)^{\frac{1}{2}}$, not being single valued in the region

$$|x - (x^2 - 1)^{\frac{1}{2}}| < |y| < |x + (x^2 - 1)^{\frac{1}{2}}|,$$

cannot have an expansion in the annular region, whereas for the outer region an expansion can be obtained by the application of the operator T of (2.1) to the next result.

Unless otherwise mentioned the above remark holds good for all subsequent expansions.

Case 2. In (4.1), putting $b = 0$ and $c = -1$, we obtain $R = A + C - \nu$ and

$$G_1(x, y) = y^{\nu}(1 - 2xy + y^2)^{-\lambda-\nu/2}F[-\nu, \nu + 2\lambda; \lambda + \tfrac{1}{2}; \tfrac{1}{2}(1 - X)],$$

where $X = (x - y)/\rho$ and $\rho = (1 - 2xy + y^2)^{\frac{1}{2}}$.

This function has an expansion of the form

$$\sum_{n=0}^{\infty} c_n F[-n - \nu, n + \nu + 2\lambda; \lambda + \tfrac{1}{2}; \tfrac{1}{2}(1 - x)]y^{n+\nu}.$$

The constant is determined by putting $x = 1$. Thus

$$(4.7)\quad \rho^{-(2\lambda+\nu)}F[-\nu, \nu + 2\lambda; \lambda + \tfrac{1}{2}; \tfrac{1}{2}(1 - X)] = \sum_{n=0}^{\infty} \frac{(2\lambda + \nu)_n}{n!} F[-n - \nu, n + \nu + 2\lambda; \lambda + \tfrac{1}{2}; \tfrac{1}{2}(1 - x)]y^n,$$

where $\rho = (1 - 2xy + y^2)^{\frac{1}{2}}$, for $|y| < |x \pm (x^2 - 1)^{\frac{1}{2}}|$ and $x \neq -1$. Truesdell (**9**, p. 85, eq. 13) has an equivalent result for Associated Legendre functions.

Special cases. When $\nu = -(\lambda + \frac{1}{2})$, we obtain

$$(4.8)\quad \left(\frac{x - y + \rho}{2}\right)^{-\lambda+\frac{1}{2}} = \sum_{n=0}^{\infty} \frac{(\lambda - \frac{1}{2})_n}{n!} \times F[-n + \lambda + \tfrac{1}{2}, n + \lambda - \tfrac{1}{2}; \lambda + \tfrac{1}{2}; \tfrac{1}{2}(1 - x)]y^n.$$

When $\nu = -(\lambda - \frac{1}{2})$, we have

$$(4.9)\quad \rho^{-1}\left(\frac{x - y + \rho}{2}\right)^{-\lambda+\frac{1}{2}} = \sum_{n=0}^{\infty} \frac{(\lambda + \frac{1}{2})_n}{n!} \times F[-n + \lambda - \tfrac{1}{2}, n + \lambda + \tfrac{1}{2}; \lambda + \tfrac{1}{2}; \tfrac{1}{2}(1 - x)]y^n$$

When $\nu = n$, a positive integer, (4.7) reduces to

$$(4.10)\quad \rho^{-2\lambda-n}P_n^{(\lambda)}\left(\frac{x - y}{\rho}\right) = \sum_{k=0}^{\infty} \frac{(n + k)!}{n!\,k!} P_{n+k}^{(\lambda)}(x)y^k;$$

cf. (**6**, p. 280, eq. 23).

Case 3. In (4.1) substituting $b = w^{-1}$ and $c = -1$, we obtain

$$R = (2 - w)A - B + (1 - w)C + 2\lambda + w\nu$$

and

$$(4.11)\quad \rho^{-2\lambda-\nu}\mu^{\nu}F[-\nu, \nu + 2\lambda; \lambda + \tfrac{1}{2}; \tfrac{1}{2}(1 - X)] = \sum_{n=0}^{\infty} F(-n, -\nu; 2\lambda; w]P_n^{(\lambda)}(x)y^n,$$

where $\rho = (1 - 2xy + y^2)^{\frac{1}{2}}$, $\mu = \{1 - 2(1 - w)xy + (1 - w)^2y^2\}^{\frac{1}{2}}$, and

$$X = \frac{\rho^2 + wy(x - y)}{\mu\rho},$$

for $|y| < \min\{|x \pm (x^2 - 1)^{\frac{1}{2}}|, |\{x \pm (x^2 - 1)^{\frac{1}{2}}\}/(1 - w)|\}$.

Special cases.

$$(4.12)\quad \mu^{-1}\left\{\frac{\mu\rho + \rho^2 + wy(x - y)}{2}\right\}^{\frac{1}{2}-\lambda} = \sum_{n=0}^{\infty} F[-n, \lambda + \tfrac{1}{2}; 2\lambda; w]P_n^{(\lambda)}(x)y^n.$$

$$(4.13)\quad \rho^{-1}\left\{\frac{\mu\rho + \rho^2 + wy(x - y)}{2}\right\}^{\frac{1}{2}-\lambda} = \sum_{n=0}^{\infty} F[-n, \lambda - \tfrac{1}{2}; 2\lambda; w]P_n^{(\lambda)}(x)y^n.$$

$$(4.14)\quad \rho^{-2\lambda-n}\mu^nP_n^{(\lambda)}\left[\frac{\rho^2 + wy(x - y)}{\mu\rho}\right] = \frac{(2\lambda)_n}{n!}\sum_{m=0}^{\infty} F[-n, -m; 2\lambda; w]P_m^{(\lambda)}(x)y^m.$$

