DIFFERENTIAL ROTATION IN DEGENERATE STARS

H. M. VAN HORN

Dept. of Physics and Astronomy and C. E. Kenneth Mees Observatory, University of Rochester, Rochester, N.Y., U.S.A.

Abstract. The problem of the development of the angular velocity distribution during the final phase of gravitational contraction of a star immediately prior to the onset of degeneracy and during the subsequent cooling phase is surveyed. Processes that may affect this distribution are discussed at some length, and estimates of the timescales for redistribution of the angular momentum are given for each process. Possible effects on the evolution and observable consequences are briefly considered.

1. Introduction

Ostriker *et al.* (1966) have shown that fully degenerate stars in differential rotation can have masses considerably greater than the Chandrasekhar limit, and Ostriker and Bodenheimer (1968) have constructed models of such stars. However, they have not discussed the problem of how such an object might be formed, and it is the aim of this paper to consider this question. The importance of an investigation of this problem has been strongly emphasized in recent years: The discovery of pulsars (Hewish *et al.*, 1968) and their subsequent interpretation as rotating neutron stars (Gold, 1968) has provided a dramatic example of a class of degenerate objects which are in rapid rotation. Still more recent has been the discovery of pulsating X-ray sources with periods in the range of 1 to 5 s (Giacconi *et al.*, 1971; Tananbaum *et al.*, 1972). The regularity of the X-ray pulsations suggests that rotation provides the basic 'clock' mechanism for these sources, in analogy with the pulsars, and the short period can again be reconciled only with a small, degenerate star.

A quantitative understanding of such objects requires the study of changes in the velocity field during the evolution of a star. For simplicity I shall consider only the final stages of evolution of an initially uniformly rotating star subsequent to the exhaustion of nuclear energy sources. The evolution then consists of two rather distinct stages: the final gravitational contraction phase immediately prior to the occurrence of complete degeneracy, and the subsequent cooling phase in which gravitational energy production is almost negligible. Models of non-rotating stars in similar phases of their evolution have been constructed, e.g., by Savedoff *et al.* (1969), and I shall make use of the qualitative features of these results for guidance in cases where rotation is important.

Most of the decrease of the stellar radius occurs in the first, gravitational contraction, stage, and it is here that the differential rotation is developed. This phase of the evolution is discussed in Section 2, where some general properties of rotating objects are reviewed, and in Section 3, where the effectiveness of meridional circulations in redistributing the angular momentum is discussed. The subsequent cooling phase is considered in Section 4. Here I discuss the effect of Ekman pumping in a white dwarf with a crystallizing, solid core, and also the question of whether a similar mechanism

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is operative when the core instead consists of uniformly rotating fluid. In Section 5 a schematic picture of the evolution of differentially rotating stars is sketched, and several important problem areas that require further study are pointed out.

My principal aim in this paper is to survey the general problem of the evolution of differentially rotating, degenerate stars. I shall be primarily concerned with attempting to present a qualitative physical understanding of the evolutionary process. Most of what I shall say therefore will not be new, but it has not previously been applied to the particular problem of the final phases of stellar evolution. I shall not discuss the very considerable literature dealing with uniform density objects (which is reviewed by Lebovitz (1967) and treated in detail by Chandrasekhar (1969); see also Fujimoto (1968)) or on stars in pure rotation (reviewed by Strittmatter (1969)). I shall also ignore cases where magnetic fields or general relativity are important; the fluid dynamics problems are sufficiently complex even in Newtonian mechanics, and the inclusion of these effects would complicate the situation even further.

2. General Considerations: The Necessity of Differential Rotation

The evolution of a contracting, differentially rotating star is governed by the equation of motion of a self-gravitating, viscous fluid:

$$\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} \equiv \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\,\mathbf{v} = -\frac{1}{\varrho}\nabla p - \nabla\Phi + \frac{\eta}{\varrho}\nabla^2\mathbf{v}. \tag{1}$$

In hydrostatic equilibrium the velocity $\mathbf{v}$ is zero, and (1) reduces to the familiar equation of pressure balance, $\nabla p = -\varrho\nabla\Phi$, where Φ is the gravitational potential.

