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THE BOUSFIELD–KAN SPECTRAL SEQUENCE FOR MORAVA K-THEORY

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Abstract We construct a spectral sequence converging to the E_2-term of the Bousfield–Kan spectral sequence (BKSS) for a wide variety of homology theories. Using this, the E_2-term of the BKSS based on $K(1)$-theory for the odd spheres is computed and the unstable $K(1)$-completion is computed.

Keywords: homotopy groups; Bousfield–Kan spectral sequence; unstable Adams spectral sequence

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1. Introduction

Given a homology theory E, we can construct the Bousfield–Kan spectral sequence (BKSS). This spectral sequence computes, for a given space X, the homotopy of an appropriate completion from $E_*(X)$. Even though we can set this spectral sequence with great generality, the E_2-term turns out to be an Ext group in some non-abelian category. In practical terms, this description limits our ability to make computations. By requiring that E be a Landweber exact homology theory, and with some mild assumptions on the space X, we can relate the E_2-term to an Ext group in an abelian category, which in turn can be calculated as the homology of some sub-complex of the stable cobar complex. Although many theories do not satisfy this property, we were able to construct a spectral sequence converging to the E_2-term of the BKSS. The input to this spectral sequence can again be calculated as the homology of some unstable cobar complex. In the case of $K(1)$ and for any space X such that $K(1)_*(X)$ is cofree as a coalgebra, the main result implies that the unstable coalgebra description of E_2 is isomorphic to an unstable $K(1)_*(K(1))$-comodule description. As observed by Kuhn [**12**], this turns out to be isomorphic to the stable E_2-term. With this and the proof of convergence of the stable Adams spectral sequence, we prove convergence of the unstable analogue to the unstable $K(1)$-completion of the odd spheres. Finally, using the tower constructed by Farjoun [**10**] and our results, we provide an example of a finite H-space such that either the inverse limit of this tower is not the $K(1)$-localization or the map between this tower and the tower of the BKSS does not have a left inverse.

Hypothesis 1.1. We assume that all primes are odd and all spectra are multiplicative Ω-spectra.

Notation 1.2. If A is a ring, then $\mathrm{char}(A)$ will denote the characteristic of A and $A(n_1, n_2, \dots, n_k)$ will denote a free A-module generated by elements in dimensions $n_1, n_2, \dots, n_k$. The p-adic integers will be denoted by $\mathbb{Z}_p^{\wedge}$. If E is an Ω-spectrum, then $\underline{E}_n$ denotes the nth space of the Ω-spectrum. The category of topological spaces, the associated homotopy category and the category of E_*-modules will be denoted by $\mathcal{T}$, $\mathcal{HO}$ and $\mathcal{A}$, respectively.

2. The construction of the Bousfield–Kan spectral sequence

Let X be a space. We define a functor from $E : \mathcal{HO} \to \mathcal{HO}$ as follows. Define a space such that

$$E(X) = \Omega^{\infty}(E \wedge \Sigma^{\infty} X).$$

If $X = S^n$, then $E(S^n) = \underline{E}_n$, the nth space in the Ω-spectrum of E. It is easy to see that if E is a free E_*-module with generators x_i, then $E(X) = \prod \underline{E}_{|x_i|}$. For any space X and for $n \geqslant 0$, we also have

$$\pi_n(E(X)) \cong E_n(X),$$

where the right-hand side is a reduced E homology.

The composition $\eta : X \to \Omega^{\infty}\Sigma^{\infty}X \to E(X)$ induces the E_* Hurewicz map. Taking the homotopy fibre of this map, we get the fibre sequence $D(X) \to X \xrightarrow{\eta} E(X)$. D is a functor on $\mathcal{HO}$. We inductively define $D^{n+1}(X) = D(D^n(X))$. There is a tower

$$\begin{array}{ccc}
\vdots & & \\
\downarrow & & \\
D^2(X) & \longrightarrow & D^2(E(X)) \\
\downarrow & & \\
D(X) & \longrightarrow & D(E(X)) \\
\downarrow & & \\
X & \longrightarrow & E(X)
\end{array}$$

which fits into an exact couple that induces a spectral sequence with

$$E_1^{s,t} = \pi_{t-s}(D^s(E(X))) \quad \text{for } t - s \geqslant 0$$

and zero otherwise. We call this spectral sequence the Bousfield–Kan spectral sequence. Next, we give a description of the E_2-term.

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3. A cosimplicial description of the E_2-term

The following definitions will allow us to describe the E_2-term as the cohomotopy of some cosimplicial group.

Definition 3.1. A cosimplicial object $\boldsymbol{X}$ over a category $\mathcal{C}$ is a collection of objects $X_i \in \mathcal{C}, n \geqslant 0$, such that for each $0 \leqslant n$ there are maps $d^i : X_n \to X_{n+1}$ and $s^i : X_{n+1} \to X_n$ with $0 \leqslant i \leqslant n$ satisfying the following identities:

$$d^j d^i = d^i d^{j-1}, \qquad i < j,$$

$$s^j d^i = \begin{cases} d^i s^{j-1}, & i < j, \\ id, & i = j, j+1, \\ d^{i-1} s^j, & i > j+1, \end{cases}$$

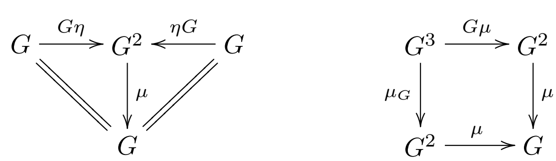
$$s^j s^i = s^{i-1} s^j, \qquad i > j.$$

In our case the category $\mathcal{C}$ will be the category $\mathcal{A}$, $\mathcal{T}$ or $\mathcal{HO}$.

Given a cosimplicial object $\boldsymbol{X}$ over $\mathcal{C}$ and a functor $F : \mathcal{C} \to \mathcal{A}$, we get a cosimplicial object over the category of abelian groups. We get a cochain complex $(F(\boldsymbol{X}))$ with $\delta^n = \sum_{i=0}^{n} (-1)^i d^i$. In the case that $\mathcal{C} = \mathcal{HO}$ and $F = \pi_*$, we call the homology of this cosimplicial group the cohomotopy of $\pi_*(\boldsymbol{X})$ and denote it by $\pi^* \pi_* \boldsymbol{X}$.