After replacing $(1-w)$ by w^{-1} for the annular region,

$$|w\{x \pm (x^2-1)^{\frac{1}{2}}\}| < |y| < |x \pm (x^2-1)^{\frac{1}{2}}|,$$

we obtain

$$(4.15) \quad (1-2wxy^{-1}+w^2y^{-2})^{\nu/2}(1-2xy+y^2)^{-\lambda-\nu/2} \times F[-\nu, \nu+2\lambda; \lambda+\tfrac{1}{2}; \tfrac{1}{2}(1+X)]$$

$$= \sum_{n=0}^{\infty} \frac{(2\lambda+\nu)_n}{n!} F[-\nu, \nu+2\lambda+n; n+1; w] \times F[-n-\nu, n+\nu+2\lambda; \lambda+\tfrac{1}{2}; \tfrac{1}{2}(1-x)]y^n$$

$$+ \sum_{n=1}^{\infty} \frac{(-\nu)_n}{n!} F[\nu+2\lambda, -\nu+n; n+1; w] \times F[n-\nu, -n+\nu+2\lambda; \lambda+\tfrac{1}{2}; \tfrac{1}{2}(1-x)]w^n y^{-n},$$

where

$$X = \frac{y\{1-(1+w)xy^{-1}+wy^{-2}\}}{\{(1-2xy+y^2)(1-2wxy^{-1}+w^2y^{-2})\}^{\frac{1}{2}}}$$

for $|w\{x \pm (x^2-1)^{\frac{1}{2}}\}| < |y| < |x \pm (x^2-1)^{\frac{1}{2}}|$ and $x \neq -1$.

Special cases.

$$(4.16) \quad \{1-2wxy^{-1}+w^2y^{-2}\}^{-\frac{1}{2}} \cdot [\tfrac{1}{2}\{(1-2xy+y^2)(1-2wxy^{-1}+w^2y^{-2})\}^{\frac{1}{2}} - \tfrac{1}{2}y\{1-(1+w)xy^{-1}+wy^{-2}\}]^{\frac{1}{2}-\lambda}$$

$$= \sum_{n=0}^{\infty} \frac{(\lambda-\frac{1}{2})_n}{n!} F[\lambda+\tfrac{1}{2}, \lambda+n-\tfrac{1}{2}; n+1; w] \times F[-n+\lambda+\tfrac{1}{2}, n+\lambda-\tfrac{1}{2}; \lambda+\tfrac{1}{2}; \tfrac{1}{2}(1-x)]y^n$$

$$+ \sum_{n=1}^{\infty} \frac{(\lambda+\frac{1}{2})_n}{n!} F[\lambda-\tfrac{1}{2}, \lambda+n+\tfrac{1}{2}; n+1; w] \times F[+n+\lambda+\tfrac{1}{2}, -n+\lambda-\tfrac{1}{2}; \lambda+\tfrac{1}{2}; \tfrac{1}{2}(1-x)]w^n y^{-n}.$$

$$(4.17) \quad (1-2wxy^{-1}+w^2y^{-2})^{n/2}(1-2xy+y^2)^{-\lambda-n/2}P_n^{(\lambda)}(X)$$

$$= \sum_{m=0}^{\infty} \frac{(n+m)!}{n!\,m!} F[-n, 2\lambda+n+m; m+1; w]P_{n+m}^{(\lambda)}(x)y^m$$

$$+ \sum_{m=1}^{n} \frac{(1-2\lambda-n)_m}{m!} F[n+2\lambda, -n+m; m+1; w]P_{n-m}^{(\lambda)}(x)w^m y^{-m},$$

where

$$X = \frac{y\{1-(1+w)xy^{-1}+wy^{-2}\}}{\{(1-2xy+y^2)(1-2wxy^{-1}+w^2y^{-2})\}^{\frac{1}{2}}}.$$

5. Generating functions annulled by $2A - B + C + 2\lambda - w$. We next examine the simultaneous equations $Lu = 0$ and $Bu = -u$; the general solution of the latter equation is $u = e^{-xy}f(y(1-x^2)^{\frac{1}{2}})$, where f is an arbitrary function.

If this is to be annulled by L, then $f(X)$ must satisfy the equation

$$X\cdot\frac{d^2f}{dX^2}+2\lambda\frac{df}{dX}+Xf=0,$$

where $X = y(1-x^2)^{\frac{1}{2}}$. Two linearly independent solutions of this are

$$F[-;\lambda+\tfrac{1}{2};-\tfrac{1}{4}X^2]$$

and

$$(-\tfrac{1}{4}X^2)^{\frac{1}{2}-\lambda}F[-;\tfrac{3}{2}-\lambda;-\tfrac{1}{4}X^2].$$

Hence the solutions of $Lu = 0$ and $(B+1)u = 0$ are

(5.1) $$e^{-xy}F[-;\lambda+\tfrac{1}{2};-\tfrac{1}{4}y^2(1-x^2)],$$
$$e^{-xy}\{\tfrac{1}{4}y^2(1-x^2)\}^{\frac{1}{2}-\lambda}F[-;\tfrac{3}{2}-\lambda;-\tfrac{1}{4}y^2(1-x^2)].$$

The first of these is analytic at $x = 1$ and we obtain

(5.2) $$e^{-xy}F\left[-;\lambda+\tfrac{1}{2};-\frac{y^2(1-x^2)}{4}\right]=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2\lambda)_n}P_n^{(\lambda)}(x)y^n;$$

(1) gives an equivalent result for Associated Legendre polynomials.