From the form of (1), it follows that viscous forces act on a timescale

$$\tau_{\text{visc}} \sim \varrho L^2/\eta\,, \tag{2}$$

where L is the characteristic length scale of the velocity field. Even for viscosities η as large as those in white dwarfs* this timescale is $\sim 10^{10}$ to 10^{14} yr if $L \sim R$, where R is a typical dimension of the star. Thus for large scale motions the fluid can be considered to be essentially inviscid. If the evolution leads to the development of small-scale structure in the velocity field, however, the viscous forces cannot be neglected. Equation (2) then provides a lower limit to the scale of length over which appreciable differential motions can be maintained: $\Delta L_{\text{min}} \sim (\eta\tau_{\text{ev}}/\varrho)^{1/2}$, where the timescale of the evolution is

$$\tau_{\text{ev}} \equiv (\mathrm{d}\ln R/\mathrm{d}t)^{-1}. \tag{3}$$

Smaller scale motions will be dissipated by viscosity in a time $\ll \tau_{\text{ev}}$. For $\tau_{\text{ev}} \sim 10^4$ yr,

* The viscosity $\eta \sim pn\lambda$, where p is the characteristic momentum transferred by particles of number density n over a mean free path λ. Thus for a non-relativistically degenerate electron gas, in which the Fermi energy is $\varepsilon_{\text{F}} = p_{\text{F}}^2/2\,m$,

$$\eta \sim p_{\text{F}} n_{\text{e}}/n_{\text{i}}(Ze^2/\varepsilon_{\text{F}})^2 \sim \frac{\hbar^5}{Zm^2e^4}\left(\frac{Z\varrho}{AH}\right)^{5/3} \sim 10^6 Z^{-1}\varrho_6{}^{5/3}\ \text{c.g.s.}$$

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which is typical of some of the more rapid phases we shall be considering, $\Delta L_{\min} \sim 10^{-3}$ to 10^{-5} R.

A second conclusion that also follows trivially from (1) is that angular momentum is conserved during any axially symmetric motion of a ring-shaped mass element in a cylindrically symmetric, inviscid fluid. Since the angular momentum per unit mass is $\mathbf{j} = \mathbf{r} \times \mathbf{v}$, the time derivative of $\mathbf{j}$ is given by (1) as

$$\frac{\mathrm{d}\mathbf{j}}{\mathrm{d}t} = \mathbf{1}_\phi \left[r \left(\frac{1}{\varrho} \frac{\partial p}{\partial z} + \frac{\partial \Phi}{\partial z} \right) - z \left(\frac{1}{\varrho} \frac{\partial p}{\partial r} + \frac{\partial \Phi}{\partial r} \right) \right], \tag{4}$$

where (r, ϕ, z) are the usual cylindrical coordinates, $\mathbf{1}_\phi$ is a unit vector in the ϕ-direction, and we have made use of the assumption of axial symmetry. Equation (4) refers to a differential mass element of volume $\mathrm{d}r \cdot r\mathrm{d}\phi \cdot \mathrm{d}z$; for a ring, $\mathrm{d}\mathbf{j}/\mathrm{d}t$ is zero by symmetry. Since the angular momentum per unit mass of a rotating ring is $r^2 \Omega \mathbf{1}_z$, where $\Omega = \Omega(r, z)$ is the angular velocity of the ring, we conclude that $r^2 \Omega$ is constant in any motion of this kind.

