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The β-transformation with a hole at 0

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Abstract. For $\beta \in (1, 2]$ the β-transformation $T_\beta : [0, 1) \to [0, 1)$ is defined by $T_\beta(x) = \beta x \pmod 1$. For $t \in [0, 1)$ let $K_\beta(t)$ be the survivor set of T_β with hole $(0, t)$ given by

$$K_\beta(t) := \{x \in [0, 1) : T_\beta^n(x) \notin (0, t) \text{ for all } n \geq 0\}.$$

In this paper we characterize the bifurcation set E_β of all parameters $t \in [0, 1)$ for which the set-valued function $t \mapsto K_\beta(t)$ is not locally constant. We show that E_β is a Lebesgue null set of full Hausdorff dimension for all $\beta \in (1, 2)$. We prove that for Lebesgue almost every $\beta \in (1, 2)$ the bifurcation set E_β contains infinitely many isolated points and infinitely many accumulation points arbitrarily close to zero. On the other hand, we show that the set of $\beta \in (1, 2)$ for which E_β contains no isolated points has zero Hausdorff dimension. These results contrast with the situation for E_2, the bifurcation set of the doubling map. Finally, we give for each $\beta \in (1, 2)$ a lower and an upper bound for the value τ_β such that the Hausdorff dimension of $K_\beta(t)$ is positive if and only if $t < \tau_\beta$. We show that $\tau_\beta \leq 1 - (1/\beta)$ for all $\beta \in (1, 2)$.

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1. *Introduction*

In recent years open dynamical systems, i.e., systems with a hole in the state space through which mass can leak away at every iteration, have received a lot of attention. Typically one wonders about the rate at which mass leaves the system and about the size and structure of the set of points that remain, called the *survivor set*. In [**Urb86**, **Urb87**] Urbański considered C^2-expanding, orientation-preserving circle maps with a hole of the form $(0, t)$. He studied the way in which the topological entropy of such a map restricted to the survivor set changes with t. To be more precise, let g be a C^2-expanding and orientation-preserving map on the circle $\mathbb{R}/\mathbb{Z} \sim [0, 1)$. For $t \in [0, 1)$ let $K_g(t)$ be the survivor set defined by

$$K_g(t) := \{x \in [0, 1) : g^n(x) \notin (0, t) \text{ for all } n \geq 0\}.$$

Urbański proved that the function $t \mapsto h_{\text{top}}(g|K_g(t))$ is a devil's staircase, where h_{top} denotes the topological entropy.

Motivated by the work of Urbański, we consider this situation for the β-transformation. Given $\beta \in (1, 2]$, the β-transformation $T_\beta : [0, 1) \to [0, 1)$ is defined by $T_\beta(x) = \beta x \pmod 1$. When $\beta = 2$, we recover the doubling map. In correspondence with [**Urb86**], set

$$K_\beta(t) := \{x \in [0, 1) : T_\beta^n(x) \notin (0, t) \text{ for all } n \geq 0\}. \tag{1.1}$$

The survivor set $K_\beta(t)$ splits naturally into two pieces, $K_\beta(t) = K_\beta^0(t) \cup K_\beta^+(t)$, where

$$\begin{aligned} K_\beta^0(t) &= \{x \in [0, 1) : \exists n \ T_\beta^n(x) = 0 \ \text{ and } T_\beta^k(x) \notin (0, t) \text{ for all } 0 \leq k < n\}, \\ K_\beta^+(t) &= \{x \in [0, 1) : T_\beta^n(x) \geq t \ \text{ for all } n \geq 0\}. \end{aligned} \tag{1.2}$$

The set $K_\beta^+(t)$ occurs in Diophantine approximation. Indeed, consider the set

$$F_\beta(t) := \{x \in [0, 1) \mid T_\beta^n(x) \geq t \text{ for all but finitely many } n \in \mathbb{N}\}$$

of points $x \geq 0$ such that 0 is badly approximable by its orbit under T_β. Then $F_\beta(t)$ can be written as a countable union of affine copies of $K_\beta^+(t)$ and thus $\dim_H F_\beta(t) = \dim_H K_\beta^+(t)$ for all $t \in [0, 1)$. The approximation properties of β-expansions have been studied by several authors. In [**LPWW14**] the authors considered the Hausdorff dimension of the set of values $\beta > 1$ for which the orbit of 1 approaches a given target value x_0 at a given speed. This work generalized that of [**PS08**], where $x_0 = 0$ and the speed is fixed. Other results on the Diophantine approximation properties of β-expansions can be found in [**Nil09**, **BW14**, **Cao14**, **GL15**, **LW16**] among others.

Note that the set-valued map $\epsilon \mapsto K_\beta(\epsilon)$ is weakly decreasing. Further on, we show that this map is locally constant almost everywhere, i.e., for almost all $t \in [0, 1)$ there exists a $\delta > 0$ such that $K_\beta(\epsilon) = K_\beta(t)$ for all $\epsilon \in [t - \delta, t + \delta]$. Such a result was also obtained by Urbański in [**Urb86**] for C^2-expanding circle maps. This fact motivates the study of the *right set valued bifurcation set* (simply called *bifurcation set*) E_β containing all parameters $t \in [0, 1)$ such that the set-valued map $\epsilon \mapsto K_\beta(\epsilon)$ is not locally constant on any right-sided neighbourhood of t, i.e.,

$$E_\beta := \{t \in [0, 1) : K_\beta(\epsilon) \neq K_\beta(t) \text{ for any } \epsilon > t\}. \tag{1.3}$$

The local structure of the sets $K_2(t)$ and E_2 was investigated in detail in [**Urb86**, **Nil09**]. The following results can be found more or less explicitly in [**Urb86**]. More recently it was shown in [**Nil09**] that these properties could also be dealt with using more elementary combinatorial methods.

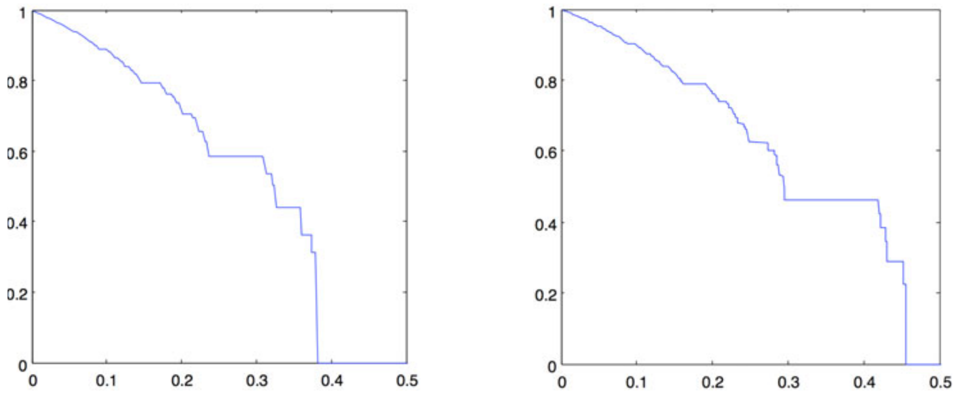

FIGURE 1. Left: the numerical plot of η_β with $\beta \approx 1.61803$ the golden ratio. Right: the numerical plot of η_β with $\beta \approx 1.83929$ the tribonacci number.

THEOREM 1.1. (Urbański [**Urb86**])

(i) *The bifurcation set E_2 is a Lebesgue null set of full Hausdorff dimension.*

(ii) *The function $\eta_2 : t \mapsto \dim_H K_2(t)$ is a devil's staircase:*
- η_2 *is decreasing and continuous on* $[0, \frac{1}{2}]$*;*
- $\eta_2'(t) = 0$ *for Lebesgue almost every* $t \in [0, \frac{1}{2}]$*;*
- $\eta_2(0) = 1$ *and* $\eta_2(\frac{1}{2}) = 0$.

(iii) *The topological closure $\overline{E_2}$ is a Cantor set.*

(iv) $\eta_2(t) > 0$ *if and only if* $t < \frac{1}{2}$.

Other results on survivor sets for the doubling map T_2 can be found in e.g. [**BY11**, **Det13**, **AB14**, **Sid14**, **GS15**, **CT17**].

An important ingredient for the proofs in [**Urb86**, **CT17**] is the fact that

$$E_2 = \{t \in [0, 1) : T_2^n(t) \geq t \text{ for all } n \geq 0\}.$$

This identity does not hold in general for $1 < \beta < 2$. Therefore, we define E_β^+ by

$$E_\beta^+ := \{t \in [0, 1) : T_\beta^n(t) \geq t \text{ for all } n \geq 0\}. \tag{1.4}$$

Note that $E_\beta^+ \subseteq E_\beta$ but in general these sets do not coincide. In this paper we consider the survivor set $K_\beta(t)$ and the bifurcation set E_β for $\beta \in (1, 2)$. We give a detailed description of the topological structure of E_β and E_β^+ and their dependence on β. Theorems A to D below list our main results. Our first result strengthens (i) and (ii) of Theorem 1.1.

THEOREM A. *Let $\beta \in (1, 2]$ and $t \in [0, 1)$.*

(i) *The bifurcation sets E_β and E_β^+ are Lebesgue null sets of full Hausdorff dimension.*

(ii) *The dimension function $\eta_\beta : t \mapsto \dim_H K_\beta(t)$ is a devil's staircase:*
- η_β *is decreasing and continuous in* $[0, 1)$*;*
- $\eta_\beta' = 0$ *Lebesgue almost everywhere in* $[0, 1)$*;*
- η_β *is not constant.*

Figure 1 shows numerical plots of the dimension functions η_β for $\beta \approx 1.61803$, the golden ratio, i.e., the real root bigger than 1 of the polynomial $x^2 - x - 1$ and for

$\beta \approx 1.83929$, the tribonacci number, i.e., the real root bigger than 1 of the polynomial $x^3 - x^2 - x - 1$.

The analogous statements of (iii) and (iv) of Theorem 1.1 for $\beta \in (1, 2)$ do not always hold. The next main theorems show that in general the topological structure of E_β differs from that of E_2 and that this structure depends on the value of β. Theorems B and C imply that (iii) of Theorem 1.1 holds only for a very small set of $\beta \in (1, 2)$.

THEOREM B. *For Lebesgue almost every $\beta \in (1, 2)$ the bifurcation sets E_β and E_β^+ contain infinitely many isolated and accumulation points arbitrarily close to zero and hence their closures are not Cantor sets. On the other hand,*

$$\dim_H(\{\beta \in (1, 2) : \exists \delta > 0 \textit{ such that } \overline{E_\beta^+} \cap [0, \delta] \textit{ is a Cantor set}\}) = 1.$$

There are also infinitely many $\beta \in (1, 2]$ such that $\overline{E_\beta^+}$ is a Cantor set. This is true, for example, for the countable family of multinacci numbers. In terms of Hausdorff dimension this set is small.

THEOREM C. *We have* $\dim_H(\{\beta \in (1, 2) : \overline{E_\beta^+}$ *is a Cantor set*$\}) = 0$.

In [**Cla16**] Clark considered the β-transformation and characterized the holes of the form (a, b) for which the survivor set $K_\beta((a, b))$ is uncountable or not. From the properties of η_β given in Theorem A it follows that for each $\beta \in (1, 2]$ there is a unique value τ_β such that $\dim_H K_\beta(t) > 0$ if and only if $t < \tau_\beta$. By (iv) of Theorem 1.1 we know that $\tau_2 = 1/2$. We have the following result on τ_β.

THEOREM D. *For each $\beta \in (1, 2]$ we have $\tau_\beta \le 1 - (1/\beta)$ and $\tau_\beta = 1 - (1/\beta)$ if and only if $\overline{E_\beta^+}$ is a Cantor set.*

In [**Nil07**] Nilsson studied the critical value $\tilde{\tau}_\beta$ for the β-transformation with holes of the form $(t, 1)$. In [**Nil07**, Proposition 7.12] he proved that for each $\beta \in (1, 2)$, $\tilde{\tau}_\beta = 1 - (1/\beta)$. Many of the proofs use the symbolic codings of the open systems T_β with hole $(t, 1)$. The main difficulty that we had to overcome in order to extend the results from the doubling map to the β-transformation is that the β-transformation is not coded by the full shift on two symbols. In fact, for most values of β, the associated symbolic system is not even sofic. This might also explain the difference between the result from Theorem D and the result from [**Nil07**, Proposition 7.12].

The paper is arranged as follows. In §2 we introduce some notation, we recall some basic properties of β-expansions and prove Theorem A. In §3 we consider the topological structure of E_β and E_β^+ and prove Theorem B. By means of Lyndon words we construct infinitely many nested basic intervals which cover the interval $(1, 2)$ up to a Lebesgue null set. We can determine all isolated points of E_β^+ by determining in which intervals it falls. The largest of these intervals are then associated to Farey words, the properties of which allow us to prove Theorem C in §4 and Theorem D in §5.

2. *Preliminaries, β-expansions and first properties of $K_\beta(t)$ and E_β*

In this section we introduce some notation about sequences that is used throughout the paper. We will recall some basic properties of β-transformations and give some basic results on $K_\beta(t)$ and E_β. We also prove Theorem A.

2.1. *Notation on sequences.* Let $\{0,1\}^{\mathbb{N}}$ be the set of sequences of 0's and 1's and let σ be the left shift on $\{0,1\}^{\mathbb{N}}$ defined by $\sigma((x_i)) = (x_{i+1})$. We use $\{0,1\}^*$ to denote the set of all finite strings of elements from $\{0,1\}$, called *words*. A word $w \in \{0,1\}^n$ is called a *prefix* of a sequence $(x_i) \in \{0,1\}^{\mathbb{N}}$ if $x_1 \ldots x_n = w$. For a word $w = w_1 \ldots w_n \in \{0,1\}^*$ we write $w^+ := w_1 \ldots w_{n-1}(w_n + 1)$ if $w_n = 0$ and we write $w^- := w_1 w_2 \ldots w_{n-1}(w_n - 1)$ if $w_n = 1$. Furthermore, we use $\overline{w}$ to denote the *reflection word* $\overline{w} := (1 - w_1)(1 - w_2) \ldots (1 - w_n)$.

Throughout the paper we use the lexicographical ordering $\prec$, $\preccurlyeq$, $\succ$ and $\succcurlyeq$ between sequences and words, which is defined as follows. For two sequences $(x_i), (y_i) \in \{0,1\}^{\mathbb{N}}$ we write $(x_i) \prec (y_i)$ or $(y_i) \succ (x_i)$ if $x_1 < y_1$, or there is an integer $m \geq 2$ such that $x_i = y_i$ for all $i < m$ and $x_m < y_m$. Moreover, we say that $(x_i) \preccurlyeq (y_i)$ or $(y_i) \succcurlyeq (x_i)$ if $(x_i) \prec (y_i)$ or $(x_i) = (y_i)$. This definition can be extended to words in the following way. For $u, v \in \{0,1\}^*$ we write $u \prec v$ if and only if $u0^\infty \prec v0^\infty$.

Let $\#A$ denote the cardinality of the set A. For a subset $\mathcal{Y} \subseteq \{0,1\}^{\mathbb{N}}$ let $\mathcal{B}_n(\mathcal{Y})$ denote the set of all words of length n that occur in a sequence in $\mathcal{Y}$. The *topological entropy* of $\mathcal{Y}$ is then given by

$$h(\mathcal{Y}) := \lim_{n\to\infty} \frac{\log \#\mathcal{B}_n(\mathcal{Y})}{n} = \inf_n \frac{\log \#\mathcal{B}_n(\mathcal{Y})}{n},$$

where the second equality holds since by the definition of $\mathcal{B}_n(\mathcal{Y})$ the sequence $(\log \#\mathcal{B}_n(\mathcal{Y}))$ is subadditive. Here and throughout the paper we will use the base-2 logarithm.

2.2. *The β-transformation and β-expansions.* Now we recall some properties of the β-transformation. Let $\beta \in (1,2]$ and let the (greedy) *β-transformation* $T_\beta : [0,1) \to [0,1)$ be given as in the introduction, i.e., $T_\beta(x) = \beta x \pmod 1$. It has a unique ergodic invariant measure that is equivalent to the Lebesgue measure (cf. [**Rén57**]). This measure is the unique measure of maximal entropy with entropy equal to $\log \beta$. For each $x \in [0,1)$ the *greedy β-expansion* of x, denoted by $b(x,\beta) = (b_i(x,\beta))$, is the sequence obtained from T_β by setting for each $i \geq 1$,

$$b_i(x,\beta) = \begin{cases} 0 & \text{if } T_\beta^{i-1}(x) \in [0, \frac{1}{\beta}), \\ 1 & \text{if } T_\beta^{i-1}(x) \in [\frac{1}{\beta}, 1). \end{cases}$$

The name greedy β-expansion stems from the fact that it is the lexicographically largest sequence $(x_i) \in \{0,1\}^{\mathbb{N}}$ satisfying

$$x = \sum_{i\geq 1} \frac{x_i}{\beta^i} =: \pi_\beta((x_i)). \tag{2.1}$$

We write $b(1,\beta)$ for the sequence $1b(\beta - 1, \beta)$.