Equations (2.5), (3.3), and (5.1) show that

$$\rho^{-2\lambda}\exp\{-w(x-y)y/\rho^2\}\ F[-;\lambda+\tfrac{1}{2};-w^2y^2(1-x^2)/4\rho^4]$$

where $\rho = (1-2xy+y^2)^{\frac{1}{2}}$, is annulled by L and

$$R = -2A+B-C-2\lambda+w.$$

Using the generating function for Laguerre polynomials **(7**, p. 100**)**, we obtain

(5.3) $$\rho^{-2\lambda}\exp\{-w(x-y)y/\rho^2\}F[-;\lambda+\tfrac{1}{2};-w^2y^2(1-x^2)/4\rho^4]$$
$$=\sum_{n=0}^{\infty}\frac{n!}{(2\lambda)_n}L_n^{(2\lambda-1)}(w)P_n^{(\lambda)}(x)y^n;$$

cf. **(8)**.

We have thus obtained in the normalized form functions which are annulled by L and $R = r_1A + r_2B + r_3C + r_4$, where the r's are constants.

6. Generating functions annulled by second-order operators. In some cases by suitable choice of a new set of variables, the equation $Lu = 0$ may be transformed into one solvable by the method of separation of variables.

Taking $X = \frac{1}{2}y(x+1)$ and $Y = \frac{1}{2}y(x-1)$ the equation $Lu = 0$ is transformed into

$$X\frac{\partial^2u}{\partial X^2}-Y\frac{\partial^2u}{\partial Y^2}+(\lambda+\tfrac{1}{2})\frac{\partial u}{\partial X}-(\lambda+\tfrac{1}{2})\frac{\partial u}{\partial Y}=0.$$

Without loss of generality, the separation constant can be taken as 1. Four linearly independent solutions are

$$(6.1)\quad \begin{cases} u_1 = F[-;\lambda+\tfrac{1}{2};\tfrac{1}{2}y(x+1)]F[-;\lambda+\tfrac{1}{2};\tfrac{1}{2}y(x-1)], \\ u_2 = \{y(x+1)\}^{\frac{1}{2}-\lambda}F[-;-\lambda+\tfrac{3}{2};\tfrac{1}{2}y(x+1)]F[-;\lambda+\tfrac{1}{2};\tfrac{1}{2}y(x-1)], \\ u_3 = \{y(x-1)\}^{\frac{1}{2}-\lambda}F[-;\lambda+\tfrac{1}{2};\tfrac{1}{2}y(x+1)]F[-;\tfrac{3}{2}-\lambda;\tfrac{1}{2}y(x-1)], \\ u_4 = \{y^2(x^2-1)\}^{\frac{1}{2}-\lambda}F[-;\tfrac{3}{2}-\lambda;\tfrac{1}{2}y(x+1)]F[-;\tfrac{3}{2}-\lambda;\tfrac{1}{2}y(x-1)]. \end{cases}$$

These functions are also annulled by

$$X\cdot\frac{\partial^2}{\partial X^2}+(\lambda+\tfrac{1}{2})\frac{\partial}{\partial X}-1=-Y(X-Y)^{-2}L+(A+\lambda+\tfrac{1}{2})B-1$$

and hence by $AB+(\lambda+\frac{1}{2})B-1$.

Of these four solutions, only the first two are analytic at $x=1$ and hence we shall be considering only these two cases. We obtain

$$(6.2)\quad F[-;\lambda+\tfrac{1}{2};\tfrac{1}{2}y(x+1)]F[-;\lambda+\tfrac{1}{2};\tfrac{1}{2}y(x-1)] = \sum_{n=0}^{\infty}\frac{1}{(2\lambda)_n(\lambda+\frac{1}{2})_n}P_n^{(\lambda)}(x)y^n.$$

Similarly,

$$(6.3)\quad \{\tfrac{1}{2}(x+1)\}^{\frac{1}{2}-\lambda}F[-;\tfrac{3}{2}-\lambda;\tfrac{1}{2}y(x+1)]F[-;\lambda+\tfrac{1}{2};\tfrac{1}{2}y(x-1)] = \sum_{n=0}^{\infty}\frac{1}{(\frac{3}{2}-\lambda)_n\, n!}F[-n+\lambda-\tfrac{1}{2},n+\lambda+\tfrac{1}{2};\lambda+\tfrac{1}{2};\tfrac{1}{2}(1-x)]y^n$$

for $x\neq-1$. Both of these equations can be obtained from (**10**, p. 148, eq. 2).