An important consequence of this is that uniform rotation can be maintained only in a homologous contraction. Consider a case in which a mass ring with initial coordinates (r, z) undergoes an axially symmetric contraction to (r', z'). From angular momentum conservation the angular velocity Ω' at the new position is given by

$$\Omega'(r', z') = \Omega(r, z) \cdot (r/r')^2 . \tag{5}$$

If the original angular velocity distribution is uniform, the new distribution will be

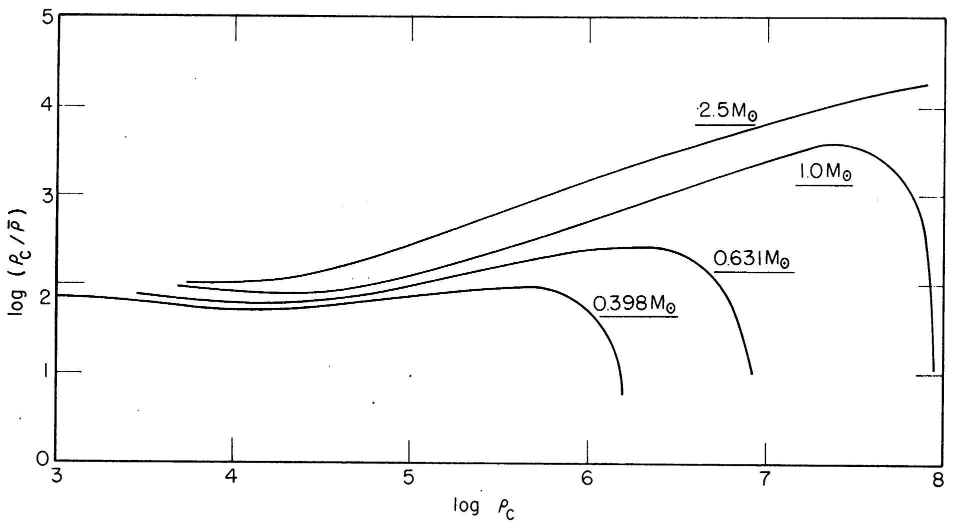

Fig. 1. The non-homologous nature of the final gravitational contraction phase, caused by copious neutrino emission from the stellar core, is indicated by the large variation of log $(\varrho_c/\bar{\varrho})$ during the evolution. The curves shown are for the non-rotating iron star models computed by Savedoff *et al.* (1969); in these stars ϱ_c increases monotonically during the evolution.

uniform also only if $r' = \alpha r$, where α is a constant; i.e. the contraction must be homologous.

The final gravitational contraction phase is distinctly non-homologous, however. In a homologous contraction the ratio $\varrho_c/\bar{\varrho}$ of central to mean density is constant; but in the non-rotating iron star models of Savedoff *et al.* (1969), this ratio can vary by many orders of magnitude, as shown in Figure 1. The more rapid contraction of the central regions is caused by the copious neutrino emission from these models. Since the neutrino losses will be qualitatively the same for a rotating star, however, we may thus conclude that this phase of the evolution will in general be non-homologous and consequently that a rotating star will develop differential rotation during this phase.

There is another consequence of non-homologous contraction that considerably complicates this simple picture, however: a non-homologous contraction generates meridional circulations. Consider a cylindrical mass-shell in a star, which we assume to be initially in uniform rotation. As the star evolves (non-homologously), the central regions contract more rapidly than the rest of the star, and the cylindrical shape of the initial mass-shell, on which Ω was constant, is not preserved. The angular velocity distribution thus develops a gradient in the z-direction as well as in the r-direction.