The set of sequences that occur as greedy β-expansions for a given β can be characterized using quasi-greedy β-expansions. For each $x \in (0, 1]$ the quasi-greedy β-expansion of x is obtained dynamically by iterating the map $\widetilde{T}_\beta : (0, 1] \to (0, 1]$ given by

$$\widetilde{T}_\beta(x) = \begin{cases} \beta x & \text{if } x \in (0, \frac{1}{\beta}], \\ \beta x - 1 & \text{if } x \in (\frac{1}{\beta}, 1]. \end{cases}$$

The only essential difference between the maps T_β and $\widetilde{T}_\beta$ is the value they take at the point $1/\beta$. For $x \in (0, 1]$ the *quasi-greedy* β-expansion $\tilde{b}(x, \beta) = (\tilde{b}_i(x, \beta))$ is then obtained by setting $\tilde{b}_i(x, \beta) = 0$ if $0 < \widetilde{T}_\beta^{i-1}(x) \le (1/\beta)$ and $\tilde{b}_i(x, \beta) = 1$ if $(1/\beta) < \widetilde{T}^{i-1}(x) \le 1$. The quasi-greedy β-expansion of 1 plays a crucial role in what follows. For $\beta \in (1, 2]$ write

$$\alpha(\beta) := \tilde{b}(1, \beta).$$

Note that if $b(x, \beta) = b_1 \ldots b_n 0^\infty$ with $b_n = 1$, then $\tilde{b}(x, \beta) = b_1 \ldots b_n^- \alpha(\beta)$. On the other hand, if $b(x, \beta)$ does not end with 0^∞, then $b(x, \beta) = \tilde{b}(x, \beta)$. The following characterization of $\alpha(\beta)$ can be found in [**KL07**, Theorem 2.3]. Let $\mathcal{Q} \subset \{0, 1\}^{\mathbb{N}}$ be the set of sequences $(a_i) \in \{0, 1\}^{\mathbb{N}}$ not ending with 0^∞ and satisfying

$$a_{n+1}a_{n+2} \ldots \preccurlyeq a_1 a_2 \ldots \quad \text{for all } n \ge 0. \tag{2.2}$$

LEMMA 2.1. *The map $\beta \mapsto \alpha(\beta)$ is a strictly increasing bijection between the interval* $(1, 2]$ *and the set* $\mathcal{Q}$.

For a given β the sequence $\alpha(\beta)$ determines the set of all greedy β-expansions in the following way. Let Σ_β be the set of all greedy β-expansions of $x \in [0, 1)$. Then (cf. [**Par60**])

$$\Sigma_\beta = \{(x_i) \in \{0, 1\}^{\mathbb{N}} : \sigma^n((x_i)) \prec \alpha(\beta) \text{ for all } n \ge 0\}. \tag{2.3}$$

Similarly, let $\widetilde{\Sigma}_\beta$ be the set of all quasi-greedy β-expansions of $x \in (0, 1]$. Then

$$\widetilde{\Sigma}_\beta = \{(x_i) \in \{0, 1\}^{\mathbb{N}} : 0^\infty \prec \sigma^n((x_i)) \preccurlyeq \alpha(\beta) \text{ for all } n \ge 0\}.$$

The following result can be found in [**Par60**] (see also [**dVKL16**]).

LEMMA 2.2. *Let $\beta \in (1, 2]$. The map $x \mapsto b(x, \beta)$ is a strictly increasing bijection from* $[0, 1)$ *to Σ_β and is right-continuous with respect to the ordering topology on Σ_β.*

On the other hand, the map $x \mapsto \tilde{b}(x, \beta)$ is a strictly increasing bijection from $(0, 1]$ *to $\widetilde{\Sigma}_\beta$ and is left-continuous with respect to the ordering topology on $\widetilde{\Sigma}_\beta$.*

2.3. *First properties of $K_\beta(t)$ and E_β.* Let $t \in [0, 1)$ be given. Recall the definitions of the survivor set $K_\beta(t) = K_\beta^0(t) \cup K_\beta^+(t)$ from (1.1) and (1.2). We define the corresponding *symbolic survivor sets* as the sets of all greedy β-expansions of elements in the sets $K_\beta(t)$, $K_\beta^0(t)$ and $K_\beta^+(t)$, respectively. Lemma 2.2 gives the following descriptions:

$$\begin{aligned} \mathcal{K}_\beta^+(t) &= \{(x_i) \in \{0, 1\}^{\mathbb{N}} : b(t, \beta) \preccurlyeq \sigma^n((x_i)) \prec \alpha(\beta)\ \forall n \ge 0\}, \\ \mathcal{K}_\beta^0(t) &= \{(x_i) \in \{0, 1\}^{\mathbb{N}} : \exists n \ge 0\ \ \sigma^n((x_i)) = 0^\infty \\ &\qquad \text{and } b(t, \beta) \preccurlyeq \sigma^k((x_i)) \prec \alpha(\beta)\ \forall 0 \le k < n\}, \\ \mathcal{K}_\beta(t) &= \mathcal{K}_\beta^+(t) \cup \mathcal{K}_\beta^0(t). \end{aligned} \tag{2.4}$$

We will often switch from $K_\beta(t)$ to $\mathcal{K}_\beta(t)$ and back. The set $K_\beta(t)$ is closed and T_β is continuous when restricted to $K_\beta(t)$. Under the metric d on $\{0, 1\}^{\mathbb{N}}$ given by

$$d((x_i), (y_i)) = \beta^{-\inf\{n\geq 1: x_n \neq y_n\}}$$

the map $\pi_\beta : (\mathcal{K}_\beta(t), \sigma) \to (K_\beta(t), T_\beta)$ is a topological conjugacy. This gives that

$$h_{\mathrm{top}}(T_\beta | K_\beta(t)) = h_{\mathrm{top}}(\mathcal{K}_\beta(t)).$$

For the bifurcation set E_β, defined in (1.3), the following description can implicitly be found in [**Urb86**].

PROPOSITION 2.3. *$E_\beta = \{t \in [0, 1) : t \in K_\beta(t)\}$ and thus $E_\beta \cap [t, 1) \subseteq K_\beta(t)$ for any $t \in (0, 1)$.*

Proof. For all $t \in (0, 1)$ we have $t \notin K_\beta(\epsilon)$ for any $\epsilon > t$. Hence, if $t \in K_\beta(t)$, then $t \in E_\beta$. Suppose that $t \notin K_\beta(t)$, i.e., there is an $N \geq 1$ such that $T_\beta^N(t) \in (0, t)$. By the right-continuity of T_β^N, there is a $\delta > 0$ such that

$$T_\beta^N(\epsilon) \in \left(T_\beta^N(t), \frac{T_\beta^N(t) + t}{2}\right) \subseteq (0, t) \quad \text{for all } \epsilon \in [t, t+\delta].$$

This implies that $K_\beta(t) \cap [t, t+\delta] = \emptyset$ and thus $K_\beta(t+\delta) \subseteq K_\beta(t) \subseteq K_\beta(t+\delta)$. We conclude that the function $\epsilon \mapsto K_\beta(\epsilon)$ is constant on $[t, t+\delta]$, so $t \notin E_\beta$. □

COROLLARY 2.4. *For each $\beta \in (1, 2]$ the set $[0, 1)\backslash E_\beta$ is open.*

Proof. Let $t \notin E_\beta$. The proof of the previous proposition then gives a $\delta_1 > 0$ such that $[t, t+\delta_1] \cap E_\beta = \emptyset$. From $t \notin K_\beta(t)$ it follows that there is an $N \geq 1$ such that $T_\beta^N(t) \in (0, t)$. Hence, $T_\beta^k(t) \neq (1/\beta)$ for any $0 \leq k \leq N$, which means that T_β^N is left-continuous in t. Then, as in the proof of Proposition 2.3, we can find a $\delta_2 > 0$ such that $[t - \delta_2, t] \cap E_\beta = \emptyset$. □

In (1.4) the set E_β^+ was defined. By the same proof as given for Proposition 2.3 we also get that E_β^+ is the bifurcation set of $K_\beta^+(t)$, i.e.,

$$E_\beta^+ = \{t \in [0, 1) : t \in K_\beta^+(t)\} = \{t \in [0, 1) : K_\beta^+(\epsilon) \neq K_\beta^+(t) \text{ for any } \epsilon > t\}.$$

As for $K_\beta(t)$ we add a third set E_β^0 of the elements in E_β that are pre-images of 0:

$$E_\beta^0 = \{t \in E_\beta : \exists n \geq 0 \ \ T_\beta^n(t) = 0\} = \{t \in [0, 1) : t \in K_\beta^0(t)\}.$$

Then $E_\beta = E_\beta^+ \cup E_\beta^0$ and $E_\beta^+ \cap E_\beta^0 = \{0\}$.

The symbolic bifurcation sets, i.e., the sets of all greedy β-expansions of elements in E_β, E_β^+ and E_β^0, can be described as follows:

$$\begin{aligned}
\mathcal{E}_\beta^+ &= \{(t_i) \in \{0, 1\}^{\mathbb{N}} : \forall n \geq 0 \ \ (t_i) \preccurlyeq \sigma^n((t_i)) \prec \alpha(\beta)\}, \\
\mathcal{E}_\beta^0 &= \{(t_i) \in \{0, 1\}^{\mathbb{N}} : \exists n \geq 0 \ \ \sigma^n((t_i)) = 0^\infty \\
&\qquad \text{and } (t_i) \preccurlyeq \sigma^k((t_i)) \prec \alpha(\beta) \text{ for all } 0 \leq k < n\}, \\
\mathcal{E}_\beta &= \mathcal{E}_\beta^+ \cup \mathcal{E}_\beta^0.
\end{aligned} \tag{2.5}$$

In the series of papers [**Rai89**, **Rai92**, **Rai94**], Raith studied invariant sets for piecewise monotone expanding maps on the interval [0, 1]. More specifically, in [**Rai94**] he removed a finite number of open intervals from [0, 1] and considered piecewise monotone expanding maps restricted to the survivor set. He then studied the dependence on the end points of the holes of the Hausdorff dimension of the survivor set and of the topological entropy of the map restricted to the survivor set. Since no $x \in [0, 1)$ has $T_\beta(x) = 1$, we can apply these results to T_β on $[0, 1)$ with the single hole $(0, t)$ removed. In particular, applying the results from [**Rai94**, Corollary 1.1 and Theorem 2] gives the following.

PROPOSITION 2.5. [**Rai94**] *Let $\beta \in (1, 2)$ be given. The maps $H_\beta : t \mapsto h_{\mathrm{top}}(\mathcal{K}_\beta(t))$ and $\eta_\beta : t \mapsto \dim_H K_\beta(t)$ are continuous on* $[0, 1)$.

In the process of proving [**Rai94**, Theorem 2] Raith proved in [**Rai94**, Lemma 3] that Bowen's dimension formula also holds in this case, i.e., the Hausdorff dimension of the survivor set is the unique zero of the pressure function. In our setting this translates to the following dimension formula:

$$\dim_H K_\beta(t) = \frac{h_{\mathrm{top}}(T_\beta | K_\beta(t))}{\log \beta}. \tag{2.6}$$

Since for any $t \in [0, 1)$ the sets $K^0_\beta(t)$ and E^0_β contain at most countably many points, we have the following properties for the sets under consideration. Let λ denote the one-dimensional Lebesgue measure.

$\dim_H K_\beta(t) = \dim_H K^+_\beta(t)$	$\dim_H K^0_\beta(t) = 0$
$\lambda(K_\beta(t)) = \lambda(K^+_\beta(t))$	$\lambda(K^0_\beta(t)) = 0$
$\dim_H E_\beta = \dim_H E^+_\beta$	$\dim_H E^0_\beta = 0$
$\lambda(E_\beta) = \lambda(E^+_\beta)$	$\lambda(E^0_\beta) = 0$
$h_{\mathrm{top}}(K_\beta(t)) = \max\{h_{\mathrm{top}}(K^+_\beta(t)), h_{\mathrm{top}}(K^0_\beta(t))\}$	

This table implies that for Theorem A(i) it is enough to consider only E_β. From Proposition 2.5 and (2.6) we also get that $t \mapsto \dim_H K^+_\beta(t)$ is continuous and that

$$h_{\mathrm{top}}(\mathcal{K}_\beta(t)) = \dim_H(K^+_\beta(t)) \log \beta.$$

The next result specifies the relations between the sets even further.

PROPOSITION 2.6. *Let $\beta \in (1, 2)$. If $t \in E^+_\beta$, then $h_{\mathrm{top}}(\mathcal{K}_\beta(t)) = h_{\mathrm{top}}(\mathcal{K}^+_\beta(t))$.*

Proof. Since $\mathcal{K}^+_\beta(t) \subseteq \mathcal{K}_\beta(t)$, it suffices to prove that $h_{\mathrm{top}}(\mathcal{K}^+_\beta(t)) \geq h_{\mathrm{top}}(\mathcal{K}_\beta(t))$. For $t = 0$ there is nothing to prove. Take $t \in E^+_\beta \setminus \{0\}$ and write $(t_i) := b(t, \beta)$. Then

$$(t_i) \preccurlyeq \sigma^n((t_i)) \prec \alpha(\beta) \quad \text{for all } n \geq 0.$$

Hence, (t_i) does not end with 0^∞ and by (2.4) we can rewrite $\mathcal{K}^0_\beta(t)$ as

$$\mathcal{K}^0_\beta(t) = \{(x_i) : \exists n \geq 0 \; \sigma^n((x_i)) = 0^\infty \text{ and } (t_i) \prec \sigma^k((x_i)) \prec \alpha(\beta) \; \forall 0 \leq k < n\}. \tag{2.7}$$

We claim that

$$|\mathcal{B}_k(\mathcal{K}^0_\beta(t))| \le \sum_{j=1}^{k+1} |\mathcal{B}_{j-1}(\mathcal{K}^+_\beta(t))|.$$

Take a word $a_1 \ldots a_k \in \mathcal{B}_k(\mathcal{K}^0_\beta(t))$ and without loss of generality suppose that it occurs as a prefix of a sequence $(x_i) \in \mathcal{K}^0_\beta(t)$, i.e., $(x_i) = a_1 \ldots a_k x_{k+1} x_{k+2} \ldots$. Let $j \ge 0$ be such that $x_j = 1$ and the tail $x_{j+1}x_{j+2} \ldots = 0^\infty$. If $j = 0$, then $(x_i) = 0^\infty$. Avoiding this trivial case we assume that $j \ge 1$ and we will prove that $x_1 \ldots x_{j-1}0 \in \mathcal{B}_j(\mathcal{K}^+_\beta(t))$. By (2.7) it follows that

$$t_1 \ldots t_{j-i} \preccurlyeq x_{i+1} \ldots x_{j-1}0 \prec \alpha_1(\beta) \ldots \alpha_{j-i}(\beta) \quad \text{for all } 0 \le i < j. \tag{2.8}$$

Let $i^* \le j$ be the smallest index such that $x_{i^*+1} \ldots x_{j-1}0 = t_1 \ldots t_{j-i^*}$. If strict inequalities in (2.8) hold for all $i < j$, then we put $i^* = j$. Note that $(t_i) \preccurlyeq \sigma^n((t_i)) \prec \alpha(\beta)$ for all $n \ge 0$. Then by the minimality of i^* it follows that

$$x_1 \ldots x_{j-1}0t_{j-i^*+1}t_{j-i^*+2} \ldots = x_1 \ldots x_{i^*}t_1t_2 \ldots \in \mathcal{K}^+_\beta(t).$$

Observe that $x_1 \ldots x_{j-1} = a_1 \ldots a_{j-1}$ if $j \le k$ and $x_1 \ldots x_k = a_1 \ldots a_k$ if $j \ge k+1$. This implies that $a_1 \ldots a_{j-1} = x_1 \ldots x_{j-1} \in \mathcal{B}_{j-1}(\mathcal{K}^+_\beta(t))$ if $j \le k$ or $a_1 \ldots a_k \in \mathcal{B}_k(\mathcal{K}^+_\beta(t))$ if $j \ge k+1$ and proves the claim.

By the claim it follows that $|\mathcal{B}_k(\mathcal{K}^0_\beta(t))| \le (k+1)|\mathcal{B}_k(\mathcal{K}^+_\beta(t))|$. Using that $\mathcal{K}_\beta(t) = \mathcal{K}^0_\beta(t) \cup \mathcal{K}^+_\beta(t)$ we have

$$|\mathcal{B}_k(\mathcal{K}_\beta(t))| \le (k+2)|\mathcal{B}_k(\mathcal{K}^+_\beta(t))| \quad \text{for all } k \ge 1.$$

Taking logarithms, dividing both sides by k and letting $k \to \infty$, we conclude that $h_{\text{top}}(\mathcal{K}_\beta(t)) \le h_{\text{top}}(\mathcal{K}^+_\beta(t))$, which gives the result. □

2.4. *The size of E_β.* The results from the previous sections are enough to prove Theorem A. We start by proving the following result, which holds for all $\beta \in (1, 2)$. It covers item (i) from Theorem A as well as part of Theorem B.

PROPOSITION 2.7. *For any $\beta \in (1, 2)$ the bifurcation set E_β is a Lebesgue null set. Furthermore,* $\dim_H(E_\beta \cap [0, \delta]) = 1$ *for any $\delta > 0$. In particular,* $\dim_H E_\beta = 1$.

Proof. For the first part of the statement, let $\beta \in (1, 2)$ and $N \in \mathbb{N}$. The ergodicity of T_β with respect to its invariant measure equivalent to the Lebesgue measure λ implies that λ-almost every $x \in [0, 1)$ is eventually mapped into the interval $(0, (1/N))$. Hence, the survivor set $K_\beta((1/N))$ is a Lebesgue null set for each $N \in \mathbb{N}$. This implies that $\lambda(E_\beta) = 0$ since by Proposition 2.3,

$$E_\beta \subseteq \bigcup_{N=1}^{\infty} K_\beta\left(\frac{1}{N}\right).$$

To prove the second part, take a large integer $N \ge 1$. Let $E_{\beta,N}$ be the set of $x \in [0, 1)$ with a greedy expansion $b(x, \beta) = (b_i(x, \beta))$ satisfying $b_1(x, \beta) \ldots b_N(x, \beta) = 0^N$ and such that the tails $b_{N+1}(x, \beta)b_{N+2}(x, \beta) \ldots$ do not contain N consecutive zeros.