The functor $E : \mathcal{HO} \to \mathcal{HO}$ of § 2 induces a cosimplical object over $\mathcal{HO}$. We use the following definition.

Definition 3.2. A triple (G, μ, η) over the category $\mathcal{C}$ is composed of a functor $G : \mathcal{C} \to \mathcal{C}$ and natural transformations $\mu : G^2 \to G$ and $\eta : 1 \to G$ such that we have the following commutative diagrams:



Using the triple (G, μ, η) we can construct a functor $\boldsymbol{G}$ from $\mathcal{C}$ to the category of cosimplicial objects over $\mathcal{C}$ as follows: let $X \in \mathcal{C}$ and define $\boldsymbol{G}(X)_n = G^{n+1}(X)$ and the maps

$$d^i = G^i \eta G^{n-i} : G^n(X) \to G^{n+1}(X) \quad \text{and} \quad s^i = G^i \mu G^{n-i} : G^{n+2}(X) \to G^{n+1}(X)$$

with $0 \leqslant i \leqslant n$.

The natural transformation $\mu : E^2 \to E$, induced by the multiplicative structure of E, together with the Heurewitz map η, makes the functor (E, μ, η) a triple in the category $\mathcal{HO}$. This in turn gives us a functor $\boldsymbol{E}$ from $\mathcal{HO}$ into cosimplicial objects over $\mathcal{HO}$.

Theorem 3.3. *Let $X \in \mathcal{T}$. Then*

$$E_2^{s,t}(X) \cong \pi^s \pi_t \boldsymbol{E}(X).$$

Proof. The proof of this theorem can be found in [**9**]. □

4. The category $\mathcal{M}(G)$ and an alternative description of the E_2-term

There is an alternative description of the E_2-term using the dual concept of triple. The advantage of this description is that it will enable us to describe the E_2-term as the target of a spectral sequence.

Definition 4.1. A cotriple (G,δ,ε) in a category $\mathcal{C}$ is a composed of functor $G:\mathcal{C}\to\mathcal{C}$ and natural transformations $\delta: G\to G^2$ and $\varepsilon: G\to 1$ such that the following diagrams commute:

$$\begin{array}{ccccc} G & \xleftarrow{G\varepsilon} & G^2 & \xrightarrow{\varepsilon_G} & G \\ & \diagdown\!\!\diagdown & \uparrow{\scriptstyle\delta} & \diagup\!\!\diagup & \\ & & G & & \end{array} \qquad \begin{array}{ccc} G & \xrightarrow{\delta} & G^2 \\ {\scriptstyle\delta}\downarrow & & \downarrow{\scriptstyle G\delta} \\ G^2 & \xrightarrow{\delta_G} & G^3 \end{array}$$

Given a cotriple G, a G-coalgebra is an object $C\in\mathcal{C}$ and a map $\psi: C\to G(C)$ such that the following diagrams commute:

$$\begin{array}{ccc} C & \xrightarrow{\psi} & G(C) \\ & \diagdown\!\!\diagdown & \downarrow{\scriptstyle\varepsilon} \\ & & C \end{array} \qquad \begin{array}{ccc} C & \xrightarrow{\psi} & G(C) \\ {\scriptstyle\psi}\downarrow & & \downarrow{\scriptstyle G\psi} \\ G(C) & \xrightarrow{\delta} & G^2(C) \end{array}$$

A map $f: C\to D$ is a G-coalgebra map if the following diagram commutes:

$$\begin{array}{ccc} C & \xrightarrow{\psi} & G(C) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle G(f)} \\ D & \xrightarrow{\psi} & G(D) \end{array}$$

We denote the category of G-coalgebras over $\mathcal{C}$ as $\mathcal{C}(G)$. If $C\in\mathcal{C}$, then $G(C)$ is a G-coalgebra with $\psi=\delta$. Given a G-coalgebra (Z,ψ), we can define a triple by setting

$$\mu = G(\varepsilon): G^2(Z)\to G(Z),$$
$$\eta = \psi : Z\to G(Z).$$

Let $\mathcal{M}$ be the category of free E_*-modules. We define a cotriple (G,δ,ε) over $\mathcal{M}$. But first we impose the following restrictions on E.

Hypothesis 4.2. We make the following assumptions.

(i) E is a multiplicative, associative, homotopy commutative, CW spectrum with unit.

(ii) $E_*(\underline{E}_k)$ is a free E_*-module for all k.

Let $M\in\mathcal{M}$ and let F be the spectrum such that $\pi_*(F)=M$. Define $G(M)=E_*(\Omega^\infty F)$. By [**3**] we know that this defines a triple over $\mathcal{M}$. With this triple we have

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the category of G-coalgebras over $\mathcal{M}$, or $\mathcal{M}(G)$. For $M \in \mathcal{M}(G)$ there is a resolution

$$G(M) \quad \overset{d^0}{\underset{d^1}{\rightrightarrows}} \quad G^2(M) \quad \overset{d^0}{\underset{d^2}{\overset{d^1}{\rightrightarrows}}} \quad \cdots$$

with codegeneracies s^i that come from the product structure of the spectrum E. We call this the G-resolution of M. When we talk about the G-resolution of X we mean the G-resolution of $E_*(X)$. Let $G(X) = G(E_*(X))$.

Applying $\mathrm{Hom}_{\mathcal{M}(G)}(E_*(S^t), -)$ to the G-resolution of X and taking the homology of this complex gives $\mathrm{Ext}^s_{\mathcal{M}(G)}(E_*(S^t), E_*(X))$. We write $\mathrm{Ext}^{s,t}_{\mathcal{M}(G)}(E_*(X))$ in place of $\mathrm{Ext}^s_{\mathcal{M}(G)}(E_*(S^t), E_*(X))$.

Theorem 4.3. *Suppose that E satisfies Hypothesis 4.2 and let X be a simply connected space such that $E_*(X) \in \mathcal{M}$. Then*

$$E_2^{s,t} = \mathrm{Ext}^{s,t}_{\mathcal{M}(G)}(E_*(X)), \quad t > s \geqslant 0.$$

This is proven in [**2**]. The authors impose an additional condition on the spectrum E (the primitives of $E_*(\underline{E}_k)$ inject into $E_*(E)$). But this condition, by [**11**], is not really necessary for the previous theorem.