Equations (2.6), (3.3), and (6.1) show that

$$(6.4)\quad \begin{cases} (1+2cxy+c^2y^2)^{-\lambda}F[-;\lambda+\tfrac{1}{2};X]F[-;\lambda+\tfrac{1}{2};Y], \\ (1+2cxy+c^2y^2)^{-\frac{1}{2}}\{b+(1+2bc)xy+c(1+bc)y^2+M\}^{\frac{1}{2}-\lambda} \\ \qquad\times F[-;\tfrac{3}{2}-\lambda;X]F[-;\lambda+\tfrac{1}{2};Y], \end{cases}$$

where

$$X=-\frac{w}{2}\left\{\frac{b+(1+2bc)xy+c(1+bc)y^2+M}{1+2cxy+c^2y^2}\right\},$$

$$Y=-\frac{w}{2}\left\{\frac{b+(1+2bc)xy+c(1+bc)y^2-M}{1+2cxy+c^2y^2}\right\},$$

with $M=[(1+2cxy+c^2y^2)\{b^2+2b(1+bc)xy+(1+bc)^2y^2\}]^{\frac{1}{2}}$, are both annulled by L and R;

$$\begin{aligned} R = {} & 3c(1+bc)A^2+bB^2+c^3(1+bc)C^2+(1+4bc)AB \\ & -c^2(3+4bc)AC+6\lambda c(1+2bc)A+(\lambda+\tfrac{1}{2})(1+4bc)B \\ & -c^2(\lambda-\tfrac{1}{2})(3+4bc)C+\lambda c(2\lambda+1)(1+2bc)+w. \end{aligned}$$

Case 1. Putting $b=-1$ and $c=0$, we have

$$R=B^2-AB-(\lambda+\tfrac{1}{2})B-w.$$

Thus

$$(6.5)\quad F[-;\lambda+\tfrac{1}{2};\tfrac{1}{2}w(1-xy+\rho)]F[-;\lambda+\tfrac{1}{2};\tfrac{1}{2}w(1-xy-\rho)] = \sum_{n=0}^{\infty}\frac{(-1)^n w^n}{(2\lambda)_n(\lambda+\tfrac{1}{2})_n}F[-;\lambda+n+\tfrac{1}{2};w]P_n^{(\lambda)}(x)y^n,$$

where $\rho = (1-2xy+y^2)^{\frac{1}{2}}$.

This is equivalent to the result of Weisner (**13**, p. 154, eq. 6.1) for Bessel functions.

Similarly

$$(6.6)\quad \{\tfrac{1}{2}(1-xy+\rho)\}^{\frac{1}{2}-\lambda}F[-;\tfrac{3}{2}-\lambda;\tfrac{1}{2}w(1-xy+\rho)] \times F[-;\lambda+\tfrac{1}{2};\tfrac{1}{2}w(1-xy-\rho)] = \sum_{n=0}^{\infty}\frac{(\lambda-\tfrac{1}{2})_n}{(2\lambda)_n}F[-;\tfrac{3}{2}-\lambda-n;w]P_n^{(\lambda)}(x)y^n,$$

where $\rho = (1-2xy+y^2)^{\frac{1}{2}}$, for $|y| < |x \pm (x^2-1)^{\frac{1}{2}}|$. An equivalent result for Bessel function is given by Weisner (**13**, p. 155, eq. 6.2).

Case 2. Putting $b = 0$ and $c = -1$, we have

$$R = 3A^2 + C^2 - AB + 3AC + 6\lambda A - (\lambda+\tfrac{1}{2})B + 3(\lambda-\tfrac{1}{2})C + \lambda(2\lambda+1) - w.$$

Thus

$$(6.7)\quad \rho^{-2\lambda}F\left[-;\lambda+\tfrac{1}{2};-\frac{wy}{2}\frac{x-y+\rho}{\rho^2}\right]F\left[-;\lambda+\tfrac{1}{2};-\frac{wy}{2}\frac{x-y-\rho}{\rho^2}\right] = \sum_{n=0}^{\infty}{}_1F_2[-n;\lambda+\tfrac{1}{2},2\lambda;w]P_n^{(\lambda)}(x)y^n$$

where $\rho = (1-2xy+y^2)^{\frac{1}{2}}$, for $|y| < |x \pm (x^2-1)^{\frac{1}{2}}|$; cf. (**3**, p. 1321, eq. 15). Similarly,

$$(6.8)\quad \rho^{-1}\left(\frac{x-y+\rho}{2}\right)^{\frac{1}{2}-\lambda}F\left[-;\frac{3}{2}-\lambda;-\frac{wy}{2}\frac{x-y+\rho}{\rho^2}\right] \times F\left[-;\lambda+\tfrac{1}{2};-\frac{wy}{2}\frac{x-y-\rho}{\rho^2}\right] = \sum_{n=0}^{\infty}\frac{(\lambda+\tfrac{1}{2})_n}{n!}{}_1F_2[-n;\lambda+\tfrac{1}{2},\tfrac{3}{2}-\lambda;w] \times {}_2F_1[-n+\lambda-\tfrac{1}{2},n+\lambda+\tfrac{1}{2};\lambda+\tfrac{1}{2};\tfrac{1}{2}(1-x)]y^n,$$

where $\rho = (1-2xy+y^2)^{\frac{1}{2}}$, for $|y| < |x \pm (x^2-1)^{\frac{1}{2}}|$ and $x \neq -1$. An equivalent result for Associated Legendre polynomials is given by Yadao (**14**, p. 120, eq. 1.3).