It immediately follows that $E_{\beta,N} \subseteq E_\beta$. Note that $K_\beta^+(1/\beta^N)$ is exactly the set of $x \in [0, 1)$ for which $b(x, \beta)$ does not have more than N consecutive zeros. Hence,

$$E_{\beta,N} = \frac{1}{\beta^N} K_\beta^+\left(\frac{1}{\beta^N}\right)$$

and thus $\dim_H E_{\beta,N} = \dim_H K_\beta^+(1/\beta^N) = \dim_H K_\beta(1/\beta^N)$. Moreover, for any $\delta > 0$, we can find a large integer N such that $E_{\beta,n} \subseteq E_\beta \cap [0, \delta]$ for all $n \geq N$. Therefore,

$$\dim_H(E_\beta \cap [0, \delta]) \geq \dim_H E_{\beta,n} = \dim_H K_\beta\left(\frac{1}{\beta^n}\right)$$

for all $n \geq N$. By continuity of the map $\eta_\beta : t \mapsto \dim_H K_\beta(t)$, letting $n \to \infty$ gives that

$$\dim_H(E_\beta \cap [0, \delta]) \geq \dim_H K_\beta(0) = \dim_H[0, 1) = 1. \qquad \square$$

Proof of Theorem A. Item (i) is given by Proposition 2.7. For item (ii), first bullet point, the fact that η_β decreases weakly immediately follows from its definition and the continuity of η_β is given by Proposition 2.5. For the second bullet point we have that the set-valued map $t \mapsto K_\beta(t)$ is locally constant Lebesgue almost everywhere since $\lambda(E_\beta) = 0$. The last bullet point follows since $\eta_\beta(0) = 1$ and for $t \geq (1/\beta)$ we completely remove the second branch from T_β, so that obviously $\dim_H(K_\beta(t)) = 0$ and $\eta_\beta(t) = 0$. $\square$

3. *Topological structure of E_β*

In this section we prove Theorem B. In fact, we prove a stronger result by specifying the set of $\beta \in (1, 2)$ for which there is a $\delta > 0$ such that $E_\beta^+ \cap [0, \delta]$ does not contain isolated points. This is the set

$$C_3 := \{\beta \in (1, 2) : \text{ the length of consecutive zeros in } \alpha(\beta) \text{ is bounded}\}. \tag{3.1}$$

From a dynamical point of view C_3 is the set of $\beta \in (1, 2)$ such that the orbit $\{\widetilde{T}_\beta^n(1)\}_{n=0}^\infty$ is bounded away from zero. Replacing $\alpha(\beta)$ in the definition of C_3 by $b(1, \beta)$ gives the set called C_3 in [**Sch97**]. In [**Sch97**] Schmeling proved that this set has zero Lebesgue measure and full Hausdorff dimension. Since the two versions of C_3 only differ by countably many points, the same holds for our set C_3 from (3.1). We prove Theorem B using Lyndon words, which we will define next.

Recall from (2.5) that

$$\mathcal{E}_\beta^+ = \{(t_i) \in \{0, 1\}^{\mathbb{N}} : (t_i) \preccurlyeq \sigma^n((t_i)) \prec \alpha(\beta) \text{ for all } n \geq 0\}.$$

In other words, any sequence in $\mathcal{E}_\beta^+$ is the lexicographically smallest sequence in Σ_β under the shift map σ. For this reason we recall the following definition (cf. [**Lot02**]).

Definition 3.1. A word $\mathbf{s}$ is called *Lyndon* if $\mathbf{s}$ is aperiodic and $\sigma^n(\mathbf{s}^\infty) \succcurlyeq \mathbf{s}^\infty$ for all $n \geq 0$.

The following lemma lists some useful properties of Lyndon words. The first and third items easily follow from the definition and we omit their proofs.

LEMMA 3.2.

(i) *$s_1 \ldots s_m$ is a Lyndon word if and only if*

$$s_{i+1} \ldots s_m \succ s_1 \ldots s_{m-i} \quad \textit{for all } 0 < i < m.$$

(ii) *If* $s_1 \dots s_m$ *is a Lyndon word, then for any* $1 \le n < m$ *with* $s_n = 0$ *the word* $s_1 \dots s_n^+$ *is also Lyndon.*

(iii) *If* v, w *are Lyndon words and* $vw \prec wv$*, then for all* $n \in \mathbb{N}$ *we have that* $v^n w$ *is a Lyndon word.*

Proof. To prove (ii), suppose that $s_n = 0$ for some $1 \le n < m$. Since 1 is a Lyndon word, the statement holds for $n = 1$. If $2 \le n < m$, then by (i) it follows that

$$s_{i+1} \dots s_n^+ \succ s_{i+1} \dots s_n \not\prec s_1 \dots s_{n-i} \quad \text{for all } 0 < i < n.$$

Therefore, again by (i) $s_1 \dots s_n^+$ is a Lyndon word, as required. □

By taking $i = m - 1$ in Lemma 3.2(i) it follows that $s_1 = 0$ and $s_m = 1$. So, any Lyndon word of length at least two starts with 0 and ends with 1. We use Lemma 3.2 to show that any isolated point in E_β^+ has a periodic greedy β-expansion.

PROPOSITION 3.3. *Let* $\beta \in (1, 2]$*. If* t *is an isolated point of* E_β^+*, then its greedy* β*-expansion* $b(t, \beta)$ *is periodic. Moreover, no element from* E_β^+ *is isolated in* E_β*.*

The proof of this proposition is based on the following two lemmas. Together they say that any point in E_β^+ with aperiodic β-expansion can be approximated from below by a sequence of points in E_β^+ that have a periodic orbit under T_β.

LEMMA 3.4. *Let* $(t_i) \in \mathcal{E}_\beta^+$ *be an aperiodic sequence. For each* $m \ge 1$ *we have*

$$(t_1 \dots t_m)^\infty \prec (t_i) \quad \textit{and} \quad (t_1 \dots t_m)^\infty \in \Sigma_\beta.$$

Proof. Let $(t_i) \in \mathcal{E}_\beta^+$ be an aperiodic sequence. Then by (2.5) we have

$$(t_i) \prec \sigma^n((t_i)) \prec \alpha(\beta) \quad \text{for all } n \ge 1. \tag{3.2}$$

Fix $m \ge 1$. By taking $n = m, 2m, \dots$ in (3.2) it follows that

$$\begin{aligned}(t_1 \dots t_m)^\infty &= t_1 \dots t_m (t_1 \dots t_m)^\infty \\ &\preccurlyeq t_1 \dots t_m t_{m+1} \dots t_{2m} (t_1 \dots t_m)^\infty \\ &\preccurlyeq t_1 \dots t_{2m} t_{2m+1} \dots t_{3m} (t_1 \dots t_m)^\infty \preccurlyeq \dots \preccurlyeq (t_i).\end{aligned}$$

Since (t_i) is not periodic, we conclude that $(t_1 \dots t_m)^\infty \prec (t_i)$.

For the second statement, (3.2) and the first part of the proposition give that

$$\sigma^n((t_1 \dots t_m)^\infty) = t_{n+1} \dots t_m (t_1 \dots t_m)^\infty \prec t_{n+1} \dots t_m t_{m+1} t_{m+2} \dots \prec \alpha(\beta)$$

for each $0 \le n < m$ and hence $(t_1 \dots t_m)^\infty \in \Sigma_\beta$. □

From [**SM94**, Proposition 2.2] we have the following lemma.

LEMMA 3.5. *Let* $(t_i) \in \mathcal{E}_\beta^+$ *be an aperiodic sequence. Then there exist infinitely many* $m \in \mathbb{N}$ *such that* $t_1 \dots t_m$ *is a Lyndon word.*

Note that both previous lemmas do not hold for $\mathcal{E}_\beta$. Let $(t_i) \in \mathcal{E}_\beta^0$ be such that $\sigma^n((t_i)) = 0^\infty$. Then for any $m > n$ we have $(t_1 \dots t_m)^\infty \succ (t_i)$, contradicting the statement of Lemma 3.4. As for the statement of Lemma 3.5, for all $m \ge 2n$ we have that $t_1 \dots t_m$ is not Lyndon.

Proof of Proposition 3.3. Let $t \in E_\beta^+$ be a point with aperiodic greedy β-expansion $b(t, \beta) = (t_i)$. Since $(t_i) \in \mathcal{E}_\beta^+$, by Lemma 3.5 there exists a sequence (m_j) such that $t_1 \ldots t_{m_j}$ is Lyndon for all $j \geq 1$. Furthermore, by Lemma 3.4 we have $(t_1 \ldots t_{m_j})^\infty \in \Sigma_\beta$ for each $j \geq 1$. Hence, for all $j \geq 1$ we have $(t_1 \ldots t_{m_j})^\infty \in \mathcal{E}_\beta^+$ and thus $\pi_\beta((t_1 \ldots t_{m_j})^\infty) \in E_\beta^+$. Letting $j \to \infty$ we conclude that $\pi_\beta((t_1 \ldots t_{m_j})^\infty) \to \pi_\beta((t_i)) = t$, which implies that t is not isolated in E_β^+.

Now assume that $t \in E_\beta^+$ has a periodic greedy β-expansion $b(t, \beta) = (t_1 \ldots t_m)^\infty$, where m is chosen minimal. We will show that t is not isolated in E_β. If $m = 1$, then we have $b(t, \beta) = 0^\infty$, i.e., $t = 0$. In this case the result trivially follows from Proposition 2.7. Now assume that $m \geq 2$. Let $a_1 \ldots a_m$ be the maximal cyclic permutation of $t_1 \ldots t_m$. Then there exists a $j \in \{0, 1, \ldots, m-1\}$ such that $a_1 \ldots a_m = t_{j+1} \ldots t_m t_1 \ldots t_j$. Note that $\sigma^n((t_1 \ldots t_m)^\infty) \prec \alpha(\beta)$ for all $n \geq 0$. Then

$$(a_1 \ldots a_m)^\infty \prec \alpha(\beta), \tag{3.3}$$

which implies that $a_1 \ldots a_m \preccurlyeq \alpha_1(\beta) \ldots \alpha_m(\beta)$. We claim that $a_1 \ldots a_m \prec \alpha_1(\beta) \ldots \alpha_m(\beta)$.

If $a_1 \ldots a_m = \alpha_1(\beta) \ldots \alpha_m(\beta)$, then (3.3) together with Lemma 2.1 gives

$$a_1 \ldots a_m \preccurlyeq \alpha_{m+1}(\beta) \ldots \alpha_{2m}(\beta) \preccurlyeq \alpha_1(\beta) \ldots \alpha_m(\beta) = a_1 \ldots a_m.$$

So, $a_1 \ldots a_{2m} = (a_1 \ldots a_m)^2$. Iterating this argument with Lemma 2.1 and (3.3) gives that $\alpha(\beta) = (a_1 \ldots a_m)^\infty$, leading to a contradiction with (3.3). This proves the claim.

For $N \in \mathbb{N}$ define the sequence $\mathbf{t}_N := (t_1 \ldots t_m)^N t_1 \ldots t_j^+ 0^\infty$. Since $t_j = 0$, the sequence $\mathbf{t}_N$ is well defined. By Lemma 3.2(iii) it follows that $\sigma^n(\mathbf{t}_N) \succ \mathbf{t}_N$ for all $0 \leq n < mN + j$. Moreover, $a_1 \ldots a_m \prec \alpha_1(\beta) \ldots \alpha_m(\beta)$. It follows that $\sigma^n(\mathbf{t}_N) \prec \alpha(\beta)$ for all $n \geq 0$. So, $\mathbf{t}_N \in \mathcal{E}_\beta^0$ for all $N \in \mathbb{N}$. Since $\pi_\beta(\mathbf{t}_N) \searrow t$ as $N \to \infty$, the point $t \in E_\beta^+$ is not isolated in E_β. □

The next proposition says that no point from $E_\beta^0 \backslash \{0\}$ can be approximated from above by elements from E_β and that a point $t \in E_\beta^0 \backslash \{0\}$ is isolated in E_β if the orbit of 1 enters $(0, t)$.

PROPOSITION 3.6. *Let $t \in E_\beta^0 \backslash \{0\}$. Then there is a $\delta > 0$ such that $E_\beta \cap [t, t+\delta] = \{t\}$. Moreover, if $\beta - 1 \notin K_\beta(t)$, then t is isolated in E_β.*

Proof. If $t \in E_\beta^0 \backslash \{0\}$, then there is a smallest $n \geq 0$ such that $T_\beta^n(t) = (1/\beta)$. By the right-continuity of T_β, there is a $\delta > 0$ such that all $\epsilon \in (t, t+\delta]$ satisfy $T_\beta^{n+1}(\epsilon) \in (0, t) \subseteq (0, \epsilon)$. Hence, $\epsilon \notin K_\beta(\epsilon)$ and thus $\epsilon \notin E_\beta$.

The first statement implies that to prove that an element from $E_\beta^0 \backslash \{0\}$ is isolated, it is enough to prove that it cannot be approximated from below. If again n is such that $T_\beta^n(t) = (1/\beta)$, then for a small enough δ we know that for any point $\epsilon \in [t-\delta, t)$ the point $T_\beta^{n+1}(\epsilon)$ is close to 1. Let m be the smallest integer such that $T_\beta^m(\beta - 1) \in (0, t)$. Then there is a $0 < \delta < t - T_\beta^m(\beta - 1)$ such that any $\epsilon \in [t-\delta, t)$ satisfies

$$T_\beta^{n+1+m+1}(\epsilon) \in (0, T_\beta^m(\beta - 1)) \subseteq (0, \epsilon).$$

Hence, $\epsilon \notin E_\beta$ and $E_\beta \cap [t-\delta, t] = \{t\}$. □

From now on we focus on the set E_β^+. We first construct subintervals of (1, 2) such that E_β^+ contains isolated points whenever β is in one of these intervals. We start with a couple of lemmas.

LEMMA 3.7. *Let $(t_i), (\alpha_i) \in \{0,1\}^{\mathbb{N}}$ be given. Suppose that there is an $m \geq 1$ such that $\alpha_m = 1$ and $\sigma^m((\alpha_i)) \preccurlyeq (t_i)$. Define the sets*

$$\mathcal{K} := \{(x_i) \in \{0,1\}^{\mathbb{N}} : (t_i) \preccurlyeq \sigma^n((x_i)) \prec (\alpha_i) \text{ for all } n \geq 0\},$$
$$\mathcal{X}_m := \{(x_i) \in \{0,1\}^{\mathbb{N}} : (t_i) \preccurlyeq \sigma^n((x_i)) \preccurlyeq (\alpha_1 \ldots \alpha_m^-)^\infty \text{ for all } n \geq 0\}.$$

Then $\mathcal{K} = \mathcal{X}_m$.

Proof. Obviously, $\mathcal{X}_m \subseteq \mathcal{K}$. We show that $\mathcal{K} \setminus \mathcal{X}_m = \emptyset$. Suppose that this is not the case and let $(x_i) \in \mathcal{K} \setminus \mathcal{X}_m$. Then there is a $j \geq 1$ such that $x_{j+1} \ldots x_{j+m} = \alpha_1 \ldots \alpha_m$. Since $(x_i) \in \mathcal{K}$, the assumption that $\sigma^m((\alpha_i)) \preccurlyeq (t_i)$ implies that

$$x_{j+m+1} x_{j+m+2} \ldots \prec \alpha_{m+1} \alpha_{m+2} \ldots \preccurlyeq (t_i),$$

which contradicts $(x_i) \in \mathcal{K}$. Hence, $\mathcal{K} \setminus \mathcal{X}_m = \emptyset$. □

Let $\beta \in (1,2)$ and $t \in [0,1)$. The previous lemma has the following consequence for $\mathcal{K}_\beta^+(t)$. If there is a smallest $m \geq 1$ such that

$$\alpha_{m+1}(\beta) \alpha_{m+2}(\beta) \ldots \preccurlyeq b(t, \beta)$$

or equivalently $\tilde{T}_\beta^m(1) \leq t$, then we can rewrite $\mathcal{K}_\beta^+(t)$ as

$$\mathcal{K}_\beta^+(t) = \{(x_i) : b(t,\beta) \preccurlyeq \sigma^n((x_i)) \preccurlyeq (\alpha_1(\beta) \ldots \alpha_m(\beta)^-)^\infty \text{ for any } n \geq 0\}.$$

Hence, any point in the survivor set $K_\beta^+(t)$ then has the property that its entire orbit lies between t and the point $\pi_\beta((\alpha_1(\beta) \ldots \alpha_m(\beta)^-)^\infty)$. We need two more lemmas. Recall the definition of the set $\mathcal{Q}$ from (2.2) as the set of sequences that occur as $\alpha(\beta)$ for some $\beta \in (1, 2]$.

