24 Effective actions in ten dimensions

In ten dimensions, supersymmetry greatly restricts the allowed particle content and effective actions of theories with massless fields. Without gauge interactions there are only two consistent possibilities. These correspond to the low-energy limits of the IIA and IIB theories. These have $N = 2$ supersymmetry (they have 32 conserved supercharges). Because the symmetry is so restrictive, we can understand a great deal about the low-energy limits of these theories without making any detailed computations. We can even make exact statements about the non-perturbative behavior of these theories. This is familiar from our studies of field theories in four dimensions with more than four supercharges. In ten dimensions, supersymmetric gauge theories have $N = 1$ supersymmetry (16 supercharges). Classically, specification of the gauge group completely specifies the terms in the effective action with up to two derivatives. Quantum mechanically, only the gauge groups $O(32)$ and $E_8 \times E_8$ are possible.

24.1 Eleven-dimensional supergravity

Rather than start with these ten-dimensional theories, it is instructive to start in eleven dimensions. Eleven is the highest dimension where one can write a supersymmetric action (in higher dimensions, spins higher than 2 are required). This fact by itself has focused much attention on this theory. But it is also known that the theory in eleven dimensions has a connection with string theory. As we will see later, if one takes the strong coupling limit of the Type IIA string theory, one obtains a theory whose low-energy limit is eleven-dimensional supergravity.

The particle content of the eleven-dimensional theory is simple: there is a graviton, g_{MN} (44 degrees of freedom) and a three-index antisymmetric tensor field, C_{MNO} (84 degrees of freedom); here $M, N, O = 0, \dots, 9$ are space–time indices. There is also a gravitino, ψ_M. This has 16×8 degrees of freedom. We have, as usual, counted degrees of freedom by considering a theory in nine dimensions, remembering that g_{MN} is symmetric and traceless and that the basic spinor representation in nine dimensions is sixteen-dimensional (it combines the two eight-dimensional spinors of $O(8)$).

The Lagrangian for the eleven-dimensional theory, in addition to the Ricci scalar, involves a field strength for the three-index field, C_{MNO}. The corresponding field strength F_{MNOP} is completely antisymmetric in *its* indices, similar to the field strength of electrodynamics:

https://doi.org/10.1017/9781009290883.031 Published online by Cambridge University Press

$$F_{MNOP} = \frac{3!}{4!}\left(\partial_M C_{NOP} - \partial_N C_{MOP} + \cdots\right)$$
$$= \frac{3!}{4!}\sum_P (-1)^P \partial_M C_{NOP}, \tag{24.1}$$

where the sum is over all permutations and the factor $(-1)^P$ is ± 1 depending on whether the permutation is even or odd. It is convenient to describe such antisymmetric tensor fields in the language of differential forms. For the reader unfamiliar with these, an introduction is provided later, in Section 26.1. For now we note that antisymmetric tensors with p indices are p-forms. The operation of taking the curl, as in Eq. (24.1), takes a p-form to a $(p+1)$-form. It is denoted by the symbol d and is called the *exterior derivative.* In terms of forms, Eq. (24.1) can be written compactly as

$$F = dC. \tag{24.2}$$

The theory has a gauge invariance:

$$C \to C + d\Lambda, \quad C_{MNO} \to \frac{2}{3!}\sum_P (-1)^P \partial_M \Lambda_{NO} \tag{24.3}$$

where Λ is a two-form.

We will not need the complete form of the action. The bosonic terms are

$$\mathcal{L}_{\text{bos}} = -\frac{1}{2\kappa^2}\sqrt{g}R - \frac{1}{48}\sqrt{g}F^2_{MNPQ} - \frac{\sqrt{2}\kappa}{3456}\epsilon^{M_1\ldots M_{11}}F_{M_1\ldots M_4}F_{M_5\ldots M_8}C_{M_9\cdots M_{11}}. \tag{24.4}$$