In practice, the previous characterizations of the E_2-term are of little use. The problem is that it does not provide an explicit way to produce elements. If we assume that E is a Landweber exact homology theory, like $E(n)$, K and BP, then we can express the E_2-term as the homology of some sub-complex of the stable cobar complex (see [**2**]).

We would like to study the BKSS based on Morava K-theories. Unfortunately, these are not Landweber exact. In these cases a more complicated object is needed to extract information about the E_2-term.

5. The f-primitive functors

Let $\mathcal{CO}$ be the subcategory of $\mathcal{M}$ consisting of (graded) coalgebras without unit. For $n \geqslant 0$, let R^nP be the derived functor of the primitives functor [**5**]. If $f : C \to D$ is a coalgebra map, we write $[f_*]_i$ for $R^iP(f)$.

Definition 5.1. Suppose $f : C \to C$ is a coalgebra map. Then

$$R^iP_f(C) = \mathrm{coker}([f_*]_i).$$

We call these the f-primitive functors.

Lemma 5.2. *Let $\{C_\alpha\}$ be a collection of coalgebras and $f_\alpha : C_\alpha \to C_\alpha$ be coalgebra maps. Then*

(i) *the map $R^iP(C_\alpha) \to R^iP_{f_\alpha}(C_\alpha)$ is onto for all i;*

(ii) *$R^iP_f(\otimes C_\alpha) = \oplus R^iP_{f_\alpha}(C_\alpha)$, where $f = \otimes f_\alpha$.*

Proof. Part (i) follows from the definitions. Since R^iP takes tensor products to direct sums and $\operatorname{coker}(\oplus f_\alpha) \cong \oplus \operatorname{coker}(f_\alpha)$, part (ii) follows. □

Definition 5.3. We say that C is f-nice if $R^iP_f(C) = 0$ for $i > 1$.

If f is the zero map, then f-nice is just nice in the sense of [**5**]. It follows from the definitions that if C is nice, then it is f-nice for all f.

Suppose $\operatorname{char}(A) = p$ and that C is a Hopf algebra over A. Then the pth-power map (the Frobenius) π is a coalgebra map. Let $C(x_n)$ be the coalgebra with a single generator of degree n and $T(x_{2n})$ be the coalgebra with generators x_{2ni} for $i \geqslant 1$ and

$$\Delta(x_{2nm}) = \sum_{i+j=m} \binom{i+j}{i} x_{2ni} \otimes x_{2nj}.$$

This latter coalgebra is dual to a divided power algebra. We can calculate the π-primitive functors of the coalgebras $C(x_n)$ and $T(x_{2n})$ of [**4**]. Let D be a cofree cocommutative coalgebra.

Lemma 5.4. *Suppose A is a ring of characteristic p and C is a Hopf algebra over A. Then:*

(i) $R^iP_\pi(C(x_{2n})) = R^iP(C(x_{2n}))$ *for all* i;

(ii) $R^0P_\pi(T(x_{2n})) = A(x_{2n})$, $R^1P_\pi(T(x_{2n})) = A(x_{2np})$ *and* $R^iP_\pi(T(x_{2n})) = 0$ *for* $i > 0$;

(iii) $R^iP_\pi(D) = R^iP(D)$ *for* $i > 0$.

Proof. Since $\pi(x) = 0$ in $C(x_n)$, the induced map is zero. This gives the first case.

For the second case, let $B(x_{2n})$ be the coalgebra of the bipolynomial algebra with generators $\{x_{2np^s} \mid s \geqslant 0\}$ with $x_{2n}^{p^s}$ primitive. The only non-trivial derived functor of this coalgebra is the zero-derived functor. This is just the module of primitives of $B(x_{2n})$ which gives $R^0P(B(x_{2n})) = A(2n, 2np, \dots)$. There is an injective extension sequence $T(x_{2n}) \to B(y_{2n}) \to B(z_{2np})$. We have the following commutative diagram:

$$\begin{array}{ccccc}
A(2np, 2np^2, \dots) & = & R^0P(B(z_{2np})) & \xrightarrow{\cong} & R^1P(T(x_{2n})) \\
\pi\downarrow & & \pi\downarrow & & \pi\downarrow \\
A(2np, 2np^2, \dots) & = & R^0P(B(z_{2np})) & \xrightarrow{\cong} & R^1P(T(x_{2n}))
\end{array}$$

where the horizontal isomorphisms come from the proof of [**4**, Proposition 3.3(iv), § 3]. The vertical map on the left takes the generator of dimension $2np^s$ to the generator of dimension $2np^{s+1}$. The result follows by taking cokernels.

For (iii), recall that if D is cofree, then $R^iP(D) = 0$ for $i > 0$. □

Definition 5.5. Suppose that $E_*(\underline{E}_n)$ is a coalgebra for all n. We say E is f-nice if $E_*(\underline{E}_n)$ is f-nice for all n.

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Suppose that $\mathrm{char}(E_*) = p$ and there is a Kunneth isomorphism

$$E_*\left(\prod \underline{E}_{n_\alpha}\right) \cong \otimes E_*(\underline{E}_{n_\alpha}).$$

Then $E_*(\underline{E}_n)$ is a Hopf algebra for all n and the Frobenius is a well-defined map on $E_*(\underline{E}_n)$.

We would like to apply the P_f-derived functors to the G-resolution of X. But, d^0 may not commute with f. Fortunately, since for any X we have $G(X) \cong \bigotimes_{n\in\Sigma} E_*(\underline{E}_n)$, it is sufficient to require that the following hypothesis is satisfied.

Hypothesis 5.6. $d^0(f(x)) \in \mathrm{Im}(f)$, where $d^0 : G^n(X) \to G^{n+1}(X)$ for all $n \geqslant 0$.

From now on we assume that the homology theory E satisfies Hypothesis 5.6. In this case, f induces a cosimplicial map from the G-resolution of X to itself. With this we can apply all previous results on the P_f derived functors.