LEMMA 3.8. *Let $(a_1 \ldots a_m)^\infty \in \mathcal{Q}$ with minimal period m. Then*

$$a_{i+1} \ldots a_m^+ \preccurlyeq a_1 \ldots a_{m-i} \quad \text{for all } 0 < i < m.$$

Proof. Let $\beta \in (1,2)$ be such that $\alpha(\beta) = (a_1 \ldots a_m)^\infty$. Then $b(1,\beta) = a_1 \ldots a_m^+ 0^\infty$. Hence, for each $0 < i < m$ we have $b(T_\beta^i(1), \beta) = a_{i+1} \ldots a_m^+ 0^\infty$ and $T_\beta^i(1) < 1$. The result then follows from Lemma 2.2. □

Note that for any non-periodic word $b_1 \ldots b_m \in \{0,1\}^*$ there is a $0 \leq j \leq m-1$ such that $b_{j+1} \ldots b_m b_1 \ldots b_j$ is the smallest among its cyclic permutations and therefore Lyndon. We denote this word by $\mathbf{S}(b_1 \ldots b_m)$ and call it the *Lyndon word for* $b_1 \ldots b_m$. Similarly, there is a $0 \leq k \leq m-1$ such that $b_{k+1} \ldots b_m b_1 \ldots b_k$ is the largest among its cyclic permutations. We denote this by word by $\mathbf{L}(b_1 \ldots b_m)$. In what follows we will sometimes use the property that for any word $b_1 \ldots b_m \in \{0,1\}^m$ and any sequence $(x_i) \in \{0,1\}^{\mathbb{N}}$,

$$\sigma^n((x_i)) \succcurlyeq b_1 \ldots b_m 0^\infty \text{ for all } n \geq 0 \quad \Longleftrightarrow \quad \sigma^n((x_i)) \succcurlyeq (b_1 \ldots b_m)^\infty \text{ for all } n \geq 0. \tag{3.4}$$

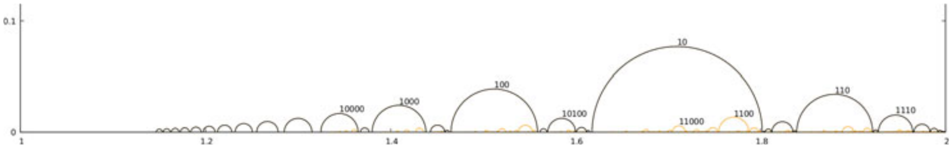

FIGURE 2. Some of the basic intervals $(\beta_L, \beta_R]$. The numbers near the arches indicate the words $a_1 \dots a_m$ such that $\alpha(\beta_L) = (a_1 \dots a_m)^\infty$. The intervals that are not contained in any other interval are the Farey intervals. They are the ones for which $a_1 \dots a_m$ is a Farey word. The arches corresponding to Farey intervals are shown in black, the lighter coloured arches correspond to words that are Lyndon, but not Farey.

LEMMA 3.9. *Let $s_1 \dots s_m$ be a Lyndon word and write $a_1 \dots a_m = \mathbf{L}(s_1 \dots s_m)$. Let $0 \le j < m$ be such that $s_1 \dots s_m = a_{j+1} \dots a_m a_1 \dots a_j$ and set*

$$\mathcal{Z}_m := \{(x_i) \in \{0, 1\}^{\mathbb{N}} : s_1 \dots s_m 0^\infty \preccurlyeq \sigma^n((x_i)) \preccurlyeq (a_1 \dots a_m)^\infty \ \forall n \ge 0\}.$$

(i) *If $(x_i) \in \mathcal{Z}_m$ has prefix $a_{j+1} \dots a_m$, then $(x_i) = (s_1 \dots s_m)^\infty$.*
(ii) *If $(x_i) \in \mathcal{Z}_m$ has prefix $a_1 \dots a_j$, then $(x_i) = (a_1 \dots a_m)^\infty$.*

Proof. Since the proofs of (i) and (ii) are similar, we only give the proof of (i). Let $a_{j+1} \dots a_m x_1 x_2 \dots \in \mathcal{Z}_m$. Then

$$s_1 \dots s_m 0^\infty \preccurlyeq \sigma^n(a_{j+1} \dots a_m x_1 x_2 \dots) \preccurlyeq (a_1 \dots a_m)^\infty \quad \text{for all } n \ge 0. \tag{3.5}$$

In particular,

$$a_{j+1} \dots a_m x_1 \dots x_j \succcurlyeq s_1 \dots s_m = a_{j+1} \dots a_m a_1 \dots a_j,$$

which gives

$$x_1 \dots x_j \succcurlyeq a_1 \dots a_j.$$

On the other hand, by taking $n = m - j$ in (3.5) we get $x_1 \dots x_m \preccurlyeq a_1 \dots a_m$. Hence,

$$x_1 \dots x_j = a_1 \dots a_j \quad \text{and} \quad x_{j+1} \dots x_m \preccurlyeq a_{j+1} \dots a_m.$$

Again, by (3.5) now with $n = m$ we have $x_{j+1} \dots x_m \succcurlyeq s_1 \dots s_{m-j} = a_{j+1} \dots a_m$. Therefore, $x_1 \dots x_m = a_1 \dots a_m$. By iteration we conclude that

$$a_{j+1} \dots a_m x_1 x_2 \dots = (a_{j+1} \dots a_m a_1 \dots a_j)^\infty = (s_1 \dots s_m)^\infty,$$

as required. □

We now construct infinitely many nested intervals $(\beta_L, \beta_R]$ such that E_β^+ has isolated points whenever $\beta \in (\beta_L, \beta_R]$. Figure 2 shows some of these intervals. We will later show that these basic intervals cover the whole interval $(1, 2)$ up to a set of zero Lebesgue measure.

PROPOSITION 3.10. *Let $s_1 \dots s_m$ be a Lyndon word and write $a_1 \dots a_m = \mathbf{L}(s_1 \dots s_m)$. Then both $(a_1 \dots a_m)^\infty$ and $a_1 \dots a_m^+(s_1 \dots s_m)^\infty$ belong to $\mathcal{Q}$ and hence there are uniquely defined bases $\beta_L, \beta_R \in (1, 2]$ such that $\alpha(\beta_L) = (a_1 \dots a_m)^\infty$ and $\alpha(\beta_R) = a_1 \dots a_m^+(s_1 \dots s_m)^\infty$. Moreover:*

(i) *$(s_1 \dots s_m)^\infty \in \Sigma_\beta$ if and only if $\beta > \beta_L$;*
(ii) *if $\beta \in (\beta_L, \beta_R]$, then $\pi_\beta((s_1 \dots s_m)^\infty)$ is an isolated point of E_β^+;*
(iii) *if $\beta > \beta_R$, then $\pi_\beta((s_1 \dots s_m)^\infty)$ is not an isolated point of E_β^+.*

Proof. Let β_L be as in the proposition. First we show that the interval $(\beta_L, \beta_R]$ is well defined, i.e., β_R exists and that $\beta_L < \beta_R$. We use the characterization from Lemma 2.1, so it suffices to show that the sequence $\mathbf{a} = a_1 \ldots a_m^+(s_1 \ldots s_m)^\infty \in \mathcal{Q}$, i.e., it satisfies $\sigma^n(\mathbf{a}) \preccurlyeq \mathbf{a}$ for all $n \geq 0$. Since $s_1 \ldots s_m$ is a Lyndon word, any word of length $1 \leq n \leq m-1$ occurring in $a_1 \ldots a_m = \mathbf{L}(s_1 \ldots s_m)$ is lexicographically larger than or equal to $s_1 \ldots s_n$. Combining this with Lemmas 3.8 and 3.2(i) gives

$$a_{n+1} \ldots a_m^+ s_1 \ldots s_n \preccurlyeq a_1 \ldots a_{m-n} a_{m-n+1} \ldots a_m \prec a_1 \ldots a_m^+$$

for all $0 < n < m$. So, $\sigma^n(\mathbf{a}) \prec \mathbf{a}$ for each $0 < n < m$. Moreover, since

$$\sigma^n((s_1 \ldots s_m)^\infty) \preccurlyeq (a_1 \ldots a_m)^\infty \prec a_1 \ldots a_m^+(s_1 \ldots s_m)^\infty$$

for all $n \geq 0$, we get $\sigma^n(\mathbf{a}) \prec \mathbf{a}$ for all $n \geq 1$ and thus $\mathbf{a} \in \mathcal{Q}$. Lemma 2.1 then implies that $\mathbf{a}$ is indeed the quasi-greedy expansion of 1 for some base β_R, i.e., $\alpha(\beta_R) = a_1 \ldots a_m^+(s_1 \ldots s_m)^\infty$. Since $\alpha(\beta_L) \prec \alpha(\beta_R)$, Lemma 2.1 also gives that $\beta_R > \beta_L$. Hence, the interval $(\beta_L, \beta_R]$ is well defined.

Let $1 \leq j \leq m-1$ be such that

$$s_1 \ldots s_m = a_{j+1} \ldots a_m a_1 \ldots a_j.$$

For (i), note that if $\beta \leq \beta_L$, then $(s_1 \ldots s_m)^\infty \notin \Sigma_\beta$ since

$$\sigma^j((s_1 \ldots s_m)^\infty) = (a_1 \ldots a_m)^\infty \succcurlyeq \alpha(\beta).$$

For $\beta \in (\beta_L, \beta_R]$ it follows immediately that $(s_1 \ldots s_m)^\infty \in \Sigma_\beta$ since $s_1 \ldots s_m$ is the smallest permutation of $a_1 \ldots a_m$ and $(a_1 \ldots a_m)^\infty \prec \alpha(\beta)$.

For (ii), let $\beta \in (\beta_L, \beta_R]$ and set $t = \pi_\beta((s_1 \ldots s_m)^\infty)$. Then $b(t, \beta) = (s_1 \ldots s_m)^\infty \in \mathcal{E}_\beta^+$, so $t \in E_\beta^+$. By Lemma 2.2 and since t has a periodic β-expansion, there exists a small $\delta > 0$ such that for any $x \in [t-\delta, t+\delta]$ the greedy expansion $b(x, \beta)$ has prefix $s_1 \ldots s_m$. By Lemma 3.7 it follows that

$$\begin{aligned}
\mathcal{K}_\beta^+(t-\delta) &\subseteq \{(x_i) : s_1 \ldots s_m 0^\infty \preccurlyeq \sigma^n((x_i)) \prec a_1 \ldots a_m^+(s_1 \ldots s_m)^\infty \ \forall n \geq 0\} \\
&= \{(x_i) : (s_1 \ldots s_m)^\infty \preccurlyeq \sigma^n((x_i)) \prec a_1 \ldots a_m^+(s_1 \ldots s_m)^\infty \ \forall n \geq 0\} \\
&= \{(x_i) : (s_1 \ldots s_m)^\infty \preccurlyeq \sigma^n((x_i)) \preccurlyeq (a_1 \ldots a_m)^\infty \ \forall n \geq 0\} \\
&= \{(x_i) : s_1 \ldots s_m 0^\infty \preccurlyeq \sigma^n((x_i)) \preccurlyeq (a_1 \ldots a_m)^\infty \ \forall n \geq 0\},
\end{aligned} \tag{3.6}$$

where we have used the fact from (3.4) in the first and last equalities. Since for any $x \in [t-\delta, t+\delta]$ the greedy expansion $b(x, \beta)$ begins with $s_1 \ldots s_m$, by Lemma 3.9(i) and (3.6) we obtain that

$$K_\beta^+(t-\delta) \cap [t-\delta, t+\delta] \subseteq \{t\}.$$

Since $t \in E_\beta^+ \cap [t-\delta, t+\delta] \subseteq K_\beta^+(t-\delta) \cap [t-\delta, t+\delta]$, we conclude that t is isolated in E_β^+ for any $\beta \in (\beta_L, \beta_R]$.

For (iii), let $\beta > \beta_R$ and again set $t = \pi_\beta((s_1 \ldots s_m)^\infty)$. We construct a sequence $(\mathbf{t}_n)$ in $\mathcal{E}_\beta^+$ such that $\mathbf{t}_n \searrow (s_1 \ldots s_m)^\infty$ in the order topology as $n \to \infty$. Let

$$\mathbf{t}_n := ((s_1 \ldots s_m)^n s_1 \ldots s_{m-j}^+)^\infty = ((a_{j+1} \ldots a_m a_1 \ldots a_j)^n a_{j+1} \ldots a_m^+)^\infty. \tag{3.7}$$

We claim that there is an $N \in \mathbb{N}$ such that $\mathbf{t}_n \in \mathcal{E}_\beta^+$ for all $n > N$. By Lemma 3.2(ii) and (iii) it follows that $\mathbf{t}_n$ is Lyndon. Left to show is that $\mathbf{t}_n \in \Sigma_\beta$. Note that the largest permutation of $\mathbf{t}_n$ is given by

$$\begin{aligned}\mathbf{d}_n &= (a_1 \ldots a_m^+ (a_{j+1} \ldots a_m a_1 \ldots a_j)^{n-1} a_{j+1} \ldots a_m)^\infty \\ &= (a_1 \ldots a_m^+ (s_1 \ldots s_m)^{n-1} s_1 \ldots s_{m-j})^\infty.\end{aligned}$$

For $\beta > \beta_R$ either $\alpha_1(\beta) \ldots \alpha_m(\beta) \succ a_1 \ldots a_m^+$ or there exists an $N \geq 1$ such that $\alpha(\beta) = a_1 \ldots a_m^+ (s_1 \ldots s_m)^{N-1} b_1 \ldots b_m$ with $b_1 \ldots b_m \succ s_1 \ldots s_m$. In the first case obviously $\mathbf{d}_n \prec \alpha(\beta)$. In the second case we have $\mathbf{d}_n \prec \alpha(\beta)$ for all $n > N$. Hence, $\mathbf{t}_n \in \Sigma_\beta$ for all $n > N$.

We have found a sequence $(\mathbf{t}_n) \subseteq \mathcal{E}_\beta^+$ decreasing to $b(t, \beta) = (s_1 \ldots s_m)^\infty$ as $n \to \infty$ and, accordingly, a sequence $(\pi_\beta(\mathbf{t}_n)) \subseteq E_\beta^+$ decreasing to $t = \pi_\beta((s_1 \ldots s_m)^\infty)$ as $n \to \infty$. Therefore, t is not isolated in E_β^+. □

Recall from (3.1) that C_3 is the set of $\beta \in (1, 2)$ such that the length of consecutive zeros in the quasi-greedy expansion $\alpha(\beta)$ is bounded.

THEOREM 3.11. *If $\beta \in (1, 2) \backslash C_3$, then both $E_\beta \cap [0, \delta]$ and $E_\beta^+ \cap [0, \delta]$ contain infinitely many isolated points and infinitely many accumulation points for all $\delta > 0$.*

Proof. By Proposition 2.7 it follows that $E_\beta \cap [0, \delta]$ and $E_\beta^+ \cap [0, \delta]$ contain infinitely many accumulation points for all $\delta > 0$, so we focus on the isolated points. Fix $\beta \in (1, 2) \backslash C_3$. Then $\alpha(\beta)$ contains consecutive zeros of arbitrary length. Hence, $\alpha(\beta)$ is not periodic and the orbit of 1 under $\widetilde{T}_\beta$ will come arbitrarily close to 0. This implies that for any $t > 0$, $\beta - 1 \notin K_\beta(t)$ and thus by Proposition 3.6 any $t \in E_\beta^0 \backslash \{0\}$ will be isolated in E_β. Note that for any $n \geq 1$ we have $(1/\beta^n) \in E_\beta^0$. This gives the statement for E_β.

To prove that E_β^+ contains infinitely many isolated points arbitrarily close to 0, we construct by induction a sequence of intervals $(\beta_{L,k}, \beta_{R,k})$, $k \geq 1$, such that $\beta \in (\beta_{L,k}, \beta_{R,k})$ for all $k \geq 1$, where $(\beta_{L,k}, \beta_{R,k})$ is defined as in Proposition 3.10. Write

$$\alpha(\beta) = 1^{l_1} 0^{m_1} 1^{l_2} 0^{m_2} \ldots 1^{l_k} 0^{m_k} \ldots. \tag{3.8}$$

Since $\alpha(\beta)$ does not end with 0^∞, we have $m_k \in \{1, 2, \ldots\}$ for all $k \geq 1$. Furthermore, from $\beta \notin C_3$ we get $\sup_{k \geq 1} m_k = \infty$.

Set $i_0 = 1$ and let $i_1 > i_0$ be the smallest index for which $m_{i_1} > m_1$. Set $\mathbf{a}_1 := 1^{l_1} 0^{m_1} \cdots 1^{l_{i_1} - 1} 0$. Note that $\alpha(\beta)$ begins with $\mathbf{a}_1^+$ and by Lemma 2.1 $\sigma^n(\alpha(\beta)) \preccurlyeq \alpha(\beta)$ for all $n \geq 0$. This implies that $\sigma^n(\mathbf{a}_1^\infty) \preccurlyeq \mathbf{a}_1^\infty$ for all $n \geq 0$. So, by Lemma 2.1 the sequence $\mathbf{a}_1^\infty$ is the quasi-greedy expansion of 1 for some base $\beta_{L,1}$, i.e., $\alpha(\beta_{L,1}) = \mathbf{a}_1^\infty$. Note that the word $\mathbf{a}_1$ contains consecutive zeros of length at most m_1. So, the Lyndon word $\mathbf{s}_1 = s_1 \ldots s_{l_1 + m_1 + \cdots + l_{i_1}}$ for $\mathbf{a}_1$ begins with $0^{m_1} 1$. Again, one can check that $\sigma^n(\mathbf{a}_1^+ \mathbf{s}_1^\infty) \preccurlyeq \mathbf{a}_1^+ \mathbf{s}_1^\infty$ for all $n \geq 0$. So, there exists $\beta_{R,1} \in (1, 2)$ such that $\alpha(\beta_{R,1}) = \mathbf{a}_1^+ \mathbf{s}_1^\infty$. By using $m_{i_1} > m_1$ and (3.8) it follows that

$$\alpha(\beta_{L,1}) = \mathbf{a}_1^\infty = (1^{l_1} 0^{m_1} \cdots 1^{l_{i_1} - 1} 0)^\infty \prec 1^{l_1} 0^{m_1} \cdots 1^{l_{i_1}} 0 \cdots = \alpha(\beta)$$

and

$$\alpha(\beta_{R,1}) = \mathbf{a}_1^+ \, \mathbf{s}_1^\infty = 1^{l_1} 0^{m_1} \cdots 1^{l_{i_1}} 0^{m_1} 1 \cdots \succ 1^{l_1} 0^{m_1} \cdots 1^{l_{i_1}} 0^{m_{i_1}} 1 \cdots = \alpha(\beta).$$

By Lemma 2.1 we have $\beta \in (\beta_{L,1}, \beta_{R,1})$. Moreover, by Proposition 3.10 we have that $\pi_\beta(\mathbf{s}_1^\infty)$ is an isolated point of E_β^+. Now we pick i_k using i_{k-1}. Let $i_k > i_{k-1}$ be the smallest index such that $m_{i_k} > m_{i_{k-1}}$. Then by the definitions of $i_1, \ldots, i_{k-1}$ it follows that $m_{i_k} > m_j$ for all $j < i_k$. Set $\mathbf{a}_k := 1^{l_1}0^{m_1} \cdots 1^{l_{i_k}-1}0$. Then the block $\mathbf{a}_k$ contains consecutive zeros of length at most $m_{i_{k-1}}$. So, the Lyndon word $\mathbf{s}_k = s_1 \ldots s_{l_1+m_1+\cdots+l_{i_k}}$ for $\mathbf{a}_k$ begins with $0^{m_{i_{k-1}}}1$. By the same argument as above we can find two bases $\beta_{L,k}, \beta_{R,k} \in (1, 2)$ such that

$$\alpha(\beta_{L,k}) = \mathbf{a}_k^\infty = (1^{l_1}0^{m_1} \cdots 1^{l_{i_k}-1}0)^\infty \prec 1^{l_1}0^{m_1} \cdots 1^{l_{i_k}}0 \cdots = \alpha(\beta),$$
$$\alpha(\beta_{R,k}) = \mathbf{a}_k^+\,\mathbf{s}_k^\infty = 1^{l_1}0^{m_1} \cdots 1^{l_{i_k}}0^{m_{i_{k-1}}}1 \cdots \succ 1^{l_1}0^{m_1} \cdots 1^{l_{i_k}}0^{m_{i_k}}1 \cdots = \alpha(\beta).$$

Therefore, $\beta \in (\beta_{L,k}, \beta_{R,k})$ and by Proposition 3.10 we have that $\pi_\beta(\mathbf{s}_k^\infty)$ is an isolated point of E_β^+.