Note. There is a computational error in Yadao's result. The correct version is

$$\frac{\rho^{-1}}{\Gamma(1-m)}\left(\frac{x-t+\rho}{x-t-\rho}\right)^{m/2} F\left[-;1-m;-\frac{ty(x-t-\rho)}{2\rho^2}\right]$$
$$\times F\left[-;1+m;-\frac{ty(x-t+\rho)}{2\rho^2}\right]$$
$$=\sum_{n=0}^{\infty}\frac{(1-m)_n}{n!}F[-n;1-m,1+m;y]P_n^m(x)t^n.$$

In the general case, from (6.4) we have

$$G_1(x,y) = (1+2cxy+c^2y^2)^{-\lambda}F[-;\lambda+\tfrac{1}{2};X]F[-;\lambda+\tfrac{1}{2};Y]$$

and

$$G_2(x,y) = (1+2cxy+c^2y^2)^{-\frac{1}{2}}\left\{\frac{b+(1+2bc)xy+c(1+bc)y^2+M}{2b}\right\}^{\frac{1}{2}-\lambda}$$
$$\times F[-;\tfrac{3}{2}-\lambda;X]F[-;\lambda+\tfrac{1}{2};Y],$$

where

$$X = \frac{w}{2}\left\{\frac{b+(1+2bc)xy+c(1+bc)y^2+M}{1+2cxy+c^2y^2}\right\},$$

$$Y = \frac{w}{2}\left\{\frac{b+(1+2bc)xy+c(1+bc)y^2-M}{1+2cxy+c^2y^2}\right\},$$

and $M = [(1+2cxy+c^2y^2)\{b^2+2b(1+bc)xy+(1+bc)^2y^2\}]^{\frac{1}{2}}$. These give

(6.9) $$G_1(x,y) = \sum_{n=0}^{\infty} c_n P_n^{(\lambda)}(x)y^n,$$

where

$$c_n = \sum_{m=0}^{n}\frac{(-1)^n(-n)_m}{(2\lambda)_m(\lambda+\frac{1}{2})_m}\frac{w^m c^{n-m}}{m!}F[-;\lambda+m+\tfrac{1}{2};wb]$$

and

(6.10) $$G_2(x,y) = \sum_{m=0}^{\infty} c_n P_n^{(\lambda)}(x)y^n,$$

where

$$c_n = \sum_{m=0}^{n}\frac{(-1)^{m+n}(\lambda-\frac{1}{2})_m(-n)_m}{(2\lambda)_m\, m!}b^{-m}c^{n-m}F[-;\tfrac{3}{2}-\lambda-m;wb].$$

7. Functions annulled by $AB+(\lambda+\frac{1}{2})B-A^2-2\lambda A+\nu(2\lambda+\nu)$. If we choose the new variables as $X=\rho-y$ and $Y=\rho+y$, where

$$\rho = (1-2xy+y^2)^{\frac{1}{2}},$$

the equation $Lu=0$ is transformed into

$$(1-X^2)\frac{\partial^2 u}{\partial x^2}-(1-Y^2)\frac{\partial^2 u}{\partial Y^2}-(2\lambda+1)X\frac{\partial u}{\partial X}+(2\lambda+1)Y\frac{\partial u}{\partial Y}=0.$$

Selecting $\nu(2\lambda + \nu)$ for the separation constant, the above equation has four linearly independent solutions:

$$
\begin{aligned}
&F[-\nu, \nu + 2\lambda; \lambda + \tfrac{1}{2}; \tfrac{1}{2}(1 - X)]F[-\nu, \nu + 2\lambda; \lambda + \tfrac{1}{2}; \tfrac{1}{2}(1 - Y)],\\
&(1 - X)^{\frac{1}{2}-\lambda}F[-\nu, \nu + 2\lambda; \lambda + \tfrac{1}{2}; \tfrac{1}{2}(1 - Y)]\\
&\qquad\times F[-\nu - \lambda + \tfrac{1}{2}, \nu + \lambda + \tfrac{1}{2}; \tfrac{3}{2} - \lambda; \tfrac{1}{2}(1 - X)],\\
(7.1)\quad &(1 - Y)^{\frac{1}{2}-\lambda}F[-\nu, \nu + 2\lambda; \lambda + \tfrac{1}{2}; \tfrac{1}{2}(1 - X)]\\
&\qquad\times F[-\nu - \lambda + \tfrac{1}{2}; \nu + \lambda + \tfrac{1}{2}; \tfrac{3}{2} - \lambda; \tfrac{1}{2}(1 - Y)],\\
&\{(1 - X)(1 - Y)\}^{\frac{1}{2}-\lambda}F[-\nu - \lambda + \tfrac{1}{2}, \nu + \lambda + \tfrac{1}{2}; \tfrac{3}{2} - \lambda; \tfrac{1}{2}(1 - X)]\\
&\qquad\times F[-\nu - \lambda + \tfrac{1}{2}, \nu + \lambda + \tfrac{1}{2}; \tfrac{3}{2} - \lambda; \tfrac{1}{2}(1 - Y)].
\end{aligned}
$$

These functions are also annulled by

$$
\begin{aligned}
&(1 - X^2)\frac{\partial^2}{\partial X^2} - (2\lambda + 1)X\frac{\partial}{\partial X} + \nu(2\lambda + \nu)\\
&\qquad = \frac{XY + 2Y - 1}{2(Y - X)}L + AB + (\lambda + \tfrac{1}{2})B - A^2 - 2\lambda A + \nu(2\lambda + \nu)
\end{aligned}
$$

and hence by $AB + (\lambda + \frac{1}{2})B - A^2 - 2\lambda A + \nu(2\lambda + \nu)$.