Definition 5.7. Let $R^s_q P_f(X)$ be the homology of the following cochain complex:

$$R^q P_f(G(X)) \;\begin{smallmatrix}\to\\ \to\end{smallmatrix}\; R^q P_f(G^2(X)) \;\begin{smallmatrix}\to\\ \to\\ \to\end{smallmatrix}\; \cdots$$

(the s homology of the q derived functors of the f-primitives applied to the G-resolution of X).

We are interested in the case where f is the Frobenius. In this case, Hypothesis 5.6 can be rephrased as follows: d^0 takes pth powers to pth powers. In this case there is a condition on the spaces in the Ω-spectrum that will guarantee that Hypothesis 5.6 is satisfied.

Lemma 5.8. *Let $\sigma : E_*(\underline{E}_m) \to E_*(E)$ be the stabilization map. Suppose* $\ker(\sigma)$ *is the set of decomposable elements and x^p is primitive if and only if x is primitive. Then Hypothesis 5.6 is satisfied.*

Proof. Let $I_n = QG^n(M)$ for $n \geqslant 0$. Recall that $d^0 = \eta_*$. Since the stabilization map σ commutes with differentials, we have

$$\sigma(d^0(xy)) = d^0(\sigma(xy)) = 0,$$

and by hypothesis we have $d^0(xy) \in I^2_{n+1}$. Now, if x is primitive, $\pi(x)$ is primitive. Since d^0 is a coalgebra map, $d^0(x^p)$ is decomposable and primitive. But the only decomposable primitives are of the form y^p for some primitive y. □

6. The (generalized) composite functor spectral sequence

We can define a map $\sigma_\pi : P_\pi E_*(\underline{E}_n) \to E_*(E)$ as follows: let $x \in P_\pi E_*(\underline{E}_n)$ and suppose that x' is a representative in $PE_*(\underline{E}_n)$. Then $\sigma_\pi(x) = \sigma(x')$. Since σ kills decomposable elements, σ_π is well defined. We impose the following condition.

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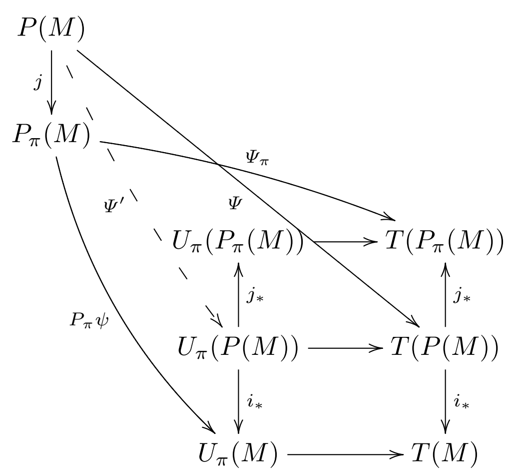
Hypothesis 6.1. The map σ_π is injective.

Remark 6.2. This is true if the hypotheses of Lemma 5.8 are satisfied.

Theorem 6.3. *Suppose that Hypothesis 6.1 is satisfied and let $U_\pi = P_\pi G$. Then U_π is the functor of a cotriple on $\mathcal{M}$.*

Proof. In [**3**, § 7], the authors define a cotriple (T, δ, ε) on $\mathcal{M}$ using the structure of the Hopf algebroid $(E_*, E_*(E))$. The stabilization map induces a map of cotriples $G \xrightarrow{\sigma} T$. From § 5 we know that if (M, ψ) is a G-coalgebra, then M is an $E_*(E)$-comodule and $P(M)$ is a sub-$E_*(E)$-comodule of M with map $\Psi : P(M) \to E_*(E)$. We have an induced $E_*(E)$-comodule $P_\pi(M)$. Let Ψ_π be the induced comodule map, $j : P(M) \to P_\pi(M)$ be the projection and $i : P(M) \to M$ be the injection. Then we have a commutative diagram:



where i_* is an injection, the rightmost j_* is a surjection and the horizontal maps are injections by Hypothesis 6.1. Since Ψ is induced from σ, there is a well-defined map $\psi_\pi : P_\pi(M) \to U_\pi(P_\pi(M))$. If we now let $M = G(N)$ for $N \in \mathcal{M}$, we have a map $\delta_\pi : U_\pi(N) \to U_\pi^2(N)$. We let the composition $U_\pi(N) \to T(N) \xrightarrow{\varepsilon} N$ be ε_π. Since $U_\pi \to T$ is an injection that commutes with the cotriple structure of T, $(U_\pi, \delta_\pi, \varepsilon_\pi)$ is a cotriple. □

With this result we can construct the category of U_π-coalgebras and state the following theorem.

Theorem 6.4. *Suppose that the homology theory E satisfies Hypotheses 5.6 and 6.1. There is then a spectral sequence*

$$\bar{E}_2^{m,n,t} = \mathrm{Ext}_{U_\pi}^{m,t}(R_0^n P_\pi(X)) \Rightarrow E_2^{n+m,t}(X)$$

(converging to the E_2-term of the BKSS based on E-theory).

Proof. Fix a $t \geqslant 0$. We form, for $n, m \geqslant 0$, the double complex concentrated at degree t:

$$D^{m,n,t} = D^{m,n}(X)_t = U_\pi^n P_\pi G^{m+1}(X)_t.$$

For each fixed n we have

$$D^{*,n,t} = D^{*,n}(X)_t = U_\pi^n P_\pi(\boldsymbol{G}(X)_t),$$

where the maps are induced by the G-resolution of X. Next we fix m. We have

$$D^{m,*,t} = D^{m,*}(X)_t = \boldsymbol{U_\pi}(P_\pi G^{m+1}(X))_t.$$

This is just the P_π-complex for $P_\pi G^m(X)$. Fixing m and taking homology we get

$$\bar{E}_1^{m,n,t} = \mathrm{Ext}_{U_\pi}^{m,t}(U_\pi(G^m(X))) = \begin{cases} G^m(X)_t & \text{if } n = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Taking homology again we have $E_2^{m,t}(X)$. If we fix n and take homology, we find that

$$\bar{E}_1^{m,n,t} = U_\pi^m R_0^n P_\pi(X)_t,$$

and taking homology again we find the following:

$$\bar{E}_2^{m,n,t} = \mathrm{Ext}_{U_\pi}^{m,t}(R_0^n P_\pi(X)).$$

□

This tells us that knowledge of the functors $R_0^m P_\pi(X)$ is sufficient to give us the E_2-term of the BKSS based on E. We now need information about these objects. We have the following theorem, which gives us a way to approach these objects.