The last of these is a Chern–Simons term. It respects the gauge invariance of Eq. (24.3) if one integrates by parts. Such terms can arise in field theories with odd dimensions; in (2+1)-dimensional electrodynamics, for example, they play an interesting role. The fermionic terms include covariant derivative terms for the gravitino as well as couplings to F and various four-fermion terms. The supersymmetry transformation laws have the structure

$$\delta e^A_M = \frac{\kappa}{2}\bar{\eta}\Gamma^A\psi_M, \tag{24.5}$$

$$\delta A_{MNP} = -\frac{\sqrt{2}}{8}\bar{\eta}\Gamma_{[MN}\psi_{P]}, \tag{24.6}$$

$$\delta\psi_M = \frac{1}{\kappa}D_M\eta + (F\eta \text{ terms}). \tag{24.7}$$

Here e^A_M is the *vielbein* field and the covariant derivative is constructed from the spin connection (discussed in Section 17.6).

24.2 The IIA and IIB supergravity theories

The eleven-dimensional fields are functions of the coordinates $x_0, \ldots, x_{10}$. We obtain the IIA supergravity theory (the low-energy limit of the Type IIA string) if we truncate the

https://doi.org/10.1017/9781009290883.031 Published online by Cambridge University Press

eleven-dimensional supergravity theory to ten dimensions, i.e. if we simply eliminate the dependence on x_{10}. We need to relabel the fields as well, since it is not appropriate to have a 10 index. So we take the components of g with ten-dimensional indices to be the ten-dimensional metric. Then $g_{10\,10}$ is a ten-dimensional scalar, which we call ϕ, and $g_{10\,\mu}$ is a ten-dimensional vector, which corresponds to the Ramond–Ramond vector of the IIA string theory. Note that $C_{10\,\mu\nu} = B_{\mu\nu}$ is a two-index antisymmetric tensor field in ten dimensions (corresponding to the two-index tensor we found in the NS–NS sector). The gravitino decomposes into two ten-dimensional gravitinos, and two spin-$1/2$ particles. With $H = dB$, the bosonic terms in the ten-dimensional action for the NS–NS fields are

$$\mathcal{L}_{\text{bos}} = -\frac{1}{2\kappa^2}R - \frac{3}{4}\phi^{-3/2}H^2_{\mu\nu\rho} - \frac{9}{16\kappa^2}\left(\frac{\partial_\mu\phi}{\phi}\right)^2. \tag{24.8}$$

The IIB theory is not obtained in this way. But, from string theory, we can see that the NS–NS action must be the same as in the Type IIA theory. The reason is that in the NS–NS sector the vertex operators of the IIA and IIB theories are the same, so the scattering amplitudes – and hence the effective action – are the same as well.

24.3 Ten-dimensional supersymmetric Yang–Mills theory

From our studies of the heterotic string we know the field content of this theory. There is a metric, an antisymmetric tensor field (which we again call $B_{\mu\nu}$), a scalar ϕ and the gauge fields, A^a_μ. The Lagrangian for g, B and ϕ is the same as in the Type II theories. The gauge terms are

$$\mathcal{L}_{\text{YM}} = -\frac{\phi^{-3/4}}{4g^2}F^2_{\mu\nu} - \frac{1}{2}\bar{\chi}^a(D_M\chi)^a. \tag{24.9}$$

It turns out that there is another crucial modification in the Yang–Mills case. The field strength H_{MNO} is not simply the curl of B_{MN} but contains an additional contribution, which closely resembles the Chern–Simons term we encountered in our study of instantons in four-dimensional Yang–Mills theory:

$$H = dB - \frac{\kappa}{\sqrt{2}}\omega_3 \tag{24.10}$$

(the notation will be thoroughly explained in Chapter 26), with

$$\omega_3 = A^aF^a - \frac{1}{3}gf_{abc}A^aA^bA^c = A^a dA^a + \frac{2}{3}gf_{abc}A^aA^bA^c. \tag{24.11}$$

There is also a gravitational term, with a similar form. This extra term plays an important role in understanding anomaly cancellation. In four dimensions we will see that it leads to the appearance of axions in the low-energy theory.