Such a distribution is not stable, however. For an inviscid fluid which we assume to be stationary (i.e. $\partial/\partial t \to 0$), the curl of (1) gives

$$\operatorname{curl}\left[(\mathbf{v}\cdot\nabla)\,\mathbf{v}\right] = -\,\mathbf{1}_\phi r\,\frac{\partial\Omega^2}{\partial z} = \nabla p \times \nabla\varrho/\varrho^2\,, \tag{6}$$

where for a state of pure rotation $\mathbf{v} = \mathrm{r}\Omega\mathbf{1}_\phi$. Surfaces of constant pressure are thus not parallel to surfaces of constant ϱ. Since the net force on a stationary mass element is parallel to ∇p, as shown by (1), displacement of a mass element parallel to a surface $p =$ constant does no work. Because ∇p is not parallel to $\nabla\varrho$, however, the displaced element is of different density than its surroundings and thus experiences a net buoyancy force that accelerates it parallel to ∇p. The sense of the force is such as to restore parallelism of surfaces of constant p and constant ϱ; i.e. to drive $\partial\Omega/\partial z$ to zero. Stable configurations are thus implicit barytropes (objects in which $p = p(\varrho)$).

From (1) and (6) we estimate that z-gradients of Ω should be eliminated on a time-scale of order

$$\tau_z \sim \Omega^{-1}\cdot\left(\frac{L}{\Omega}\frac{\partial\Omega}{\partial z}\right)^{-1/2}, \tag{7}$$

where $L \sim R$ is the characteristic scale of the meridional circulation currents. Thus if the evolution were so rapid as to produce $\partial\Omega/\partial z \sim \Omega/R$, we would expect very strong circulation currents that would reorganize the angular velocity distribution onto cylinders in a time $\sim\Omega^{-1}$. (However, see Section 3.) In reality, of course, this means that evolution can only lead to the development of very small gradients, of order $\partial\Omega/\partial z \sim (\Omega/R)/(\Omega\tau_{ev})^2$, as Durisen (1972) has also pointed out. Even for an evolutionary timescale as short as $\sim 10^4$ yr and a rotation period as long as $\sim$days, this only gives $\partial\Omega/\partial z \sim 10^{-13}\Omega/R$. The final pre-white-dwarf contraction phase thus may indeed produce differential rotation on cylinders, as envisaged by Ostriker, Bodenheimer, and Lynden-Bell.

3. Meridional Circulation

We have seen that the final contraction phase of stellar evolution can be expected to generate gradients of the angular velocity distribution. However, if we are to conclude that the star maintains $\nabla\Omega\neq 0$, we must show that angular momentum transport cannot restore uniform rotation within the timescale τ_{ev} of the evolution. We must thus investigate the timescale of the meridional circulations. This problem has been discussed in considerable detail by Mestel (1965, 1970) and others, and I shall merely summarize Mestel's treatment, but in somewhat more general terms.

Perhaps the best-known and most thoroughly studied example of meridional circulation is that driven by the failure of strict radiative equilibrium in a rotating star. The timescale for this circulation, which was originally studied by Eddington (1929) and by Sweet (1950) and which now bears their names, can be crudely estimated by the following rather general argument:

The equation of energy conservation in a stellar interior is given by

$$T\frac{\mathrm{d}s}{\mathrm{d}t}=\varepsilon_n-\varepsilon_\nu-\frac{1}{\varrho}\operatorname{div}\mathbf{H}, \tag{8}$$

where $T\mathrm{d}s/\mathrm{d}t$ is the rate of increase of the heat content per unit mass of the stellar matter, and ε_{n}, ε_ν, and div $\mathbf{H}/\varrho$ are, respectively, the nuclear energy production rate, the neutrino loss rate, and the net thermal energy flux from the mass element. In most calculations, this equation is applied only to the radiative zone of a stationary stellar model, in which case it reduces to the simpler form div $\mathbf{H}=0$.

The thermal energy flux is given by

$$\mathbf{H}=-\frac{4ac}{3}\frac{T^3}{\kappa\varrho}\nabla T \tag{9}$$

and may include electron conduction processes as well as the ordinary radiative flux.