Theorem 6.5. *Suppose that the homology theory E satisfies Hypotheses 5.6 and 6.1. There is then a spectral sequence*

$$E_2^{i,j} = R_i^j P_\pi(X) \Rightarrow R^{i+j} P_\pi(E_*(X)).$$

Proof. Consider the following double complex:

$$D^{i,j} = P_\pi S^{j+1} G^{i+1}(X).$$

Fixing i first, we have

$$D^{i,*} = P_\pi \tilde{\boldsymbol{S}}(G^{i+1}(X)).$$

This is just the functor P_π applied to the S-resolution for $G^{i+1}(X)$.

Now fixing j, we have

$$D^{*,j} = P_\pi S^{j+1} \boldsymbol{G}(X).$$

This map is induced by the G-resolution of M. Fixing i again and taking homology, we have

$$E_1^{i,j} = \begin{cases} R^{i+1} P_\pi(G^{j+1}(X)) & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

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Taking homology again we have $R_i^j P_\pi(X)$. Let us now fix j. We get

$$E_1^{i,j} = \begin{cases} P_\pi S^{j+1}(E_*(X)) & \text{if } i = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Homology again gives $R^j P_\pi(E_*(X))$. □

Corollary 6.6. *Suppose that the homology theory E satisfies Hypotheses 5.6 and 6.1. Suppose also that E is π-nice. There is then a long exact sequence*

$$\cdots \to R_0^k P_\pi(X) \to R^k P_\pi(E_*(X)) \to R_1^{k-1} P_\pi(X) \to R_0^{k+1} P_\pi(X) \to \cdots .$$

Proof. Since $E_*(\underline{E}_k)$ is nice, and since $G^r(X)$ is just a tensor product of these Hopf algebras, for $k > 1$, we have $R_k^j P_\pi(X) = 0$. We get a spectral sequence with just two rows. This becomes a long exact sequence. □

Theorem 6.7. *Suppose that $R^q P_\pi(E_*(X)) = 0$ for $q \geqslant n$:*

(i) $R_0^0 P_\pi(X) \cong P_\pi(E_*(X))$,

(ii) $R_0^q P_\pi(X) \cong R_1^{q+2} P_\pi(X)$ *for $q \geqslant n$,*

(iii) $R_0^1 P_\pi(X)$ *injects into $R^1 P_\pi(E_*(X))$ and, if $E_*(X)$ is cofree, then $R_0^1 P_\pi(X)$ is trivial.*

Proof. This follows from Corollary 6.6. □

Remark 6.8. If f is the zero map, then $P_f = P$, and Hypothesis 6.1 is merely the requirement that σ is injective on the primitives. This is satisfied by all p-local Landweber theories. Also, Hypothesis 5.6 is not necessary if E is Landweber.

7. The Morava K-theory case

We want to apply the previous results to $K(n)$. But first we need to make sure that Hypotheses 5.6 and 6.1 are satisfied. By Lemma 5.8 and Remark 6.2 it is sufficient that $\ker(\sigma)$ is the set of decomposable elements and that the only primitives are x and the pth powers of x where x is primitive.

Lemma 7.1. *If $E = K(n)$, then $\ker(\sigma)$ is the set of decomposable elements and x^p is primitive if and only if x is primitive. Also $K(n)$ is π-nice.*

Proof. By [**14**], $\Gamma_{n,m} = K(n)_*(\underline{K(n)}_m)$ has generators $a^I \circ b_{(0)}^{j_0} \circ b^J \circ e_1^\varepsilon$ with $\varepsilon = 0, 1$, $i_k = 0, 1$, $0 \leqslant j_k < p^n$, $j_0 < p^n - 1$. These elements stabilize to $\tau^I b^J$. So σ is injective on the indecomposables. Using the Hopf algebra structure computed in [**14**] one can conclude that x^p is primitive if and only if x is primitive.

To prove that $K(n)$ is π-nice is sufficient to prove that each $K(n)_*(\underline{K(n)}_k)$ is nice. By [**14**] we know that $K(n)_*(\underline{K(n)}_k)$ is a tensor product of exterior, truncated and polynomial Hopf algebras. The exterior algebra only contributes zero-derived functors, the (primitively generated) polynomial algebra is a tensor product of coalgebras of the

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type $T(x_{2n})$ (see [**4**]) and this has primitive dimension 1. So the only unknown is the truncated algebra. But by [**14**, Theorem 2.1] this is just a divided power algebra mod p and this is cofree. □

Remark 7.2. Hypothesis 6.1 is not satisfied by $E(1) \bmod p$. The element

$$v_1 b_{(0)}^{p-1} \circ b_{(1)}^{n} \circ e_1 - b_{(1)}^{n} \circ e_1 \circ [v_1] \in E(1)_*(\underline{E}(1)_{2n+1})$$

is non-zero unstably but it is in $\ker(\sigma)$. This does not happen in $K(1)$ because we have the extra relation $v_1 b_{(0)}^{p-1} \circ b_{(1)}^{n} \circ e_1 = b_{(1)}^{n} \circ e_1 \circ [v_1]$.

8. Applications to $K(1)$

We apply all the previous results to $K(1)$. In this case we can say much more.