We shall be considering the first two cases only. We obtain

$$
\begin{aligned}
(7.2)\quad &F[-\nu, \nu + 2\lambda; \lambda + \tfrac{1}{2}; \tfrac{1}{2}(1 - \rho + y)]F[-\nu, \nu + 2\lambda; \lambda + \tfrac{1}{2}; \tfrac{1}{2}(1 - \rho - y)]\\
&\qquad = \sum_{n=0}^{\infty} \frac{(-\nu)_n(\nu + 2\lambda)_n}{(\lambda + \frac{1}{2})_n(2\lambda)_n} P_n^{(\lambda)}(x)y^n,
\end{aligned}
$$

where $\rho = (1 - 2xy + y^2)^{\frac{1}{2}}$, for $|y| < |x \pm (x^2 - 1)^{\frac{1}{2}}|$; cf. **(2**, p. 945, eq. **17)**.

Special case.

$$
(7.3)\quad P_n^{(\lambda)}(\rho - y)P_n^{(\lambda)}(\rho + y) = \left\{\frac{(2\lambda)_n}{n!}\right\}^2 \times \sum_{m=0}^{n} \frac{(-n)_m(n + 2\lambda)_m}{(\lambda + \frac{1}{2})_m(2\lambda)_m} P_m^{(\lambda)}(x)y^m.
$$

Next we obtain

$$
\begin{aligned}
(7.4)\quad &\left(\frac{1 - \rho + y}{2y}\right)^{\frac{1}{2}-\lambda} F[-\nu - \lambda + \tfrac{1}{2}, \nu + \lambda + \tfrac{1}{2}; \tfrac{3}{2} - \lambda; \tfrac{1}{2}(1 - \rho + y)]\\
&\qquad\times F[-\nu, \nu + 2\lambda; \lambda + \tfrac{1}{2}; \tfrac{1}{2}(1 - \rho - y)]\\
&= \sum_{n=0}^{\infty} \frac{(-\nu - \lambda + \frac{1}{2})_n(\nu + \lambda + \frac{1}{2})_n}{(\frac{3}{2} - \lambda)_n\, n!}\\
&\qquad\times F[-n - \tfrac{1}{2} + \lambda, n + \tfrac{1}{2} + \lambda; \lambda + \tfrac{1}{2}; \tfrac{1}{2}(1 - x)]y^n,
\end{aligned}
$$

where $\rho = (1 - 2xy + y^2)^{\frac{1}{2}}$, for $|y| < |x \pm (x^2 - 1)^{\frac{1}{2}}|$ and $x \neq -1$.

From (2.5), (3.4), and (7.1) we obtain

$$
(7.5)\quad \left\{
\begin{aligned}
&(1 + 2cxy + c^2y^2)^{-\lambda}F[-\nu, \nu + 2\lambda; \lambda + \tfrac{1}{2}; \tfrac{1}{2}(1 - X)]\\
&\qquad\times F[-\nu, \nu + 2\lambda; \lambda + \tfrac{1}{2}; \tfrac{1}{2}(1 - Y)],\\
&(1 + 2cxy + c^2y^2)^{-\lambda}(1 - X)^{\frac{1}{2}-\lambda} F[-\nu - \lambda + \tfrac{1}{2}, \nu + \lambda + \tfrac{1}{2};\\
&\qquad \tfrac{3}{2} - \lambda; \tfrac{1}{2}(1 - X)]F[-\nu, \nu + 2\lambda; \lambda + \tfrac{1}{2}; \tfrac{1}{2}(1 - Y)],
\end{aligned}
\right.
$$

where

$$X = \left\{\frac{(1+wb)^2 + 2(1+wb)(c+w+wbc)xy + (c+w+wbc)^2y^2}{1+2cxy+c^2y^2}\right\}^{\frac{1}{2}} + w\left\{\frac{b^2 + 2b(1+bc)xy + (1+bc)^2y^2}{1+2cxy+c^2y^2}\right\}^{\frac{1}{2}}$$

and

$$Y = \left\{\frac{(1+wb)^2 + 2(1+wb)(c+w+wbc)xy + (c+w+wbc)^2y^2}{1+2cxy+c^2y^2}\right\}^{\frac{1}{2}} - w\left\{\frac{b^2 + 2b(1+bc)xy + (1+bc)^2y^2}{1+2cxy+c^2y^2}\right\}^{\frac{1}{2}},$$

and that these are annulled by L and R;

$$\begin{aligned} R = {} & \{w + 3c(1+bc)(1+2wbc)\}A^2 + b(1+bw)B^2 \\ & + c^2(1+bc)(c+w+wbc)C^2 - \{1 - 2(1+wb)(1+2bc)\}AB \\ & - c\{c + 2(1+bc)(c+w+wbc)\}AC + 2\lambda\{w + 3c(1+bc) \\ & + 6wbc(1+bc)\}A + (\lambda + \tfrac{1}{2})\{1 + 2b(2c+w+2wbc)\}B \\ & + c(\lambda - \tfrac{1}{2})\{c - 2(1+2bc)(2c+w+2wbc)\}C \\ & + \lambda c(2\lambda+1)\{1 + 2b(c+w+wbc)\} - \nu w(2\lambda+\nu). \end{aligned}$$

Case 1. Putting $b = -1$ and $c = 0$, we have

$$R = wA^2 - (1-w)B^2 + (1-2w)AB + 2\lambda wA + (\lambda + \tfrac{1}{2})(1-2w)B - \nu w(2\lambda+\nu).$$