24.4 Coupling constants in string theory

The Standard Model is defined, in part, by specifying a set of coupling constants. The fact that there are so many parameters is one of the reasons we have given that the model is not satisfactory as some sort of ultimate description of nature. In our discussion of string interactions we introduced a coupling constant g_s. There is one such constant for each of the string theories we have introduced, bosonic, Type I, Types IIA and IIB and heterotic, as well as for non-supersymmetric strings. But the idea that string theory possesses a free parameter is, it turns out, an illusion. By changing the expectation value of the dilaton field, we can change the value of the coupling. This is similar to phenomena we observed in four-dimensional supersymmetric gauge theories. In situations with a great deal of supersymmetry there will be no potential, perturbative or non-perturbative, for this field and the choice of coupling will correspond to a choice of vacuum. But, in vacua in which supersymmetry is broken, we would expect that dynamical effects would fix the value of this and any other moduli. The coupling constants of the low-energy theory would then be determined fully in ways which, in principle, one could understand and eventually hope to calculate. In the next few sections we explain this connection between coupling constants and fields.

24.4.1 Couplings in closed-string theories

When we constructed vertex operators we saw that we could include a coupling constant g_s in the definition of the vertex operator. In the heterotic string the same coupling enters in all vertices. This is a consequence of unitarity. At tree level, for example, we saw that scattering amplitudes factorize near poles of the S-matrix; if one introduced independent couplings for each vertex operator, the amplitudes would not factorize correctly. As a result all amplitudes can be expressed in terms of a single parameter. In the heterotic string theory this means that there is a calculable relation between the gravitational constant and the Yang–Mills coupling. To work out this coupling, one needs to calculate the three-point interactions for three gravitons and for three gauge bosons carefully (see the exercises at the end of the chapter). The results are necessarily of the form

$$\kappa_{10}^2 = ag^2(2\alpha')^4, \quad g_{\rm YM}^2 = bg^2(2\alpha')^3. \tag{24.12}$$

The calculation yields $a = 1/4$, $b = 1$.

A similar analysis in the Type I theory gives a relation between the open-string and closed-string couplings and a relation between the gauge and gravitational couplings.

In both theories we see that the string scale is smaller than the Planck scale:

$$M_s = (g_s)^{1/4} M_p. \tag{24.13}$$

This is a satisfying result. It means that if we think of M_s as the cutoff on the gravity theory, gravitational loops are suppressed by powers of g_s.

https://doi.org/10.1017/9781009290883.031 Published online by Cambridge University Press

24.4.2 The coupling is not a parameter in string theory

So far, in all the string theories it would appear that there is an adjustable, dimensionless, parameter. As we said earlier this is not really the case; the reason can be traced to the *dilaton*. Classically, in all the string theories we have studied the dilaton has no potential, so its expectation value is not fixed. In the next two short subsections we will demonstrate that changing the expectation value of the dilaton changes the effective coupling. In four dimensions with $N > 1$ supersymmetry (and automatically in dimensions greater than five) there is no potential for the dilaton, so the question of the value of the coupling is equivalent to a choice among degenerate vacuum states. Without supersymmetry (or with $N \leq 1$ supersymmetry in four dimensions), one expects quantum mechanical effects to generate a potential for the dilaton, and the value of the coupling is then a *dynamical* question.

24.4.3 Effective Lagrangian argument

Perhaps the simplest way to understand the role of the dilaton is to examine the ten-dimensional effective action. We start with the case of the heterotic string in ten dimensions. We can redefine ϕ as $g^{-2}\kappa^{3/2}\phi'$, eliminating g everywhere in the action. Note that since $\kappa \propto g$ this means that $\phi' \sim g^{1/2}$. Then we can do a Weyl rescaling

$$g_{\mu\nu} = \phi^{-1} g_{\mu\nu}. \tag{24.14}$$

This puts a common power of ϕ in front of the action, ϕ^{-4} and is consistent with g being the string loop parameter, since effectively we have a factor g^{-2} at the front.