Corollary 8.1. *Suppose that $R^i P_\pi(K(1)_*(X)) = 0$ for $i \geqslant n$. Then, for $i \geqslant n+1$,*

$$R_0^i P_\pi(X) = 0.$$

Proof. Suppose $i > n+1$ and $R_0^i P_\pi(X) \neq 0$ and we have a generator x. Since we can get new generators from x by multiplying by v_1^k, we consider $R^1 P_\pi(G^{i+2}(X)) \otimes \mathbb{Z}_p$. By the universal coefficients theorem and Lemma 7.1, we know that the only part contributing first-derived functors is the polynomial part. This is of the form $\otimes_{\alpha \in \Lambda} T(x_{2n_\alpha})$ for some index set Λ. The generator x_{2n_α} is of the form $a_0 \circ b^{J_\alpha}$ with dimension $2(1 + \sum j_{i_\alpha} p^i)$ with $j_0 < p-1$ and $n_\alpha = 1 + \sum j_i p^i$. Since $R^1 P(T(x_{2n_\alpha})) = \mathbb{Z}_p(2n_\alpha p)$, we have generators of degree $2p(1 + \sum j_{i_\alpha} p^i)$. Now let us look at generators for

$$R^0 P_\pi(G^{i+1}(X)) \otimes \mathbb{Z}_p = P_\pi(G^{i+1}(X)) \otimes \mathbb{Z}_p.$$

This has generators $a_{(0)} \circ b^K \circ e_1^\varepsilon$ and $b_{(0)} \circ b^K \circ e_1^\varepsilon$ of degree $2(1 + \sum_{k \geqslant 0} k_i p^i) + \varepsilon$ with $k_0 < p-1$ in both cases. By Theorem 6.7(i) the dimensions of the generators have to agree. So $\varepsilon = 0$. But the generators of the first case were divisible by p and neither of the other two are. □

In fact this result implies the following theorem.

Theorem 8.2. *Let X be an H-space and suppose that $K(1)_*(X)$ is cofree. Then*

$$E_2^{s,t}(X) \cong \operatorname{Ext}_{U_\pi}^{s,t}(P_\pi(K(1)_*(X))).$$

Proof. Since $K(1)_*(X)$ is cofree, by 8.1 we know that we have only zero-derived functors of P_π, and so the spectral sequence of Theorem 6.4 collapses and the result follows. □

Corollary 8.3. *Suppose that $K(1)_*(X)$ is cofree. Then*

$$E_2(X) \cong E_2^{\mathcal{S}}(\Sigma^\infty X),$$

where the object on the right is the E_2-term of the stable Adams spectral sequence.

Proof. The category U_π is just the category of unstable $K(1)_*(K(1))$-comodules. By [**12**], there are no unstable $K(n)_*(K(n))$-comodules. Since we have a collapse to the zero line of the spectral sequence of Theorem 6.4, then the two E_2-terms have to agree. □

Remark 8.4. From the previous corollary, the E_2-terms of the stable spectral sequence for the sphere and the unstable spectral sequence for the odd sphere agree. This does not happen with the even sphere. By Lemma 5.4, $R^1P_\pi(S^{2n})$ is a $K(1)_*$-module with a generator in dimension $4n$. This leads to a long exact sequence:

$$\cdots \to \mathrm{Ext}^{s,t}_{U_\pi}(S^{2n}) \to \mathrm{Ext}^{s,t}_{\mathcal{M}(G)}(S^{2n}) \to \mathrm{Ext}^{s-1,t}_{U_\pi}(S^{4n}) \to \cdots,$$

where the last map on the left has bidegree $(2,0)$.

We now develop a way of computing $\mathrm{Ext}_{U_\pi}(M)$ for all $K(n)$. With this we can calculate specific elements in the $K(n)$-based spectral sequence.

9. Calculation of $U_\pi(M)$ for Morava K-theories

Our knowledge of $\mathrm{Ext}_{U_\pi}(M)$ depends on $U^n_\pi(M)$ and the differentials.

From [**15**] we know that

$$K(n)_*(K(n)) = \Lambda[\tau_0,\dots,\tau_{n-1}] \otimes K(n)_* \otimes_{BP_*} \frac{BP_*(BP)}{(v_n t_i^{p^n} - v_n^{p^i} t_i)}.$$

As in [**3**], we use the basis consisting of h_n instead of t_n. Since the τ_k comes from the dual of the Steenrod algebra, this implies that the canonical anti-isomorphism is given by

$$\tau_k + \sum_{i=0}^{k} t_{k-i}^{p^i} c(\tau_i) = 0$$

or $\tau_k = c(\tau_k)$ mod decomposables. Let $\beta_k = c(\tau_k)$. Then the exterior part of Γ_n is generated by β_i with $0 \leqslant i \leqslant n-1$.

Notation 9.1. We define

$$<_i = \begin{cases} < & \text{for } i = 0, \\ \leqslant & \text{for } i = 1. \end{cases}$$

Theorem 9.2. *Let $\Gamma_n = K(n)_*(K(n))$ and M be a $K(n)_*$-module. Then the following conditions apply.*

(i) $U_\pi(M) = \mathrm{span}_{K(n)_*}\{\beta^I h^J \otimes m \in \Gamma_n \otimes M \mid l(I) + 2l(J) <_{i_0} |m|\}$.

(ii) $U_\pi(M)$ *injects into the stable cobar complex.*

(iii) *Suppose that M is an unstable Γ-comodule, with coaction $\Psi : M \to U_\pi(M)$. The differential is then given by*

$$d([\gamma_1 \mid \cdots \mid \gamma_n]m) = [1 \mid \gamma_1 \mid \cdots \mid \gamma_n]m + \sum_{j=1}^{n}(-1)^j[\gamma_1 \mid \cdots \mid \gamma'_j \mid \gamma''_j \mid \cdots \mid \gamma_n] + (-1)^{n+1}\sum[\gamma_1 \mid \cdots \mid \gamma\prime_n]m',$$

where $\Psi(\gamma_j) = \sum \gamma' \otimes \gamma''$ and $\Psi(m) = \sum \gamma' \otimes m'$.

Proof. The primitive elements in $K(n)_*(\underline{K(n)}_m)$ are

(i) exterior: $[v_n^k] \circ a^I \circ b_0^{j_0} \circ b^J \circ e_1$ with $i_0 = 1$ or $j_0 = 1$;

(ii) truncated: $[v_n^k] \circ a^I \circ b_0^{j_0} \circ b^J$ with $I \neq I(1)$ and $i_0 = 1$ or $j_0 = 1$;

(iii) polynomial: $([v_n^k] \circ a^{I(1)} \circ b_0^{j_0} \circ b^J)^{*p^k}$, $k \geqslant 1$.

The cases (i) and (ii) are also indecomposable. In case (iii) the only element that is indecomposable is when $k = 1$. All of the cases suspend to $\beta^I h^J \otimes v_n^k i_m$. We have, for all cases,

$$l(I) + 2j_0 + 2l(J) - kq_n = m$$

or

$$l(I) + 2l(J) \leqslant l(I) + 2j_0 + 2l(I) = m + kq_n = |v_n^k i_m|,$$

where $q_n = |v_n|$. The inequality on the left-hand side is strict if $j_0 = 1$ and is an equality if $i_0 = j_0 = 1$. This proves (i).