By induction we construct a sequence of intervals $(\beta_{L,k}, \beta_{R,k})$, $k \geq 1$, such that $\beta \in (\beta_{L,k}, \beta_{R,k})$ for all $k \geq 1$. Moreover, the points $\pi_\beta(\mathbf{s}_k^\infty)$ are isolated in E_β^+. Note that $\mathbf{s}_k$ begins with a block $0^{m_{i_{k-1}}}1$ for any $k \geq 1$ and $m_{i_{k-1}}$ strictly increases to ∞ as $k \to \infty$. This implies that $E_\beta^+ \cap [0, \delta]$ contains infinitely many isolated points for any $\delta > 0$. □

THEOREM 3.12. *For $\beta \in C_3$ there is a $\delta > 0$ such that $E_\beta^+ \cap [0, \delta]$ has no isolated points.*

Proof. Fix $\beta \in C_3$. Then the length of consecutive zeros in $\alpha(\beta)$ is bounded by some large integer M. Set $\delta = (1/\beta^{M+3}) = \pi_\beta(0^{M+2}10^\infty)$. To show that $E_\beta^+ \cap [0, \delta]$ has no isolated points, suppose on the contrary that t is an isolated point of $E_\beta^+ \cap [0, \delta]$. By Proposition 3.3 it follows that the greedy β-expansion $b(t, \beta)$ of t is periodic, namely

$$b(t, \beta) = (t_1 \ldots t_m)^\infty \in \mathcal{E}_\beta^+$$

with minimal period m. Moreover, $t_1 \ldots t_m$ is Lyndon. For $m = 1$ we get that $t = 0$, which by Proposition 2.7 is not isolated in E_β^+. Let $m \geq 2$ and let $a_1 \ldots a_m = \mathbf{L}(t_1 \ldots t_m)$. Then $(a_1 \ldots a_m)^\infty \in \mathcal{Q}$, so by Lemma 2.1 it is the quasi-greedy expansion of 1 for some base β_L, i.e., $\alpha(\beta_L) = (a_1 \ldots a_m)^\infty$. By Proposition 3.10 it follows that $\beta \in (\beta_L, \beta_R]$, where β_R is the unique base satisfying

$$\alpha(\beta_R) = a_1 \ldots a_m^+(t_1 \ldots t_m)^\infty.$$

Hence,

$$(a_1 \ldots a_m)^\infty \prec \alpha(\beta) \preccurlyeq a_1 \ldots a_m^+(t_1 \ldots t_m)^\infty. \tag{3.9}$$

Since $t \leq \delta = \pi_\beta(0^{M+2}10^\infty)$, we have $(t_1 \ldots t_m)^\infty = b(t, \beta) \preccurlyeq 0^{M+2}10^\infty$. So, $t_1 \ldots t_m$ begins with $M + 2$ consecutive zeros and $a_1 \ldots a_m$ contains $M + 2$ consecutive zeros. By (3.9) we conclude that $\alpha(\beta)$ contains $M + 1$ consecutive zeros, leading to a contradiction with our hypothesis that the number of consecutive zeros in $\alpha(\beta)$ is bounded by M. □

Proof of Theorem B. The first part of the statement follows from Proposition 2.7 and Theorem 3.11 since $\lambda(C_3) = 0$ by the results from [**Sch97**]. The fact from [**Sch97**] that $\dim_H C_3 = 1$ together with Theorem 3.12 gives the last part of the result. □

4. *When E_β^+ does not have isolated points*

In this section we prove Theorem C, which states that the set of $\beta \in (1, 2)$ for which E_β^+ has no isolated points is rather small; it has zero Hausdorff dimension. The theorem is obtained by showing that the intervals $(\beta_L, \beta_R]$ introduced in the previous section cover all but a Hausdorff dimension zero part of the interval $(1, 2)$. Figure 2 suggests that the basic intervals are nested. In Proposition 4.1 below we prove that this is indeed the case. Subsequently, we identify those intervals $(\beta_L, \beta_R]$ that are not contained in any other basic interval, which turn out to be the ones given by a specific subset of the Lyndon words, called Farey words.

PROPOSITION 4.1. *Let $I_1 = (\beta_L, \beta_R]$ and $I_2 = (\tilde{\beta}_L, \tilde{\beta}_R]$ be two different basic intervals. If $I_1 \cap I_2 \neq \emptyset$, then $I_1 \subset I_2$ or $I_2 \subset I_1$.*

Proof. Suppose that $I_1 = (\beta_L, \beta_R]$ is parameterized by the word $a_1 \ldots a_m$ and $I_2 = (\tilde{\beta}_L, \tilde{\beta}_R]$ is parameterized by the word $b_1 \ldots b_n$, i.e.,

$$\alpha(\beta_L) = (a_1 \ldots a_m)^\infty, \quad \alpha(\beta_R) = a_1 \ldots a_m^+ (s_1 \ldots s_m)^\infty;$$
$$\alpha(\tilde{\beta}_L) = (b_1 \ldots b_n)^\infty, \quad \alpha(\tilde{\beta}_R) = b_1 \ldots b_n^+ (t_1 \ldots t_n)^\infty,$$

where $s_1 \ldots s_m = \mathbf{S}(a_1 \ldots a_m)$ and $t_1 \ldots t_n = \mathbf{S}(b_1 \ldots b_n)$ are the Lyndon words for $a_1 \ldots a_m$ and $b_1 \ldots b_n$, respectively. Since $I_1 \cap I_2 \neq \emptyset$, by symmetry we may assume that $\tilde{\beta}_L \in I_1 = (\beta_L, \beta_R]$. We are going to show that $\tilde{\beta}_R < \beta_R$, which by Lemma 2.1 is equivalent to showing that

$$b_1 \ldots b_n^+ (t_1 \ldots t_n)^\infty \prec a_1 \ldots a_m^+ (s_1 \ldots s_m)^\infty. \tag{4.1}$$

Since $\beta_L < \tilde{\beta}_L \leq \beta_R$, by Lemma 2.1 it follows that

$$(a_1 \ldots a_m)^\infty \prec (b_1 \ldots b_n)^\infty \preccurlyeq a_1 \ldots a_m^+ (s_1 \ldots s_m)^\infty. \tag{4.2}$$

We claim that $n > m$.

- If $n < m$, then by (4.2) we have $b_1 \ldots b_n = a_1 \ldots a_n$. Write $m = un + r$ with $u \geq 1$ and $1 \leq r \leq n$. By Lemma 3.8 and (4.2) it follows that $a_1 \ldots a_{un} = (b_1 \ldots b_n)^u$ and $b_1 \ldots b_r = a_1 \ldots a_r = a_{un+1} \ldots a_m^+$, so

  $$a_1 \ldots a_m = (b_1 \ldots b_n)^u b_1 \ldots b_r^-.$$

  By using that $s_1 \ldots s_m = \mathbf{S}(a_1 \ldots a_m)$ we obtain that

  $$\begin{aligned} &a_1 \ldots a_m^+ (s_1 \ldots s_m)^\infty \\ &\quad = (b_1 \ldots b_n)^u b_1 \ldots b_r (s_1 \ldots s_m)^\infty \\ &\quad \preccurlyeq (b_1 \ldots b_n)^u b_1 \ldots b_r (b_{r+1} \ldots b_n b_1 \ldots b_r^- (b_1 \ldots b_n)^{u-1} b_1 \ldots b_r)^\infty \\ &\quad \prec (b_1 \ldots b_n)^\infty, \end{aligned}$$

  leading to a contradiction with (4.2).
- If $n = m$, then by (4.2) we have $b_1 \ldots b_m = a_1 \ldots a_m$ or $b_1 \ldots b_m = a_1 \ldots a_m^+$. Both cases contradict (4.2).

Therefore, we find that $n > m$. Write $n = km + j$ with $k \geq 1$ and $1 \leq j \leq m$. By (4.2) we have

$$b_1 \dots b_n \preccurlyeq a_1 \dots a_m^+ (s_1 \dots s_m)^{k-1} s_1 \dots s_j.$$

From

$$s_{j+1} \dots s_m s_1 \dots s_j \preccurlyeq a_1 \dots a_m \prec a_1 \dots a_m^+$$

one can easily see that

$$(a_1 \dots a_m^+ (s_1 \dots s_m)^{k-1} s_1 \dots s_j)^\infty \succ a_1 \dots a_m^+ (s_1 \dots s_m)^\infty \succcurlyeq (b_1 \dots b_n)^\infty.$$

So, $b_1 \dots b_n \neq a_1 \dots a_m^+ (s_1 \dots s_m)^{k-1} s_1 \dots s_j$ and hence

$$b_1 \dots b_n^+ \preccurlyeq a_1 \dots a_m^+ (s_1 \dots s_m)^{k-1} s_1 \dots s_j. \tag{4.3}$$

If strict inequality holds in (4.3), then (4.1) follows immediately and we are done. Suppose that the equality holds in (4.3). We split the proof of (4.1) into the following two cases.

(I) $1 \leq j \leq (m/2)$. Since $s_1 \dots s_m$ is a Lyndon word it follows that

$$s_1 \dots s_j^- \prec s_1 \dots s_j \preccurlyeq s_{j+1} \dots s_{2j}.$$

Furthermore, $t_1 \dots t_n$ is the Lyndon word for

$$b_1 \dots b_n = a_1 \dots a_m^+ (s_1 \dots s_m)^{k-1} s_1 \dots s_j^-.$$

Then

$$\begin{aligned}(t_1 \dots t_n)^\infty &\preccurlyeq (s_1 \dots s_j^- a_1 \dots a_m^+ (s_1 \dots s_m)^{k-1})^\infty \\ &\prec (s_{j+1} \dots s_{2j} s_{2j+1} \dots s_m s_1 \dots s_j)^\infty.\end{aligned}$$

By (4.3) this proves (4.1), as required.

(II) $(m/2) < j \leq m$. Since $s_1 \dots s_m$ and $t_1 \dots t_n$ are both Lyndon words, by Lemma 3.2(i) it follows that

$$\begin{aligned}(t_1 \dots t_n)^\infty &\preccurlyeq (s_1 \dots s_{m-j} s_{m-j+1} \dots s_j^- a_1 \dots a_m^+ (s_1 \dots s_m)^{k-1})^\infty \\ &\prec (s_{j+1} \dots s_m s_1 \dots s_j)^\infty.\end{aligned}$$

Again we have established (4.1). □

4.1. *Farey words.* The set of Farey words is constructed recursively as follows. Let F_0 be the ordered set containing the two words 0 and 1, i.e., $F_0 := (0, 1)$. For each $n \geq 1$, $F_n = (v_1, \dots, v_{2^n+1})$ is the ordered set obtained from $F_{n-1} = (w_1, \dots, w_{2^{n-1}+1})$ by

$$\begin{aligned} v_{2i-1} &:= w_i \qquad &&\text{for } 1 \leq i \leq 2^{n-1} + 1, \\ v_{2i} &:= w_i w_{i+1} \qquad &&\text{for } 1 \leq i \leq 2^{n-1}, \end{aligned}$$

where $w_i w_{i+1}$ denotes the concatenation of the words w_i and w_{i+1}. For example,

$$F_0 = (0, 1), \quad F_1 = (0, 01, 1), \quad F_2 = (0, 001, 01, 011, 1).$$

Then a word $w \in \{0, 1\}^*$ is a *Farey word* if there is an $n \geq 0$ such that $\omega \in F_n$. For each $n \geq 0$ the words in F_n are listed from left to right in a lexicographically increasing order (cf. [**CIT18**, Lemma 2.2]). In particular, no Farey word is periodic. Let

$$\mathcal{F} := \bigcup_{n \geq 0} F_n \backslash \{0, 1\}$$

be the set of *non-degenerate* Farey words. Clearly, any $w_1 \dots w_m \in \mathcal{F}$ has $w_1 = 0 = 1 - w_m$. It is well known that Farey words are *balanced*, i.e., if for $i = 0, 1$ we use $|u|_i$ to denote the number of occurrences of the symbol i in the word u, then any $w \in \mathcal{F}$ has the property that for any two subwords u and v of w of the same length and $i = 0, 1$, $||u|_i - |v|_i| \le 1$. We recall from [**CIT18**, Proposition 2.3] the following definition.

Definition 4.2. Let $w = w_1 \dots w_m \in \mathcal{F}$. A decomposition $w = uv$ is called the *standard factorization* of w if u and v are both Farey words.

By the construction of F_n the standard factorization of a non-degenerate Farey word $w_1 \dots w_m$ is *unique*. We list some properties of Farey words. The proofs can be found in [**CIT18**, Propositions 2.8 and 2.9].

(f1) For $w_1 \dots w_m \in \mathcal{F}$ both $w_1 \dots w_{m-1}0$ and $1w_2 \dots w_m$ are palindromes, i.e.,

$$w_2 \dots w_{m-1} = w_{m-1} \dots w_2.$$

(f2) Suppose that $w_1 \dots w_m \in \mathcal{F}$ has standard factorization $(w_1 \dots w_{m_1})(w_{m_1+1} \dots w_m)$. The lexicographically largest cyclic permutation of $w_1 \dots w_m$ is given by

$$w_{m-m_1+1} \dots w_m w_1 \dots w_{m-m_1} = w_m w_{m-1} \dots w_2 w_1.$$

(f3) Suppose that $w_1 \dots w_m \in \mathcal{F}$ has standard factorization $(w_1 \dots w_{m_1})(w_{m_1+1} \dots w_m)$. Then $w_1 \dots w_m$ is a Lyndon word and its lexicographically second smallest cyclic permutation is $w_{m_1+1} \dots w_m w_1 \dots w_{m_1}$.

Recall that for $w_1 \dots w_m \in \{0, 1\}^*$, $\overline{w_1 \dots w_m} = (1 - w_1)(1 - w_2) \dots (1 - w_m)$ and note that by symmetry in the set $\mathcal{F}$,

$$w_1 \dots w_m \in \mathcal{F} \quad \Rightarrow \quad \overline{w_m \dots w_1} \in \mathcal{F}.$$

By Lemma 3.2(i) it follows that if $w_1 \dots w_m \in \mathcal{F}$, then $(\overline{w_1 \dots w_m})^\infty \in \mathcal{Q}$, i.e., $\sigma^n((\overline{w_1 \dots w_m})^\infty) \preccurlyeq (\overline{w_1 \dots w_m})^\infty$ for all $n \ge 0$. Properties (f1), (f2) and (f3) imply the following.

LEMMA 4.3. *Let $s_1 \dots s_m \in \mathcal{F}$ with $a_1 \dots a_m = \mathbf{L}(s_1 \dots s_m)$. Suppose that*

$$s_1 \dots s_m = (s_1 \dots s_{m_1})(s_{m_1+1} \dots s_m)$$

is the standard factorization of $s_1 \dots s_m$.

(i) *The words $a_1 \dots a_{m-1}1$ and $0a_2 \dots a_m$ are palindromes, i.e.,*

$$a_2 \dots a_{m-1} = a_{m-1} \dots a_2.$$

(ii) *The Lyndon word associated to $a_1 \dots a_m$ is given by*

$$a_{m-m_1+1} \dots a_m a_1 \dots a_{m-m_1} = a_m a_{m-1} \dots a_1.$$

(iii) $(a_1 \dots a_{m_1})^\infty \in \mathcal{Q}$.