We obtain, after replacing wy by $-y$,

$$(7.6) \quad F[-\nu, \nu+2\lambda; \lambda+\tfrac{1}{2}; \tfrac{1}{2}(1-X)]F[-\nu, \nu+2\lambda; \lambda+\tfrac{1}{2}; \tfrac{1}{2}(1-Y)] = \sum_{n=0}^{\infty} \frac{(-\nu)_n(\nu+2\lambda)_n}{(\lambda+\frac{1}{2})_n(2\lambda)_n} F[-\nu+n, \nu+2\lambda+n; \lambda+\tfrac{1}{2}+n; w]P_n^{(\lambda)}(x)y^n,$$

where

$$X = \{(1-w)^2 - 2(1-w)xy + y^2\}^{\frac{1}{2}} - \{w^2 + 2wxy + y^2\}^{\frac{1}{2}}$$

and

$$Y = \{(1-w)^2 - 2(1-w)xy + y^2\}^{\frac{1}{2}} + \{w^2 + 2wxy + y^2\}^{\frac{1}{2}},$$

for $|y| < \min\{|(1-w)[x \pm (x^2-1)^{\frac{1}{2}}]|, |w[x \pm (x^2-1)^{\frac{1}{2}}]|\}$.

Special case.

$$(7.7) \quad P_n^{(\lambda)}(X)P_n^{(\lambda)}(Y) = \left(\frac{(2\lambda)_n}{n!}\right)^2 \sum_{m=0}^{n} \frac{(-n)_m(n+2\lambda)_m}{(\lambda+\frac{1}{2})_m(2\lambda)_m} \times F[-n+m, n+2\lambda+m; \lambda+\tfrac{1}{2}+m; w]P_m^{(\lambda)}(x)y^m.$$

Similarly

$$(7.8)\quad \left(\frac{1-X}{2w}\right)^{\frac{1}{2}-\lambda} F[-\nu-\lambda+\tfrac{1}{2}, \nu+\lambda+\tfrac{1}{2}; \tfrac{3}{2}-\lambda; \tfrac{1}{2}(1-X)]$$
$$\times F[-\nu, \nu+2\lambda; \lambda+\tfrac{1}{2}; \tfrac{1}{2}(1-Y)]$$
$$= \sum_{n=0}^{\infty} \frac{(\lambda-\frac{1}{2})_n}{(2\lambda)_n} F[-\nu-\lambda+\tfrac{1}{2}, \nu+\lambda+\tfrac{1}{2}; \tfrac{3}{2}-\lambda-n; w] P_n^{(\lambda)}(x) y^n.$$

Case 2. Putting $b = 0$ and $c = -1$, we have

$$R = (3-w)A^2 + (1-w)C^2 - AB + (3-2w)AC + 2\lambda(3-w)A$$
$$- (\lambda+\tfrac{1}{2})B + (\lambda-\tfrac{1}{2})(3-2w)C + \lambda(2\lambda+1) + \nu w(2\lambda+\nu).$$

We obtain

$$(7.9)\quad \rho^{-2\lambda} F[-\nu, \nu+2\lambda; \lambda+\tfrac{1}{2}; \tfrac{1}{2}(1-X)] F[-\nu, \nu+2\lambda; \lambda+\tfrac{1}{2}; \tfrac{1}{2}(1-Y)]$$
$$= \sum_{n=0}^{\infty} {}_3F_2[-n, -\nu, \nu+2\lambda; \lambda+\tfrac{1}{2}, 2\lambda; w] P_n^{(\lambda)}(x) y^n$$

where

$$X = \frac{[1-2(1-w)xy+(1-w)^2y^2]^{\frac{1}{2}} + wy}{\rho},$$
$$Y = \frac{[1-2(1-w)xy+(1-w)^2y^2]^{\frac{1}{2}} - wy}{\rho}$$

and $\rho = (1-2xy+y^2)^{\frac{1}{2}}$, for $|y| < |x \pm (x^2-1)^{\frac{1}{2}}|$; cf. (**3**, p. 1319, eq. 2).

Special cases.

$$(7.10)\quad \rho^{-1}\left\{\frac{\rho(\rho^2+2wxy)^{\frac{1}{2}} + wy(x-y) + \rho^2}{2}\right\}^{\frac{1}{2}-\lambda}$$
$$= \sum_{n=0}^{\infty} F[-n, \lambda-\tfrac{1}{2}; 2\lambda; w] P_n^{(\lambda)}(x) y^n,$$

$$(7.11)\quad \rho^{-2\lambda} P_n^{(\lambda)}(X) P_n^{(\lambda)}(Y)$$
$$= \left\{\frac{(2\lambda)_n}{n!}\right\}^2 \sum_{m=0}^{\infty} {}_3F_2[-m, -n, n+2\lambda; \lambda+\tfrac{1}{2}; 2\lambda, w] P_m^{(\lambda)}(x) y^m,$$

and from the second equation of (7.5)

$$(7.12)\quad \rho^{-2\lambda}\left(\frac{X-1}{2wy}\right)^{\frac{1}{2}-\lambda} F[-\nu-\lambda+\tfrac{1}{2}, \nu+\lambda+\tfrac{1}{2}; \tfrac{3}{2}-\lambda; \tfrac{1}{2}(1-X)]$$
$$\times F[-\nu, \nu+2\lambda; \lambda+\tfrac{1}{2}; \tfrac{1}{2}(1-Y)]$$
$$= \sum_{n=0}^{\infty} \frac{(\lambda+\frac{1}{2})_n}{n!} {}_3F_2[-n, -\nu-\lambda+\tfrac{1}{2}, \nu+\lambda+\tfrac{1}{2}; \tfrac{3}{2}-\lambda, \lambda+\tfrac{1}{2}; w]$$
$$\times F[-n+\lambda-\tfrac{1}{2}, n+\lambda+\tfrac{1}{2}; \lambda+\tfrac{1}{2}; \tfrac{1}{2}(1-x)] y^n;$$

cf. (**4**, p. 81, eq. 5).