Part (ii) follows immediately from (i). Part (iii) follows from the fact that we know the differential in the stable cobar. Since $U_\pi(M)$ injects, the result follows. □

At last we see why we chose the derived functors of the π-primitives instead of just the usual derived functors of the primitives: these do not inject into the stable object (the pth powers of $[v^k] \circ a_{(0)} \circ b_{(0)}^{j_0} \circ b^I$ are killed by σ).

Let us find some elements in $E_2(S^{2n+1}; K(1))$. In this case we have $\beta_0 = c(\tau_0) = \tau_0$. The only elements in $\Gamma = K(1)_*(K(1))$ that are primitive are h_1^{pk}, $k > 0$, and τ_0. But, by [**15**], in Γ we have $h_1 v_1^p = h_1^p v_1$, so we have only to consider h_1 and τ_0. We therefore see that, for $M = K(1)_*(S^{2n+1})$, $n > 0$, E_2 is generated by h_1 and τ_0.

From now on the element $[\gamma_1 \mid \cdots \mid \gamma_n]i_{2n+1}$ will be represented in homology by $\gamma_1 \cdots \gamma_n i_{2n+1}$ or, if it is clear on which sphere we are working, $i_{2n+1} = 1$ and $\gamma_1 \cdots \gamma_n$, with the convention that $\deg(\gamma_1 \cdots \gamma_n) = 2n + 1 + \sum \deg(\gamma_i)$. Since for $K(1)$ the right action and the left actions commute, we immediately have

$$E_1^{0,t}(S^{2n+1}) = \begin{cases} \mathbb{Z}_p & \text{if } t = 2n+1+kq,\ k \in \mathbb{Z} \text{ generated by } v_1^k, \\ 0 & \text{otherwise.} \end{cases}$$

The filtration-r elements τ_0^r and $v_1^k h_1 \tau_0^{r-1}$ are non-trivial.

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10. Composition pairings in the $K(1)$ spectral sequence

By [**7**], there is also composition pairing in the spectral sequence for $t-s \geqslant 1$ and $r \geqslant 2$:

$$E_r^{s,m+t}(X) \otimes E_r^{s',t'}(S^m) \xrightarrow{\circ} E_r^{s+s',t+t'}(X).$$

Given the natural map $i : X \to X_E^\wedge$, where $X_E^\wedge$ is the completion, there is a commutative diagram:

$$\begin{array}{ccc} \pi_{t+m}X \otimes \pi_{t'}S^m & \xrightarrow{\;*\;} & \pi_{t+t'}X \\ \big\downarrow{\scriptstyle i_*\otimes i_*} & & \big\downarrow{\scriptstyle i_*} \\ \pi_{t+m}X_E^\wedge \otimes \pi_{t'}(S^m)_E^\wedge & \xrightarrow{\;\circ\;} & \pi_{t+t'}X_E^\wedge \end{array}$$

Lemma 10.1. *The composition corresponds (up to sign) to the Yoneda product in the category U_π:*

$$\begin{aligned} &\operatorname{Ext}^s_{U_\pi}(K(1)_*(S^{t+2m+1}), K(1)_*(S^{2n+1})) \otimes \operatorname{Ext}^{s'}_{U_\pi}(K(1)_*(S^{t'}), K(1)_*(S^{2m+1})) \\ &\quad \to \operatorname{Ext}^s_{U_\pi}(K(1)_*(S^{t+2m+1}), K(1)_*(S^{2n+1})) \otimes \operatorname{Ext}^{s'}_{U_\pi}(K(1)_*(S^{t+t'}), K(1)_*(S^{t+2m+1})) \\ &\qquad \to \operatorname{Ext}^{s+s'}_{U_\pi}(K(1)_*(S^{t+t'}), K(1)_*(S^{2n+1})). \end{aligned}$$

Proof. This follows from [**7**]. □

We use this result to study compositions by τ_0. The proofs of the next two results are analogous to the proof of the stable statements.

Claim 10.2. *For $k > 0$, $\tau_0^k \neq 0$.*

As in the stable case, we have an element representing multiplication by p in homotopy.

Lemma 10.3. *Suppose that the spectral sequences for X and S^{2n+1} converge. Suppose also that $x \in E_2$ survives to E_∞ and represents $\alpha \in \pi_*(X)$. Then $a_{(0)} \circ x$ represents $p\alpha \in \pi_*(X)$.*

Form this it follows that we have infinite towers, for $k \in \mathbb{Z}$, in dimensions $t-s = 2n+1+kq$, generated by $v_1^k\tau_0^s$, and towers in dimension $t-s = 2n+kq$ generated by $v_1^k\tau_0^s h_1$. The only thing missing is knowledge about the differentials.

11. Convergence of the stable Adams spectral sequence

The next result will give us the missing piece. We define $\nu(k)$ as $k = ap^{\nu(k)}$ with $p \nmid a$.

Theorem 11.1. *The stable Adams spectral sequence based on $K(1)$ of the sphere converges for $t-s > 0$ and v_1^k supports a $d_{\nu(k)+2}$ differential.*

Proof. Since the only place in which the E_2-term has classes is in dimensions $t-s = kq-1, kq$ with $k \in \mathbb{Z}$, it is sufficient to worry about those dimensions. By [**13**] we know that the $E(1)$ spectral sequence for the sphere converges and $\pi_*(S^\wedge_{E(1)})$ has a $\mathbb{Z}_{p^{\nu(k)+1}}$ generated by $\alpha_k = d_1(v_1^k)/p^{\nu(k)+1}$ in dimension $kq-1$. We have a map of ring spectra

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$j : E(1) \to K(1)$, which induces a map between the spectral sequence of these spectra and sends α_k to $v_1^{k-1}h_1$. Since $E_r^{s,t} = 0$ for any $t - s = kq - 2$, the element

$$v_1^{k-1}h_1 \in \varprojlim_s \pi_{kq-1}(\overline{K(1)}^s)$$

and since there is an onto map to $\pi_{kq-1}(S^\wedge_{K(1)})$ we have

$$v_1^{k-1}h_1 \in \pi_{kq-1}(S^\wedge_{K(1)}).$$

This implies that $1 \leqslant \operatorname{ord}(v_1^{k-1}h_1) \leqslant p^{\nu(k)+1}$. Since multiplication by τ_0 represents multiplication by p, the tower over $v_1^{k-1}h_1$ has to be killed at filtration less than or equal to $\nu(k) + 2$. So v_1^k supports a d_r differential, where $r \leqslant \nu(k) + 2$.