Proof. (i) and (ii) immediately follow from (f1) and (f2), respectively. For (iii) we know that $s_1 \dots s_{m_1}$ is a Lyndon word and therefore $(s_1 \dots s_{m_1})^\infty \preccurlyeq \sigma^n((s_1 \dots s_{m_1})^\infty)$ for all $n \in \mathbb{N}$. This gives $(a_1 \dots a_{m_1})^\infty \succcurlyeq \sigma^n((a_1 \dots a_{m_1})^\infty)$ for all $n \in \mathbb{N}$. □

For Farey words we obtain a strengthened version of Lemma 3.9, which will be useful in the proofs of Theorems C and D. We define a family $\{\Psi_p\}$ of substitutions first. For each $p \geq 1$ set

$$\Psi_p(0) = 0^{p+1}1 \quad \text{and} \quad \Psi_p(1) = 0^p1. \tag{4.4}$$

We extend this definition to words $b_1 \ldots b_n \in \{0, 1\}^*$ by

$$\Psi_p(b_1 \ldots b_n) = \Psi_p(b_1) \ldots \Psi_p(b_n)$$

and similarly for sequences in $\{0, 1\}^{\mathbb{N}}$. One easily shows that τ_k preserves the lexicographical ordering $\{0, 1\}^{\mathbb{N}}$: for any two sequences $(b_i), (d_i) \in \{0, 1\}^{\mathbb{N}}$ we have

$$(b_i) \preccurlyeq (d_i) \Leftrightarrow \Psi_p(b_i) \preccurlyeq \Psi_p(d_i). \tag{4.5}$$

PROPOSITION 4.4. *Let $w = s_1 \ldots s_m \in \mathcal{F}$. Then setting*

$$\mathcal{Z}_w := \{(x_i) \in \{0, 1\}^{\mathbb{N}} : w0^\infty \preccurlyeq \sigma^n((x_i)) \preccurlyeq (s_m \ldots s_1)^\infty \textit{ for all } n \geq 0\}$$

we have that $\mathcal{Z}_w := \{\sigma^j(w^\infty) : 0 \leq j < m\}$; in particular, $\#\mathcal{Z}_m = m$.

Proof. It is clear that $\{\sigma^j(w^\infty) : 0 \leq j < m\} \subseteq \mathcal{Z}_w$. The other inclusion we prove by induction on the level of the Farey words. For $w = 01$ the statement is trivial. Let $n \geq 2$ be given and assume that the statement is true for all non-degenerate Farey words of F_j, $j < n$. Let $w = s_1 \ldots s_m \in F_n$. Note that if $w = 0^{m-1}1$ or $w = 01^{m-1}$, then the statement is obviously true, so we exclude this case. Since all Farey words are balanced there is a p such that w is of the form

$$w = 0^{p+1}10^{p_1}1 \ldots 0^{p_N}10^p1 \quad \text{or} \quad w = 01^p01^{p_1} \ldots 01^{p_N}01^{p+1}$$

for some $N \in \mathbb{N} \cup \{0\}$, where $p_1 \ldots p_N \in \{p, p+1\}^N$ is a palindrome. Assume that $w = 0^{p+1}10^{p_1}1 \ldots 0^{p_N}10^p1$; the proof for the other case is similar. Recall the substitution Ψ_p defined in (4.4). There is a word $v = 0t_1 \ldots t_N1 \in \{0, 1\}^*$ with $\Psi_p(v) = w$. In [**CIT18**, Lemma 2.12] it is proven that v is a Farey word, so $v \in F_k$ for some $k < n$. Moreover, since $w \neq 0^{p+1}1$ we have $v \notin \{0, 1\}$. Recall that

$$s_m \ldots s_1 = 1s_2 \ldots s_{m-1}0 = 10^p10^{p_1}1 \ldots 0^{p_n}10^{p+1},$$

so that

$$\sigma((s_m \ldots s_1)^\infty) = \Psi_p((1t_1 \ldots t_N0)^\infty).$$

Let $x \in \mathcal{Z}_w$ be given. Then by the form of w any two 1's in x are separated by at least p and at most $p+1$ 0's. Assume first that $x_1 \ldots x_{p+2} = 0^{p+1}1$, so that there is a $y \in \{0, 1\}^{\mathbb{N}}$ such that $\Psi_p(y) = x$. Note that for any $r \geq 1$ there corresponds a $j \geq 1$ such that $\sigma^j(x) = \Psi_p(\sigma^r(y))$ since any digit in y corresponds to a block $0^{p+1}1$ or 0^p1 in x. From (3.4) we get that

$$\Psi_p(\sigma^r(y)) = \sigma^j(x) \succcurlyeq w^\infty = \Psi_p(v^\infty),$$

which by (4.5) above implies that $\sigma^r(y) \succcurlyeq v^\infty$ for all $r \geq 0$. On the other hand, from $\sigma^j(x) \preccurlyeq (s_m \ldots s_1)^\infty$ for all $j \geq 0$ it follows that $\sigma^r(y) \preccurlyeq (1t_2 \ldots t_N0)^\infty$ for all $r \geq 0$. Hence, $y \in \mathcal{Z}_v$ and by the induction hypothesis there is an $\ell \in \{0, 1, \ldots, N\}$ such that $y = \sigma^\ell(v^\infty)$. This implies that

$$x = \Psi_p(y) = \sigma^i(w^\infty),$$

where

$$i = \begin{cases} 0 & \text{if } \ell = 0, \\ p+2 & \text{if } \ell = 1, \\ (p+2) + (p_1+1) + (p_2+1) + \cdots + (p_{\ell-1}+1) & \text{if } 2 \le \ell \le N. \end{cases} \tag{4.6}$$

If x is such that $x_1 \ldots x_{j+1} = 0^j 1$ for some $0 \le j \le p$, then there is a $y \in \{0, 1\}^{\mathbb{N}}$ such that $\Psi_p(y) = \sigma^{j+1}(x)$ and by the same arguments as above we get that

$$x = 0^j 1 \sigma^{j+1}(x) = 0^j 1 \Psi_p(y) = 0^j 1 \sigma^i(w^\infty) = \sigma^{i'}(w^\infty),$$

where, in view of (4.6), $i' \in \{0, 1, \ldots, m-1\}$ is defined by

$$i' = \begin{cases} m - j - 1 & \text{if } i = 0, \\ i - j - 1 & \text{otherwise.} \end{cases}$$

This completes the proof. □

4.2. *Farey intervals.* We now use the Farey words to identify the basic intervals $(\beta_L, \beta_R]$ that are not contained in any other basic interval.

Definition 4.5. Let $s_1 \ldots s_m \in \mathcal{F}$ with $a_1 \ldots a_m = \mathbf{L}(s_1 \ldots s_m)$ and let γ_L and γ_R be given by the quasi-greedy expansions $\alpha(\gamma_L) = (a_1 \ldots a_m)^\infty$ and $\alpha(\gamma_R) = a_1 \ldots a_m^+(a_m a_{m-1} \ldots a_1)^\infty$, respectively. Then the interval $J_{a_1 \ldots a_m} = (\gamma_L, \gamma_R]$ is called the *Farey interval* generated by $a_1 \ldots a_m$.

The following lemma is used to show that the Farey intervals are the maximal basic intervals.

LEMMA 4.6. *Let $w = s_1 \ldots s_m \in \mathcal{F}$ and let $a = a_1 \ldots a_m = \mathbf{L}(s_1 \ldots s_m)$. If $a_n = 1$ for some $1 \le n \le m$, then*

$$(\mathbf{S}(a_1 \ldots a_n^-))^\infty \prec w^\infty.$$

Proof. We will prove this lemma by induction on the level of the Farey words. For the word 01 the statement is clear. Let $k \ge 2$ be given and assume that the statement holds for all non-degenerate Farey words in F_j with $j < k$. Let $w = s_1 \ldots s_m \in F_k$. If $w = 0^{m-1}1$ or $w = 01^{m-1}$, then the statement obviously holds. Otherwise, in view of the fact that any Farey word is balanced, w must have the form

$$w = 0^{p+1} 1 0^{p_1} 1 0^{p_2} \ldots 1 0^{p_N} 1 0^p 1 \quad \text{or} \quad w = 0 1^p 0 1^{p_1} 0 1^{p_2} \ldots 0 1^{p_N} 0 1^{p+1}$$

for some $p \in \mathbb{N}$ and $N \in \mathbb{N} \cup \{0\}$, where $p_1 \ldots p_N \in \{p, p+1\}^N$ is a palindrome. We split the proof into the following two cases.
(I) $w = 0^{p+1} 1 0^{p_1} 1 0^{p_2} \ldots 1 0^{p_N} 1 0^p 1$. Then

$$a = \mathbf{L}(w) = 1 0^p 1 0^{p_1} 1 0^{p_2} \ldots 1 0^{p_N} 1 0^{p+1} =: 1 0^{p_0} 1 0^{p_1} 1 0^{p_2} \ldots 1 0^{p_N} 1 0^{p_{N+1}}. \tag{4.7}$$

Let Ψ_p be the substitution map from (4.4). Then by (4.7) there exists a word $v = t_0 t_1 \ldots t_N t_{N+1} = 1 t_1 \ldots t_N 0$ such that

$$\sigma(a^\infty) = (\Psi_p(v))^\infty.$$

By [**CIT18**, Lemma 2.12] it follows that $v = \mathbf{L}(0t_1 \ldots t_N 1)$ and $0t_1 \ldots t_N 1 \in F_i$ for some $i < k$. Let $1 \leq n \leq m$ be such that $a_n = 1$. Then there is a $0 \leq j \leq N+1$ such that

$$a_1 \ldots a_n^- = 10^{p_0}10^{p_1} \ldots 10^{p_{j-1}}10^{p_j+1}.$$

Observe that $p_j \in \{p, p+1\}$. If $p_j = p+1$, then the Lyndon word $\mathbf{S}(a_1 \ldots a_n^-)$ begins with $0^{p+2}1$ and w begins with $0^{p+1}1$. This implies that $(\mathbf{S}(a_1 \ldots a_n^-))^\infty \prec w^\infty$. If $p_j = p$, then $t_j = 1$ and

$$(\mathbf{S}(a_1 \ldots a_n^-))^\infty = (\mathbf{S}(\Psi_p(1t_1 \ldots t_j^-)))^\infty.$$

By the induction hypothesis it follows that

$$(\mathbf{S}(1t_1 \ldots t_j^-))^\infty \prec (0t_1 \ldots t_N 1)^\infty.$$

Since the map Ψ_p preserves the lexicographical ordering (see (4.5)) this gives

$$\begin{aligned}(\mathbf{S}(a_1 \ldots a_n^-))^\infty &= (\mathbf{S}(\Psi_p(1t_1 \ldots t_j^-)))^\infty = \Psi_p(\mathbf{S}(1t_1 \ldots t_j^-))^\infty) \\ &\prec \Psi_p((0t_1 \ldots t_N 1)^\infty) = w^\infty.\end{aligned}$$

(II) $w = 01^p01^{p_1}01^{p_2} \ldots 01^{p_N}01^{p+1}$. Then the largest cyclic permutation of w is

$$a = \mathbf{L}(w) = 1^{p+1}01^{p_1}01^{p_2}0 \ldots 1^{p_N}01^p0 =: 1^{p_0}01^{p_1}01^{p_2}0 \ldots 1^{p_N}01^{p_{N+1}}0. \tag{4.8}$$

Define the substitution map $\hat{\Psi}_p$ by

$$\hat{\Psi}_p(0) = 01^p \quad \text{and} \quad \hat{\Psi}_p(1) = 01^{p+1}$$

and extend it to words and sequences in the usual way. One easily shows that $\hat{\Psi}_p$ preserves the lexicographical ordering. Then by (4.8) there exists a word $v = t_0t_1 \ldots t_Nt_{N+1} = 1t_1 \ldots t_N0$ such that

$$\sigma^{m-1}(a^\infty) = \hat{\Psi}_p(v^\infty).$$

Furthermore, by [**CIT18**, Lemma 2.12] it follows that $v = \mathbf{L}(0t_1 \ldots t_N 1)$ and $0t_1 \ldots t_N 1 \in F_i$ for some $i < k$. Let $1 \leq n \leq m$ be such that $a_n = 1$. Then by (4.8) there exist $0 \leq j \leq N+1$ and $0 < \ell \leq p_j$ such that

$$a_1 \ldots a_n^- = 1^{p_0}01^{p_1}01^{p_2}0 \ldots 1^{p_{j-1}}01^{p_j-\ell}0.$$

Observe that $p_j \in \{p, p+1\}$. Then $0 \leq p_j - \ell \leq p$. If $p_j - \ell < p$, then $\mathbf{S}(a_1 \ldots a_n^-)$ begins with $01^{p_j-\ell}0$ and w begins with 01^p. So, $(\mathbf{S}(a_1 \ldots a_n^-))^\infty \prec w^\infty$. If $p_j - \ell = p$, then $p_j = p+1$ and $t_j = 1$. Since $0t_1 \ldots t_N 1$ is a non-degenerate Farey word in F_i with $i < k$ by the induction hypothesis we have

$$(\mathbf{S}(1t_1 \ldots t_j^-))^\infty \prec (0t_1 \ldots t_N 1)^\infty.$$

Since the map $\hat{\Psi}_p$ preserves the lexicographical ordering it follows that

$$\begin{aligned}(\mathbf{S}(a_1 \ldots a_n^-))^\infty &= (\mathbf{S}(\hat{\Psi}_p(1t_1 \ldots t_j^-)))^\infty = \hat{\Psi}_p((\mathbf{S}(1t_1 \ldots t_j^-))^\infty) \\ &\prec \hat{\Psi}_p((0t_1 \ldots t_N 1)^\infty) = w^\infty.\end{aligned}$$

This completes the lemma. □

PROPOSITION 4.7. *Each Farey interval is a maximal basic interval.*

Proof. By Proposition 4.1 the basic intervals are nested, so it suffices to prove that a Farey interval cannot be contained in any other basic interval. Let $(\gamma_L, \gamma_R]$ be a Farey interval generated by a Farey word $s_1 \ldots s_m$ and let $a_1 \ldots a_m = \mathbf{L}(s_1 \ldots s_m)$. Then

$$\alpha(\gamma_L) = (a_1 \ldots a_m)^\infty \quad \text{and} \quad \alpha(\gamma_R) = a_1 \ldots a_m^+ (s_1 \ldots s_m)^\infty.$$

Suppose on the contrary that there exists another basic interval $(\beta_L, \beta_R]$ such that $(\gamma_L, \gamma_R] \subsetneq (\beta_L, \beta_R]$. Assume that $(\beta_L, \beta_R]$ is generated by the Lyndon word $t_1 \ldots t_n$ and let $b_1 \ldots b_n = \mathbf{L}(t_1 \ldots t_n)$. Then

$$\alpha(\beta_L) = (b_1 \ldots b_n)^\infty \quad \text{and} \quad \alpha(\beta_R) = b_1 \ldots b_n^+ (t_1 \ldots t_n)^\infty.$$

So, by using $\beta_L < \gamma_L \leq \beta_R$ it follows that

$$(b_1 \ldots b_n)^\infty \prec (a_1 \ldots a_m)^\infty \preccurlyeq b_1 \ldots b_n^+ (t_1 \ldots t_n)^\infty. \tag{4.9}$$

By the same argument as in the proof of Proposition 4.1 we obtain $m > n$.

Now we claim that $a_1 \ldots a_n = b_1 \ldots b_n^+$. By (4.9) it follows that $b_1 \ldots b_n \preccurlyeq a_1 \ldots a_n \preccurlyeq b_1 \ldots b_n^+$. So, it suffices to prove that $a_1 \ldots a_n \neq b_1 \ldots b_n$. Suppose that $a_1 \ldots a_n = b_1 \ldots b_n$. Write $m = kn + j$ with $k \geq 1$ and $1 \leq j \leq n$. Note that $a_1 \ldots a_m$ is the largest cyclic permutation of a Farey word. Then $a_{i+1} \ldots a_m \prec a_1 \ldots a_{m-i}$ for all $i < m$. So,

$$a_1 \ldots a_m \preccurlyeq (a_1 \ldots a_n)^k a_{m-j+1} \ldots a_m \prec (a_1 \ldots a_n)^k a_1 \ldots a_j = (b_1 \ldots b_n)^k b_1 \ldots b_j,$$

leading to a contradiction with (4.9). This establishes the claim.

By the claim it follows that $a_n = 1$ and $t_1 \ldots t_n = \mathbf{S}(b_1 \ldots b_n) = \mathbf{S}(a_1 \ldots a_n^-)$. Since $s_1 \ldots s_m$ is a non-degenerate Farey word and $a_1 \ldots a_m = \mathbf{L}(s_1 \ldots s_m)$, by Lemma 4.6 it follows that

$$(t_1 \ldots t_n)^\infty \prec (s_1 \ldots s_m)^\infty \preccurlyeq (a_{n+1} \ldots a_m a_1 \ldots a_n)^\infty.$$

Again by the claim we conclude that

$$b_1 \ldots b_n^+ (t_1 \ldots t_n)^\infty \prec (a_1 \ldots a_m)^\infty.$$

This leads to a contradiction with (4.9). □

Proposition 3.10 states that for any $\beta \in J_{a_1 \ldots a_m}$ the set E_β^+ contains an isolated point. So, the set of $\beta \in (1, 2)$ for which E_β^+ has no isolated points is a subset of $(1, 2) \backslash \bigcup_{s_1 \ldots s_m \in \mathcal{F}} J_{s_m \ldots s_1}$. Suppose on the other hand that $\beta \in (1, 2) \backslash \bigcup_{s_1 \ldots s_m \in \mathcal{F}} J_{s_m \ldots s_1}$. From Proposition 3.3 we know that any isolated point t of E_β^+ must have a periodic β-expansion $b(t, \beta)$. To such a β-expansion we can relate a basic interval $(\beta_L, \beta_R]$ as in Proposition 3.10. From the maximality of the Farey intervals and Proposition 3.10 we can then deduce that t is not isolated for E_β^+. Thus, the set of $\beta \in (1, 2)$ for which E_β^+ has no isolated points is in fact equal to the set

$$(1, 2) \Big\backslash \bigcup_{s_1 \ldots s_m \in \mathcal{F}} J_{s_m \ldots s_1}.$$

To prove Theorem C it is therefore enough to prove that this set has Hausdorff dimension zero. We do so by relating each Farey interval $J_{a_1\ldots a_m}$ to another interval $I_{\overline{a_1\ldots a_m}}$ associated to the doubling map and using known results for the union $\bigcup I_{\overline{a_1\ldots a_m}}$.