In the general case, from (7.5) we have

$$G_1(x, y) = (1 + 2cxy + c^2y^2)^{-\lambda}F[-\nu, \nu + 2\lambda; \lambda + \tfrac{1}{2}; \tfrac{1}{2}(1 - X)] \times F[-\nu, \nu + 2\lambda; \lambda + \tfrac{1}{2}; \tfrac{1}{2}(1 - Y)],$$

$$G_2(x, y) = (1 + 2cxy + c^2y^2)^{-\lambda}\left(\frac{X - 1}{2bw}\right)^{\frac{1}{2}-\lambda} \times F[-\nu - \lambda + \tfrac{1}{2}, \nu + \lambda + \tfrac{1}{2}; \tfrac{3}{2} - \lambda; \tfrac{1}{2}(1 - X)] \times F[-\nu, \nu + 2\lambda; \lambda + \tfrac{1}{2}; \tfrac{1}{2}(1 - Y)],$$

where

$$X = \left\{\frac{(1 - wb)^2 + 2(1 - wb)(c - w - wbc)xy + (c - w - wbc)^2y^2}{1 + 2cxy + c^2y^2}\right\}^{\frac{1}{2}} - w\left\{\frac{b^2 + 2b(1 + bc)xy + (1 + bc)^2y^2}{1 + 2cxy + c^2y^2}\right\}^{\frac{1}{2}}$$

and

$$Y = \left\{\frac{(1 - wb)^2 + 2(1 - wb)(c - w - wbc)xy + (c - w - wbc)^2y^2}{1 + 2cxy + c^2y^2}\right\}^{\frac{1}{2}} + w\left\{\frac{b^2 + 2b(1 + bc)xy + (1 + bc)^2y^2}{1 + 2cxy + c^2y^2}\right\}^{\frac{1}{2}}.$$

In these cases we have

$$G_1(x, y) = \sum_{n=0}^{\infty} c_n P_n^{(\lambda)}(x)y^n, \tag{7.13}$$

where

$$c_n = \sum_{m=0}^{n} \frac{(-1)^n(-n)_m}{(2\lambda)_m} \frac{(-\nu)_m(\nu + 2\lambda)_m}{(\lambda + \frac{1}{2})_m} \frac{w^m c^{n-m}}{m!} \times F[-\nu + m, 2\lambda + \nu + m, \lambda + \tfrac{1}{2} + m; wb$$

and

$$G_2(x, y) = \sum_{n=0}^{\infty} c_n P_n^{(\lambda)}(x)y^n, \tag{7.14}$$

where

$$c_n = \sum_{m=0}^{n} (-1)^{n+m} \frac{(\lambda - \frac{1}{2})_m(-n)_m}{(2\lambda)_m} \frac{b^{-m} c^{n-m}}{m!} \times F[-\nu - \lambda + \tfrac{1}{2}, \nu + \lambda + \tfrac{1}{2}; \tfrac{3}{2} - \lambda - m; wb]$$

I take this opportunity to express my sincere thanks to Dr. L. Weisner, my dissertation supervisor, for having suggested this problem to me and for all the help and encouragement he gave me. My thanks are also due to the Graduate School, University of New Brunswick, and to the National Research Council, Ottawa, for financial help.

References

1. E. L. Bloh, *On an expansion of Bessel functions in a series of Legendre functions* (from Math. Revs., *16* (1955), 587).
2. F. Brafman, *Generating functions of Jacobi and related polynomials*, Proc. Amer. Math. Soc., *2* (1951), 942–949.
3. ——— *An ultraspherical generating function*, Pacific J. Math., *7* (1957), 1319–1323.
4. ——— *A generating function for associated Legendre polynomials*, Quart. J. Math., *8* (1957), 81–83.
5. L. Carlitz, *Some generating functions for the Jacobi polynomials*, Boll. Un. Math. Ital., *16* (1961), 150–155.
6. E. D. Rainville, *Special functions* (New York, 1960).
7. G. Szego, *Orthogonal polynomials*, Colloquium Publications 23 (Amer. Math. Soc., New York, 1959).
8. L. Toscano, *Funzione generatice dei prodotti di polinomi di Laguerre con gli ultrasferici* (from Math. Revs., *12* (1951), 333).
9. C. Truesdell, *A unified theory of special functions* (Princeton, 1948).
10. G. N. Watson, *Theory of Bessel functions* (Cambridge, 1952).
11. L. Weisner, *Group theoretic origin of certain generating functions*, Pacific J. Math., *5* (1955), 1033–1039.
12. ——— *Generating functions for Hermite functions*, Can. J. Math., *11* (1959), 141–147.
13. ——— *Generating functions for Bessel functions*, Can. J. Math., *11* (1959), 148–155.
14. G. M. Yadao, *Generating functions for associated Legendre polynomials*, Quart. J. Math., *14* (1963), 120–122.

University of New Brunswick, Fredericton, N.B.