We prove $\nu(k) + 2 = r$ by induction on d_n. For $k = 1$, we have, by the previous paragraph, $d_2(v_1) = \tau_0 h_1$ and, using the derivation rule, we have $d_2(v_1^k\tau_0^w) = kv_1^{k-1}\tau_0^{w+1}h_1$. Suppose now that

$$d_m(v_1^{kp^{m-2}}\tau_0^w) = kv_1^{kp^{m-2}-1}\tau_0^{w+m-1}h_1 \quad \text{for } m < n.$$

By the induction hypothesis we know that $d_{n-1}(v_1^{p^{n-2}}) = pv_1^{p^{n-2}-1}\tau_0^{n-2}h_1 = 0$. So the smallest differential in which $v_1^{p^{n-2}}$ is non-zero is $d_n = d_{\nu(n-2)+2}$. The general result follows from using the derivation rule. □

Corollary 11.2. *The BKSS based on $K(1)$ for the sphere converges completely and, for $m \geqslant 1$, the completion in the sense of [**1**] is given by*

$$\pi_m((S^{2n+1})^\wedge) = \begin{cases} \mathbb{Z}_{p^{\nu(k)+1}} & \text{for } m = 2n + kq,\ k \in \mathbb{Z} - \{0\}, \\ \mathbb{Z}_p^\wedge & \text{for } m = 2n+1, 2n, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The differentials on v_1^k for $k \geqslant 0$ can be deduced by the stable differentials. For $k \leqslant 0$, [**7**] says that d_r is a derivation for $r > 1$. We have the following formulae:

$$0 = d_r(1) = d_r(v_1^k v_1^{-k}) = d_r(v_1^k)v_1^{-k} + v_1^k d_r(v_1^{-k})$$

or $d_r(v_1^{-k}) = -v_1^{-2k}d_r(v_1^k)$. Since we have a vanishing line, by [**8**, § IX] we know that the spectral sequence converges completely. □

12. Some remarks on the general $K(n)$ spectral sequence

Although we have not been able to prove that the $R_0^i P_\pi(X)$ vanish when the groups $R^i P_\pi(K(n)_*(X))$ vanish for $n > 1$, we have the following theorem.

Theorem 12.1. *The zero line of the spectral sequence of Theorem 6.4 for $K(n)$ injects into the stable Adams spectral sequence.*

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Proof. The spectral sequence of Theorem 6.4 has an edge homomorphism:

$$\mathrm{Ext}^{m,t}_{U_\pi}(P_\pi K(n)_*(X)) = \bar{E}_2^{m,0,t} \to E_2^{m,t}(X).$$

Since this construction commutes with stabilization, we have a commutative diagram:

$$\begin{array}{ccccc} \mathrm{Ext}^{m,t}_{U_\pi}(P_\pi K(n)_*(X)) & \longrightarrow & \mathrm{Ext}^{m,t}_{\mathcal{M}(G)}(K(n)_*(X)) & = & E_2^{m,t}(X) \\ \cong\downarrow & & \sigma\downarrow & & \sigma\downarrow \\ \mathrm{Ext}^{m,t'}_{\Gamma_n}(K(n)_*(X)) & \xrightarrow{\cong} & \mathrm{Ext}^{m,t'}_{\Gamma_n}(K(n)_*(\Sigma^\infty X)) & = & E_2^{m,t'}(\Sigma^\infty X) \end{array}$$

where $\Gamma_n = K(n)_*(K(n))$. □

13. The $K(1)$-completion of S^{2n+1} and its relation to the work of Farjoun

In [**10**], Farjoun defines a tower $\{Y_n(X)\}$ under X as follows. Let $E(X)$ be the functor defined in § 2 and let $Y_1(X) = E(X)$. Define

$$Y_n(X) = \mathrm{fb}\left[Y_{n-1}(X) \to E\left(\frac{Y_{n-1}(X)}{X}\right)\right],$$

where fb denotes the homotopy fibre. The null homotopy from X into the cofibre gives a map $X \to Y_n(X)$. Let $Y_\infty(X) = \varprojlim Y_n(X)$. Farjoun puts forward the following two questions.

Question 13.1. Let X be an H-space of finite type. Is the natural map $X_E \to Y_\infty$ a homotopy equivalence?

Question 13.2. When does the natural map of towers $\{Y_n(X)\}_{n\in\mathbb{N}} \to \{D^n(X)\}_{n\in\mathbb{N}}$ have a left inverse?

Although we cannot answer Question 13.1 in the affirmative or get necessary conditions to answer Question 13.2, we can deduce the following from our work.

Lemma 13.3. *Let $E = K(1)$. Then Questions 13.1 and 13.2 cannot be true at the same time for $X = S^{2n+1}$.*

Proof. Suppose that if X is an H-space of finite type, this implies that the natural map $X_E \to Y_\infty$ is a homotopy equivalence and the map of towers from Question 13.2 has a left inverse. We then have an injective map

$$\pi_*(S^{2n+1}_{K(1)}) \to \pi_*(K(1)^\wedge(S^{2n+1})).$$

The $K(1)$-localization of the odd spheres was calculated in [**6**]. We have

$$\pi_i(S^{2n+1}_{K(1)}) = \begin{cases} \mathbb{Z}_{p^{\nu(k)+1}} & \text{for } i = 2n+qk, 2n-1+kq,\ k \in \mathbb{Z}-\{0\}, \\ \mathbb{Z}_p^\wedge & \text{for } i = 2n, 2n-1, \\ \mathbb{Z}_p^\wedge \oplus \mathbb{Z}_p^\wedge & \text{for } i = 2n-1, \\ 0 & \text{otherwise.} \end{cases}$$

Comparing this with the result of Corollary 11.2, we see that the map cannot be injective. □

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