Recall that the doubling map is given by $T_2(x) = 2x \pmod 1$ and that $\pi_2 : \{0, 1\}^{\mathbb{N}} \to [0, 1]$ is the projection map defined in (2.1). Set

$$E_D := \left\{x \in \left[0, \tfrac{1}{2}\right) : T_2^n(x) \in \left[x, x + \tfrac{1}{2}\right] \text{ for all } n \geq 0\right\}.$$

For each Farey word $w = w_1 \ldots w_m \in \mathcal{F}$ we denote by $I_w := (q_L, q_R)$ the open interval associated to w, where

$$q_L = \pi_2((w_m w_{m-1} \ldots w_1)^\infty) - \tfrac{1}{2} \quad \text{and} \quad q_R = \pi_2((w_1 \ldots w_m)^\infty).$$

The interval $I_w = (q_L, q_R)$ is well defined since by (f1) it follows that

$$\begin{aligned} q_L &= \pi_2(0 w_{m-1} w_{m-2} \ldots w_1 (w_m w_{m-1} \ldots w_1)^\infty) \\ &= \pi_2(w_1 w_2 \ldots w_{m-1} 0 (w_m w_{m-1} \ldots w_1)^\infty) < \pi_2((w_1 \ldots w_m)^\infty) = q_R. \end{aligned}$$

In [**CIT18**] we find the following result.

PROPOSITION 4.8. [**CIT18**, Proposition 2.14]

(i) *Each I_w is a connected component of* $(0, \frac{1}{2}) \backslash E_D$. *Moreover,*

$$\left(0, \frac{1}{2}\right) \Big\backslash E_D = \bigcup_{w \in \mathcal{F}} I_w.$$

(ii) $\dim_H E_D = 0$.

Recall that by Lemma 2.1 the function $\alpha : \beta \mapsto \alpha(\beta)$ is a strictly increasing bijection from $(1, 2]$ to $\mathcal{Q}$. Moreover, $\pi_2 : \{0, 1\}^{\mathbb{N}} \to (0, 1]$ is a strictly increasing bijection if we remove from $\{0, 1\}^{\mathbb{N}}$ all sequences ending with 0^∞. Since such sequences do not occur as quasi-greedy expansions of 1 and since the first digit $\alpha_1(\beta)$ equals 1 for any $\beta \in (1, 2)$, the map

$$\phi : (1, 2) \to \left(\frac{1}{2}, 1\right), \ \beta \mapsto \pi_2(\alpha(\beta)) = \sum_{i=1}^{\infty} \frac{\alpha_i(\beta)}{2^i}$$

is strictly increasing as well. The image $\phi((1, 2))$ is a proper subset of $(\frac{1}{2}, 1)$.

LEMMA 4.9.

$$\phi\left((1, 2) \Big\backslash \bigcup_{s_1 \ldots s_m \in \mathcal{F}} J_{s_m \ldots s_1}\right) \subseteq \left(\frac{1}{2}, 1\right) \Big\backslash \bigcup_{s_1 \ldots s_m \in \mathcal{F}} (1 - I_{s_1 \ldots s_m}) = 1 - E_D.$$

Proof. Let $s_1 \ldots s_m \in \mathcal{F}$ with $a_1 \ldots a_m = \mathbf{L}(s_1 \ldots s_m)$. Note that

$$q_R = \pi_2((\overline{a_1 \ldots a_m})^\infty) = \sum_{n \geq 1} \frac{1}{2^n} - \pi_2((a_1 \ldots a_m)^\infty) = 1 - \phi(\gamma_L).$$

Moreover, by Lemma 4.3(i) and (ii) it follows that

$$\alpha(\gamma_R) = a_1 \ldots a_m^+ (a_m a_{m-1} \ldots a_1)^\infty = 1 a_{m-1} a_{m-2} \ldots a_1 (a_m a_{m-1} \ldots a_1)^\infty.$$

Then

$$\begin{aligned}\phi(\gamma_R) &= \pi_2(1a_{m-1}a_{m-2}\ldots a_1(a_m a_{m-1}\ldots a_1)^\infty)\\ &= \tfrac{1}{2} + \pi_2((a_m a_{m-1}\ldots a_1)^\infty) = \tfrac{1}{2} + (1 - \pi_2((\overline{a_m a_{m-1}\ldots a_1})^\infty))\\ &= 1 - \big(\pi_2((\overline{a_m a_{m-1}\ldots a_1})^\infty) - \tfrac{1}{2}\big) = 1 - q_L.\end{aligned}$$

Since ϕ is strictly increasing and bijective from $(1, 2)$ to $\phi((1, 2))$ this implies that

$$\phi^{-1}((1 - q_R, 1 - q_L)) = (\gamma_L, \gamma_R).$$

By Proposition 4.8(i) this gives the result. □

Finally, to determine the Hausdorff dimension of $(1, 2)\setminus\bigcup_{s_1\ldots s_m\in\mathcal{F}} J_{s_m\ldots s_1}$, we prove that the inverse $\phi^{-1} : \pi_2\circ\alpha((1, 2)) \to (1, 2)$ is Hölder continuous and combine this with the following well-known result: if $f : (X, \rho_1) \to (Y, \rho_2)$ is a c-Hölder continuous map between two metric spaces (X, ρ_1) and (Y, ρ_2), then $\dim_H f(X) \le (1/c)\dim_H X$.

Lemma 4.10. *For any integer $N \ge 2$ the function ϕ^{-1} is c-Hölder continuous with $c = (\log(1 + 1/N)/\log 4)$ on the set $\phi([1 + (1/N), 2))$.*

Proof. Fix $N \ge 2$ and let $\beta_1, \beta_2 \in [1 + (1/N), 2)$ with $\beta_1 < \beta_2$. Then $\alpha(\beta_1) \prec \alpha(\beta_2)$. Let n be the positive integer such that

$$\alpha_1(\beta_1)\ldots\alpha_{n-1}(\beta_1) = \alpha_1(\beta_2)\ldots\alpha_{n-1}(\beta_2) \quad\text{and}\quad \alpha_n(\beta_1) < \alpha_n(\beta_2). \tag{4.10}$$

By using $1 = \pi_{\beta_1}(\alpha(\beta_1)) = \pi_{\beta_2}(\alpha(\beta_2))$ and (4.10) it follows that

$$\begin{aligned}0 < \beta_2 - \beta_1 &= \beta_2\sum_{j=1}^\infty \frac{\alpha_j(\beta_2)}{\beta_2^j} - \beta_1\sum_{j=1}^\infty \frac{\alpha_j(\beta_1)}{\beta_1^j}\\ &\le \sum_{j=1}^\infty \frac{\alpha_j(\beta_2)}{\beta_2^{j-1}} - \sum_{j=1}^\infty \frac{\alpha_j(\beta_1)}{\beta_2^{j-1}}\\ &= \sum_{j=n}^\infty \frac{\alpha_j(\beta_2) - \alpha_j(\beta_1)}{\beta_2^{j-1}} \le \sum_{j=n}^\infty \frac{1}{(1 + \frac{1}{N})^{j-1}} = N\left(1 + \frac{1}{N}\right)^{2-n}.\end{aligned} \tag{4.11}$$

On the other hand, by (4.10) we also have

$$\begin{aligned}\pi_2(\alpha(\beta_2)) - \pi_2(\alpha(\beta_1)) &= \sum_{j=1}^\infty \frac{\alpha_j(\beta_2) - \alpha_j(\beta_1)}{2^j} = \sum_{j=n}^\infty \frac{\alpha_j(\beta_2) - \alpha_j(\beta_1)}{2^j}\\ &\ge \frac{1}{2^n} - \sum_{j=n+1}^\infty \frac{\alpha_j(\beta_1)}{2^j} \ge \frac{1}{2^n(2^n - 1)} > \frac{1}{4^n},\end{aligned} \tag{4.12}$$

where the second inequality follows by Lemma 2.1 and the fact that

$$\alpha_{n+1}(\beta_1)\alpha_{n+2}(\beta_1)\cdots \preccurlyeq \alpha_1(\beta_1)\alpha_2(\beta_1)\cdots \preccurlyeq (1^{n-1}0)^\infty.$$

Combining (4.11) and (4.12) we conclude that

$$\begin{aligned}|\pi_2(\alpha(\beta_2)) - \pi_2(\alpha(\beta_1))| \ge \frac{1}{4^n} &= \left(1 + \frac{1}{N}\right)^{-(\log 4)/(\log(1+1/N))n}\\ &\ge (N(1 + 1/N)^2)^{-(\log 4)/(\log(1+1/N))}\\ &\quad\times |\beta_2 - \beta_1|^{(\log 4)/(\log(1+1/N))}.\end{aligned}$$

□

Proof of Theorem C. By Lemma 4.9 the only thing left to show is that $\dim_H \phi^{-1}(1-E_D)=0$. This follows from Lemma 4.10 and Proposition 4.8(ii) in the following way:

$$\begin{aligned}0 \le \dim_H \phi^{-1}(1-E_D) &= \dim_H \left(\bigcup_{N\ge 2}\left(\phi^{-1}(1-E_D)\cap\left[1+\frac{1}{N},2\right)\right)\right)\\ &= \sup_{N\ge 2}\dim_H \phi^{-1}\left((1-E_D)\cap\phi\left(\left[1+\frac{1}{N},2\right)\right)\right)\\ &\le \sup_{N\ge 2}\frac{\log 4}{\log(1+1/N)}\dim_H\left((1-E_D)\cap\phi\left(\left[1+\frac{1}{N},2\right)\right)\right)\\ &\le \sup_{N\ge 2}\frac{\log 4}{\log(1+1/N)}\dim_H(1-E_D)=\sup_{N\ge 2}\frac{\log 4}{\log(1+1/N)}\dim_H E_D=0. \qquad\square\end{aligned}$$

5. *The critical points of the dimension function*

Since the map $\eta_\beta : t\mapsto \dim_H K_\beta(t)$ is a decreasing, continuous function with $\eta_\beta(0)=1$ and $\eta_\beta(1/\beta)=0$, there is a unique value τ_β such that $\dim_H K_\beta(t)>0$ if and only if $t<\tau_\beta$. Determining the value of τ_β would extend the results from **[Cla16]** for holes of the form $(0,t)$. For $\beta=\gamma_L$ equal to the left end point of one of the Farey intervals, we show below that $\tau_\beta = 1-(1/\beta)$. This result is based on the following lemma.

LEMMA 5.1. *Let $s_1\ldots s_m\in\mathcal{F}$ with $a_1\ldots a_m=\mathbf{L}(s_1\ldots s_m)$. Let $1\le j\le m$ be such that $s_1\ldots s_m=a_{j+1}\ldots a_m a_1\ldots a_j$. For each $N\ge 1$ define the sequence $\mathbf{t}_N\in\{0,1\}^{\mathbb{N}}$ by*

$$\mathbf{t}_N := (0a_2\ldots a_m(a_1\ldots a_m)^N a_1\ldots a_j)^\infty. \tag{5.1}$$

Then for each $N\ge 1$, $\mathbf{t}_N\prec\mathbf{t}_{N+1}$. Furthermore, any sequence $\mathbf{t}$ that is a concatenation of blocks of the form

$$0a_2\ldots a_m(a_1\ldots a_m)^k a_1\ldots a_j,\qquad k\ge N,$$

satisfies $\mathbf{t}_N\preccurlyeq\sigma^n(\mathbf{t})\prec(a_1\ldots a_m)^\infty$ for all $n\ge 0$. In particular, we have for each $n\ge 0$ that

$$\mathbf{t}_N\preccurlyeq\sigma^n(\mathbf{t}_N)\prec(a_1\ldots a_m)^\infty.$$

Proof. By Lemma 4.3 it follows that

$$s_1\ldots s_m = a_m a_{m-1}\ldots a_1 = 0a_2\ldots a_m^+ = a_{j+1}\ldots a_m a_1\ldots a_j. \tag{5.2}$$

This implies that for all $N\ge 1$

$$\begin{aligned}\mathbf{t}_N &= (0a_2\ldots a_m(a_1\ldots a_m)^N a_1\ldots a_j)(a_{j+1}\ldots a_m a_1\ldots a_j^-(a_1\ldots a_m)^N a_1\ldots a_j)^\infty\\ &\prec (0a_2\ldots a_m(a_1\ldots a_m)^{N+1}a_1\ldots a_j)^\infty=\mathbf{t}_{N+1},\end{aligned}$$

giving the first part of the statement. For the second part, let $\mathbf{t}$ be a sequence consisting of a concatenation of blocks of the form $0a_2\ldots a_m(a_1\ldots a_m)^k a_1\ldots a_j$ with prefix $0a_2\ldots a_m(a_1\ldots a_m)^K a_1\ldots a_j$ for some $K\ge N$. We first show that $\sigma^n(\mathbf{t})\prec(a_1\ldots a_m)^\infty$ for all $n\ge 0$. For $n=0$ the statement is clear. By Lemma 3.8 it follows that $a_{i+1}\ldots a_m\prec a_1\ldots a_{m-i}$ for each $0<i<m$. This implies that $\sigma^n(\mathbf{t})\prec(a_1\ldots a_m)^\infty$ for

each $\ell m < n < (\ell+1)m$, $0 \le \ell \le K$. For all other values of $n < (K+1)m + j$ we obtain the result from (5.2), which implies that

$$a_1 \ldots a_j 0 a_2 \ldots a_m = a_1 \ldots a_m a_1 \ldots a_j^- \prec a_1 \ldots a_m a_1 \ldots a_j.$$

The same arguments then give the result for any $n \ge 0$. Hence, $\sigma^n(\mathbf{t}) \prec (a_1 \ldots a_m)^\infty$ for all $n \ge 0$. We now show that $\sigma^n(\mathbf{t}) \succcurlyeq \mathbf{t}_N$ for each $n \ge 0$. Note that $\mathbf{t}$ has prefix

$$s_1 \ldots s_m^- (a_1 \ldots a_m)^K a_1 \ldots a_j.$$

For $n = 0$ the statement follows from (5.2). By (5.2) and Lemmas 3.2 and 3.8 it follows that

$$s_{i+1} \ldots s_m^- \succcurlyeq s_1 \ldots s_{m-i} \quad \text{and} \quad a_1 \ldots a_i \succ a_{m-i+1} \ldots a_m = s_{m-i+1} \ldots s_m^-$$

for all $0 < i < m$, giving the statement for all $0 < n < m$. Since $s_1 \ldots s_m$ is the Lyndon word associated to $a_1 \ldots a_m$ we obtain

$$a_{i+1} \ldots a_m a_1 \ldots a_i \succcurlyeq s_1 \ldots s_m \succ s_1 \ldots s_m^- \quad \text{for any } 0 \le i < m.$$

Since $a_1 \ldots a_j s_1 \ldots s_{m-j} = a_1 \ldots a_m$ the conclusion that $\sigma^n(\mathbf{t}) \succcurlyeq \mathbf{t}_N$ for all $n \ge 0$ follows. □

PROPOSITION 5.2. *Let* $s_1 \ldots s_m \in \mathcal{F}$ *with* $a_1 \ldots a_m = \mathbf{L}(s_1 \ldots s_m)$ *and let* $\beta \in (1, 2)$ *be such that* $\alpha(\beta) = (a_1 \ldots a_m)^\infty$. *Then* $1 - (1/\beta) \in E_\beta^0$ *and*

$$\tau_\beta = 1 - \frac{1}{\beta} = \max \overline{E_\beta^+}.$$

Proof. Since m is the minimal period of $\alpha(\beta)$ the greedy β-expansion of 1 is equal to $b(1, \beta) = a_1 \ldots a_m^+ 0^\infty$. Lemma 4.3 tells us that $a_1 \ldots a_m^+ = 1 a_{m-1} \ldots a_1$, so

$$\pi_\beta(a_m a_{m-1} \ldots a_1 0^\infty) = \pi_\beta(1 a_{m-1} \ldots a_1 0^\infty) - \frac{1}{\beta} = \pi_\beta(a_1 \ldots a_m^+ 0^\infty) - \frac{1}{\beta} = 1 - \frac{1}{\beta}.$$

Recall that $a_m a_{m-1} \ldots a_1 = 0 a_2 \ldots a_m^+$. Then by Lemma 3.8 it follows that for each $n \ge 0$, $\sigma^n(a_m a_{m-1} \ldots a_1 0^\infty) \prec (a_1 \ldots a_m)^\infty = \alpha(\beta)$ and hence $a_m a_{m-1} \ldots a_1 0^\infty$ is the greedy β-expansion of $1 - (1/\beta)$, i.e., $b(1 - (1/\beta), \beta) = a_m a_{m-1} \ldots a_1 0^\infty$. By Lemma 3.2, $b(1 - (1/\beta), \beta) \in \mathcal{E}_\beta^0$, so $1 - (1/\beta) \in E_\beta^0$.