With this rescaling it is the string scale which is fundamental. Remember that $M_{\rm p}^2 = M_{\rm s}^2/(g^2)^{1/4}$. By rescaling the metric we have rescaled lengths which were originally expressed in units of $M_{\rm p}$ in terms of $M_{\rm s}$. So we have a consistent picture. The cutoff for the effective Lagrangian is $M_{\rm s}$. All dimensional parameters in the Lagrangian are of order $M_{\rm s}$, and loops are accompanied by $g^2 \sim \phi^4$.

24.4.4 World-sheet coupling of the dilaton

As we will discuss further in the next chapter, we can define a generating functional for the S-matrix by taking the two-dimensional field theory and adding space–time fields weighted by vertex operators. So, for example, for the bosonic string we would add terms to the action of the form

$$(\eta_{\mu\nu} + h_{\mu\nu}) e^{ik\cdot x} \partial x^\mu \bar{\partial} x^\nu. \tag{24.15}$$

We can generalize this to a background metric, yielding a two-dimensional non-linear sigma model,

$$g_{\mu\nu}(x) \partial x^\mu \bar{\partial} x^\nu. \tag{24.16}$$

https://doi.org/10.1017/9781009290883.031 Published online by Cambridge University Press

Just as we can couple the graviton to the world sheet, we can also couple the dilaton to it. The dilaton turns out to couple to the two-dimensional curvature:

$$\mathcal{L}_\Phi = \frac{1}{4\pi}\int d^2\sigma\,\sqrt{h}\Phi(X)R^{(2)}. \tag{24.17}$$

In two dimensions, however, the dynamics of gravity is trivial. Indeed, if we use our usual counting rules, the graviton has less than a single degree of freedom. So, the $R^{(2)}$ factor should not generate any sensible graviton dynamics. If we go to the conformal gauge,

$$h_{\alpha\beta} = e^{\phi}\eta_{\alpha\beta}, \tag{24.18}$$

the curvature is a total divergence:

$$R^{(2)} = \partial^2\phi. \tag{24.19}$$

Thus at most this factor in the action is topological. To get some feeling for this, let us evaluate the integral in the case of a sphere. We have seen that one representation for the sphere is provided by the space CP^1. This space has one complex coordinate. It is Kahler, which means that the only non-vanishing component of g is

$$g_{z\bar{z}} = \partial_z\partial_{\bar{z}}K(z,\bar{z}), \tag{24.20}$$

where, in this case,

$$K = \ln(1+\bar{z}z). \tag{24.21}$$

So we have

$$g = \left(\frac{1}{1+\bar{z}z}\right)^2. \tag{24.22}$$

From this, we can read off ϕ,

$$\phi = 2\ln(1+\bar{z}z) = -2\ln\big(1+\sigma_x^2+\sigma_y^2\big), \tag{24.23}$$

and the integral over the curvature is

$$\frac{1}{4\pi}\int d^2\sigma\,\partial^2\big[-2\ln\big(1+\sigma_x^2+\sigma_y^2\big)\big] = 2. \tag{24.24}$$

Note that this is invariant under a constant Weyl rescaling; it is topological. It is known as the Euler character of the surface and satisfies

$$\chi = \frac{1}{4\pi}\int d^2\sigma\,\sqrt{h}R^{(2)} \tag{24.25}$$

and

$$\chi = 2(1-g). \tag{24.26}$$

In this expression, χ is known as the Euler character of the manifold and g is the genus. For the sphere, $g=0$; for the torus, $g=1$; and so on for higher-genus string amplitudes. So string amplitudes, for constant Φ, come with a factor

$$e^{-2\Phi(1-g)}. \tag{24.27}$$

Thus we can identify e^{Φ} with the string coupling constant.

https://doi.org/10.1017/9781009290883.031 Published online by Cambridge University Press

Suggested reading

Ten-dimensional effective actions were described in some detail by Green *et al.* (1987). The couplings of the dilaton in string theory were discussed in detail by Polchinski (1998).

Exercise

(1) By studying the OPEs of the appropriate vertex operators, verify Eq. (24.12). To avoid making this calculation too involved, you may want to isolate particular terms in the gravitational and Yang–Mills couplings. The required vertices in general relativity can be found in Sannan (1986).

https://doi.org/10.1017/9781009290883.031 Published online by Cambridge University Press