The quasi-greedy β-expansion of $1 - (1/\beta)$ is given by

$$\tilde{b}\left(1 - \frac{1}{\beta}, \beta\right) = 0 a_2 \ldots a_m (a_1 \ldots a_m)^\infty.$$

Now consider the sequences $\mathbf{t}_N$ from Lemma 5.1. Since $\mathbf{t}_N \preccurlyeq \sigma^n(\mathbf{t}_N) \prec (a_1 \ldots a_m)^\infty = \alpha(\beta)$ for all $n \ge 0$ we have $\mathbf{t}_N \in \mathcal{E}_\beta^+$ for each $N \ge 1$. Moreover, if we set $t_N := \pi_\beta(\mathbf{t}_N)$, then Lemma 2.2 gives that $t_N \nearrow 1 - (1/\beta)$ as $N \to \infty$. So, $\max \overline{E_\beta^+} \ge 1 - (1/\beta)$. Furthermore, the fact that any sequence of concatenations of blocks of the form $0 a_2 \ldots a_m (a_1 \ldots a_m)^k a_1 \ldots a_j$, $k \ge N$, belongs to $\mathcal{K}_\beta^+(t_N)$ implies that $h_{\text{top}}(\mathcal{K}_\beta^+(t_N)) > 0$ for all $N \ge 1$ and hence also $h_{\text{top}}(\mathcal{K}_\beta(t_N)) > 0$ for all $N \ge 1$. By the dimension formula (2.6) we then get that $\tau_\beta \ge 1 - (1/\beta)$.

On the other hand, by Lemma 4.3(ii) and Proposition 4.4 we have

$$\mathcal{K}^+_\beta\left(1-\frac{1}{\beta}\right) = \{(x_i) : a_m a_{m-1}\ldots a_1 0^\infty \preccurlyeq \sigma^n((x_i)) \prec (a_1\ldots a_m)^\infty\ \forall n \ge 0\} = \emptyset. \tag{5.3}$$

Since $E^+_\beta \cap [1-(1/\beta), 1] \subseteq K^+_\beta(1-(1/\beta))$ this implies that $\max \overline{E^+_\beta} \le 1-(1/\beta)$. It also implies that $\dim_H K_\beta(1-(1/\beta)) = 0$, which gives that $\tau_\beta \le 1-(1/\beta)$ and proves the result. □

Remark 5.3. Note that the previous lemma also implies that for any $t < 1-(1/\gamma_L)$ we have $h_{\mathrm{top}}(\mathcal{K}^+_{\gamma_L}(t)) > 0$. We will use this later on.

Next we will give a lower and an upper bound for τ_β on each Farey interval $(\gamma_L, \gamma_R]$.

LEMMA 5.4. *Let $s_1\ldots s_m \in \mathcal{F}$ with $a_1\ldots a_m = \mathbf{L}(s_1\ldots s_m)$. For each $\beta \in (\gamma_L, \gamma_R]$ set $t^* = \pi_\beta(0a_2\ldots a_m(a_1\ldots a_m)^\infty)$ and $t^\diamond = \pi_\beta(0a_2\ldots a_m^+0^\infty)$. Then $t^* \in E^+_\beta$, $t^\diamond \in E^0_\beta$ and*

$$1-\frac{1}{\beta}-\frac{1}{\beta^m}+\frac{1}{\beta(\beta^m-1)} \le t^* \le \tau_\beta \le t^\diamond < 1-\frac{1}{\beta}.$$

Proof. Take $\beta \in (\gamma_L, \gamma_R]$. Then

$$(a_1\ldots a_m)^\infty \prec \alpha(\beta) \preccurlyeq a_1\ldots a_m^+(a_m a_{m-1}\ldots a_1)^\infty.$$

We first show that $\tau_\beta \ge t^*$. By Lemmas 4.3 and 3.8 we have

$$\sigma^n(0a_2\ldots a_m(a_1\ldots a_m)^\infty) \preccurlyeq (a_1\ldots a_m)^\infty \prec \alpha(\beta) \quad \text{for all } n \ge 0.$$

Hence, $b(t^*, \beta) = 0a_2\ldots a_m(a_1\ldots a_m)^\infty$ and as in the proof of Lemma 5.1 we have that $\sigma^n(b(t^*, \beta)) \succcurlyeq b(t^*, \beta)$ for each $n \ge 0$. So, $t^* \in E^+_\beta$.

For each $t < t^*$ we have by Lemma 2.2 that $b(t, \beta) \prec 0a_2\ldots a_m(a_1\ldots a_m)^\infty$. This implies that for N large enough $b(t, \beta) \prec \mathbf{t}_N \prec (a_1\ldots a_m)^\infty \prec \alpha(\beta)$. By Lemma 5.1 it follows that $t_N \in \mathcal{K}^+_\beta(t)$ and $h_{\mathrm{top}}(\mathcal{K}_\beta(t)) \ge h_{\mathrm{top}}(\mathcal{K}^+_\beta(t)) > 0$. Thus, $\dim_H \mathcal{K}_\beta(t) > 0$ and $\tau_\beta \ge t^*$.

On the other hand, for $t^\diamond$ we have that $0a_2\ldots a_m^+0^\infty$ is admissible for any $\beta \in (\gamma_L, \gamma_R]$ and that $\sigma^n(0a_2\ldots a_m^+0^\infty) \succ 0a_2\ldots a_m^+0^\infty$ for all $0 < n < m$, so $t^\diamond \in E^0_\beta$. By Lemmas 4.3 and 3.7 we get

$$\begin{aligned}
&\mathcal{K}^+_\beta(t^\diamond)\\
&\quad\subseteq \{(x_i) : a_m a_{m-1}\ldots a_1 0^\infty \preccurlyeq \sigma^n((x_i)) \prec a_1\ldots a_m^+(a_m a_{m-1}\ldots a_1)^\infty\ \forall n \ge 0\}\\
&\quad= \{(x_i) : (a_m a_{m-1}\ldots a_1)^\infty \preccurlyeq \sigma^n((x_i)) \prec a_1\ldots a_m^+(a_m a_{m-1}\ldots a_1)^\infty\ \forall n \ge 0\}\\
&\quad= \{(x_i) : (a_m a_{m-1}\ldots a_1)^\infty \preccurlyeq \sigma^n((x_i)) \preccurlyeq (a_1\ldots a_m)^\infty\ \forall n \ge 0\}\\
&\quad= \{(x_i) : a_m a_{m-1}\ldots a_1 0^\infty \preccurlyeq \sigma^n((x_i)) \preccurlyeq (a_1\ldots a_m)^\infty\ \forall n \ge 0\}.
\end{aligned} \tag{5.4}$$

By Proposition 4.4 it follows that $\#\mathcal{K}^+_\beta(t^\diamond) < \infty$, so that $\dim_H K_\beta(t^\diamond) = 0$. This gives that $\tau_\beta \le t^\diamond$. Note that

$$\pi_{\gamma_R}(a_1a_2\ldots a_m^+(0a_2\ldots a_m)^\infty) = 1.$$

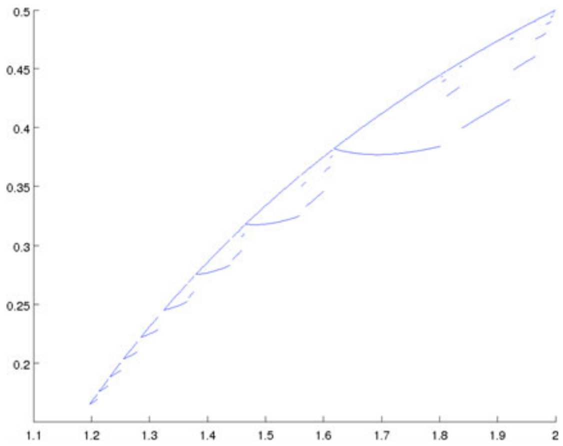

FIGURE 3. A plot of $1 - (1/\beta)$ and $1 - (1/\beta) - (1/\beta^m) + (1/\beta(\beta^m - 1))$ for basic intervals corresponding to Farey words of length m with $m \le 10$.

Then we have for each $\beta \in (\gamma_L, \gamma_R]$ that

$$t^* = \pi_\beta(0a_2 \ldots a_m(a_1a_2 \ldots a_m)^\infty)$$
$$> 1 - \frac{1}{\beta} - \frac{1}{\beta^m} + \sum_{i=1}^{\infty} \frac{1}{\beta^{im+1}} = 1 - \frac{1}{\beta} - \frac{1}{\beta^m} + \frac{1}{\beta(\beta^m - 1)}.$$

From Proposition 5.2 we know that $\pi_{\gamma_L}(0a_2 \ldots a_m^+0^\infty) = 1 - (1/\gamma_L)$. For $\beta > \gamma_L$ we have $a_1 \ldots a_m^+0^\infty \prec b(1, \beta)$, so that

$$t^\diamond = \pi_\beta(0a_2 \ldots a_m^+0^\infty) = \pi_\beta(a_1 \ldots a_m^+0^\infty) - \pi_\beta(10^\infty) < 1 - \frac{1}{\beta}. \qquad \square$$

In Figure 3 we see a plot of the lower and upper bounds for τ_β found in Lemma 5.4.

The next lemma considers the critical point τ_β for the remaining values of β, i.e., those that are not in the closure of a Farey interval.

LEMMA 5.5. *Let* $\beta \in (1, 2)\setminus \bigcup[\gamma_L, \gamma_R]$ *with the union taken over all Farey intervals. Then* $\max \overline{E_\beta^+} = \tau_\beta = 1 - (1/\beta)$.

Proof. Take $\beta \in (1, 2)\setminus \bigcup[\gamma_L, \gamma_R]$. First we show that $\tau_\beta \ge 1 - (1/\beta)$. Let $t < 1 - (1/\beta)$ with $b(t, \beta) = (b_i(t, \beta))$. Since $\dim_H((1, 2)\setminus \bigcup[\gamma_L, \gamma_R]) = 0$ there exists a sequence of Farey intervals $([\gamma_{L,k}, \gamma_{R,k}])$ such that $\gamma_{L,k} \nearrow \beta$ as $k \to \infty$. Thus, as $k \to \infty$ we have

$$\sum_{i=1}^{\infty} \frac{b_i(t, \beta)}{(\gamma_{L,k})^i} \searrow \sum_{i=1}^{\infty} \frac{b_i(t, \beta)}{\beta^i} = t \quad \text{and} \quad 1 - \frac{1}{\gamma_{L,k}} \nearrow 1 - \frac{1}{\beta}. \tag{5.5}$$

For each k we have a sequence $(\mathbf{t}_{k,N}) \subseteq \mathcal{E}^+_{\gamma_{L,k}}$ as given in (5.1). Since $\gamma_{L,k} < \beta$ we obtain for each $N, n \ge 1$ that

$$\mathbf{t}_{k,N} \preccurlyeq \sigma^n(\mathbf{t}_{k,N}) \prec \alpha(\gamma_{L,k}) \prec \alpha(\beta).$$

Hence, $\mathbf{t}_{k,N} \in \mathcal{E}_\beta^+$ for all $k \geq 1$ and $N \geq 1$. This gives that $\max \overline{E_\beta^+} \geq 1 - (1/\beta)$. Moreover, since $t < 1 - (1/\beta)$ we can find by (5.5) a sufficiently large $M \in \mathbb{N}$ such that

$$t < t_1 := \sum_{i=1}^{\infty} \frac{b_i(t, \beta)}{(\gamma_{L,M})^i} < 1 - \frac{1}{\gamma_{L,M}} < 1 - \frac{1}{\beta}.$$

Observe that $b(t, \beta) = (b_i(t, \beta))$ is a $\gamma_{L,M}$-expansion of t_1, which is lexicographically less than or equal to its greedy expansion $b(t_1, \gamma_{L,M})$. Then

$$\begin{aligned}\mathcal{K}_\beta^+(t) &= \{(x_i) : b(t, \beta) \preccurlyeq \sigma^n((x_i)) \prec \alpha(\beta)\ \forall n \geq 0\}\\ &\supseteq \{(x_i) : b(t_1, \gamma_{L,M}) \preccurlyeq \sigma^n((x_i)) \prec \alpha(\gamma_{L,M})\ \forall n \geq 0\} = \mathcal{K}_{\gamma_{L,M}}^+(t_1). \qquad (5.6)\end{aligned}$$

Since $\tau_{\gamma_{L,M}} = 1 - (1/\gamma_{L,M}) > t_1$ by Remark 5.3 we know that $h_{\mathrm{top}}(\mathcal{K}_{\gamma_{L,M}}^+(t_1)) > 0$ and together with (5.6) we then find that $h_{\mathrm{top}}(\mathcal{K}_\beta^+(t)) > 0$, which in turn implies that $\tau_\beta \geq t$. Since $t < 1 - (1/\beta)$ was taken arbitrarily we conclude that $\tau_\beta \geq 1 - (1/\beta)$.

To prove the other inequality we show that for any $t > 1 - (1/\beta)$ we have $\mathcal{K}_\beta^+(t) = \emptyset$. Take $t > 1 - (1/\beta)$. There is a sequence of Farey intervals $([\gamma_{L,k}, \gamma_{R,k}])$ such that $\gamma_{L,k} \searrow \beta$ as $k \to \infty$. Thus, when $k \to \infty$ we have

$$\sum_{i=1}^{\infty} \frac{b_i(t, \beta)}{(\gamma_{L,k})^i} \nearrow \sum_{i=1}^{\infty} \frac{b_i(t, \beta)}{\beta^i} = t \quad \text{and} \quad 1 - \frac{1}{\gamma_{L,k}} \searrow 1 - \frac{1}{\beta}.$$

Since $t > 1 - (1/\beta)$ we can find a sufficiently large $N \in \mathbb{N}$ such that

$$1 - \frac{1}{\beta} < 1 - \frac{1}{\gamma_{L,N}} < t_2 := \sum_{i=1}^{\infty} \frac{b_i(t, \beta)}{(\gamma_{L,N})^i} < t.$$

Since $\gamma_{L,N} > \beta$, $b(t, \beta)$ is the greedy $\gamma_{L,N}$-expansion of t_2, i.e., $b(t, \beta) = b(t_2, \gamma_{L,N})$. Therefore,

$$\begin{aligned}\mathcal{K}_\beta^+(t) &\subseteq \{(x_i) : b(t_2, \gamma_{L,N}) \preccurlyeq \sigma^n((x_i)) \prec \alpha(\gamma_{L,N})\ \forall n \geq 0\}\\ &= \mathcal{K}_{\gamma_{L,N}}^+(t_2) \subseteq \mathcal{K}_{\gamma_{L,N}}^+(\tau_{\gamma_{L,N}}).\end{aligned}$$

From (5.3) we conclude that $\mathcal{K}_\beta^+(t) = \emptyset$ and hence $\max \overline{E_\beta^+}$, $\tau_\beta \leq t$. Since $t > 1 - (1/\beta)$ was taken arbitrarily we have $\max \overline{E_\beta^+} = \tau_\beta = 1 - (1/\beta)$. □

Proof of Theorem D. From Proposition 5.2 and Lemmas 5.4 and 5.5 we know that for all $\beta \in (1, 2)$ we have $\tau_\beta \leq 1 - (1/\beta)$ with equality only if $\beta \in (1, 2) \setminus \bigcup (\gamma_L, \gamma_R]$. We also know that for these points $\tau_\beta = \max \overline{E_\beta^+}$.

By Proposition 3.3 we know that any isolated point of E_β^+ has a periodic greedy β-expansion $b(t, \beta)$. From Proposition 3.10 it follows that any $t \in (0, 1)$, for which $b(t, \beta) = (s_1 \ldots s_m)^\infty$ is Lyndon, is isolated in E_β^+ if and only if β lies in the basic interval associated to $(s_1 \ldots s_m)^\infty$. Since Farey intervals are maximal by Proposition 4.7, if $\beta \notin \bigcup (\gamma_L, \gamma_R]$, then E_β^+ cannot contain an isolated point and $\overline{E_\beta^+}$ is a Cantor set. □

6. *Final observations and remarks*

With the results from Theorems B and C we have shown that the situation for $\beta \in (1, 2)$ differs drastically from the situation for $\beta = 2$ that was previously investigated in [**Urb86**, **Nil09**, **CT17**]. There are still several unanswered questions.

Firstly, the structure of E_β^0 remains illusive to us. We know that $t \in E_\beta^0$ is isolated in E_β if $\beta - 1 \notin K_\beta(t)$ and in Proposition 2.6 we proved that $h_{\text{top}}(\mathcal{K}_\beta(t)) = h_{\text{top}}(\mathcal{K}_\beta^+(t))$ for any $t \in E_\beta^+$. It would be interesting to know whether $t \in E_\beta^0$ is isolated in E_β in case $\beta - 1 \in K_\beta(t)$ and to consider $h_{\text{top}}(\mathcal{K}_\beta^0(t))$ also in case $t \notin E_\beta^+$.

In the previous section we have investigated the value of the critical point τ_β of the dimension function $\eta_\beta : t \mapsto \dim_H K_\beta(t)$. We could determine this value for any β in the set $(1, 2)\setminus \bigcup(\gamma_L, \gamma_R]$. If $\beta \in (\gamma_L, \gamma_R]$ for some Farey interval $(\gamma_L, \gamma_R]$, we only have a lower and an upper bound for τ_β. With a calculation very similar to the one in (5.4) one can show that for any $\beta \in (\gamma_L, \gamma_R]$ that satisfies

$$\alpha(\beta) \prec a_1 \ldots a_m^+(0a_2 \ldots a_m)(a_1 \ldots a_m)^\infty$$

we have $\tau_\beta = t^*$. However, for larger values of $\beta \in (\gamma_L, \gamma_R]$ the situation seems more intricate. It would be interesting to consider this question further by specifying τ_β more precisely also on $\bigcup(\gamma_L, \gamma_R]$ and by analysing the behaviour of the function $\tau : \beta \mapsto \tau_\beta$.

For $\beta = 2$ it is shown in [**Urb86**] that $\dim_H(E_2 \cap [t, 1]) = \dim_H K_2(t)$. Motivated by Proposition 2.7 we conjecture the following.

CONJECTURE 6.1. *For any* $t \in [0, 1)$ *and any* $\beta \in (1, 2)$ *we have* $\dim_H(E_\beta \cap [t, 1]) = \dim_H K_\beta(t)$.

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