The necessity of meridional circulations derives from Eddington's resolution of von Zeipel's (1924) celebrated paradox: that div $\mathbf{H}\neq 0$ in a rotating star. This follows, after a rather lengthy argument, from the equation of hydro-stationary equilibrium,

$$-r\Omega^2\mathbf{1}_r=-\frac{1}{\varrho}\nabla p-\nabla\Phi. \tag{10}$$

As pointed out above, the equilibrium is unstable unless Ω is independent of z. In this case the curl of the centrifugal force term vanishes, however, and it is then derivable from a scalar potential. We may thus rewrite (10) in terms of a total (gravitational plus centrifugal) potential, ψ:

$$\frac{1}{\varrho}\nabla p=-\nabla\psi. \tag{11}$$

Since (11) shows that surfaces of constant p are everywhere parallel to surfaces of constant ψ, we have $p=p(\psi)$ and consequently, by taking the gradient, $\varrho=-(\mathrm{d}p/\mathrm{d}\psi)^{-1}=\varrho(\psi)$.

Since p and ϱ are functions only of ψ, T is also, and we can therefore write

$$\begin{aligned}\operatorname{div}\mathbf{H} &= \operatorname{div}\left[-\frac{4ac}{3}\frac{T^3}{\kappa\varrho}\frac{\mathrm{d}T}{\mathrm{d}\psi}\nabla\psi\right] = \\ &= \frac{\mathrm{d}}{\mathrm{d}\psi}\left[-\frac{4ac}{3}\frac{T^3}{\kappa\varrho}\frac{\mathrm{d}T}{\mathrm{d}\psi}\right]|\nabla\psi|^2 - \frac{4ac}{3}\frac{T^3}{\kappa\varrho}\frac{\mathrm{d}T}{\mathrm{d}\psi}\nabla^2\psi\,. \end{aligned} \tag{12}$$

This expression does not in general vanish, however. This is easy to see in the case of uniform rotation, when $\nabla^2\psi=4\pi G\varrho-2\Omega^2$, and the second term in (12) becomes a function only of ψ. In this case, since $|\nabla\psi|$ is not constant on an equipotential surface, it is evident that the first term leads to a variation of div $\mathbf{H}$ along an equipotential that is *not* cancelled by a corresponding variation of the second term. Evidently this must also be the case when $\Omega(r)$ is arbitrary; although the second term now varies also along an equipotential surface, cancellation of the variation of the first term would clearly occur only for very special forms of Ω.

Now in (8), ε_n and ε_v are functions only of ϱ and T and thus only of ψ. Because of the variation of div $\mathbf{H}$ along an equipotential surface, however, the net rate of change of the heat content of individual mass elements also varies along an equipotential, and it is this differential heating and cooling relative to the average that drives the Eddington-Sweet circulations. If we let $\langle\cdots\rangle$ denote the average over such a surface of constant ψ, we evidently must have

$$\left\langle T\frac{\mathrm{d}s}{\mathrm{d}t}\right\rangle = \varepsilon_n - \varepsilon_v - \frac{1}{\varrho}\langle\operatorname{div}\mathbf{H}\rangle\,.$$

Hence, subtracting from (8), we obtain

$$T\frac{\mathrm{d}s}{\mathrm{d}t} - \left\langle T\frac{\mathrm{d}s}{\mathrm{d}t}\right\rangle = -\frac{1}{\varrho}\left[\operatorname{div}\mathbf{H} - \langle\operatorname{div}\mathbf{H}\rangle\right]. \tag{13}$$

To lowest order, the difference between $T\,\mathrm{d}s/\mathrm{d}t$ and the term $\langle T\,\mathrm{d}s/\mathrm{d}t\rangle$ (which is, e.g., the negative of the average gravitational energy generation rate on an equipotential, in a contracting, non-degenerate star) is simply the convective term

$$T\mathbf{v}\cdot\nabla s = \mathbf{v}\cdot\mathbf{1}_\psi T\frac{\mathrm{d}s}{\mathrm{d}\psi}|\nabla\psi|\,, \tag{14}$$

where $\mathbf{1}_\psi$ is a unit vector in the direction of $\nabla\psi$, and T, s are functions of ψ only. Equations (12), (13), and (14), together with the general result $\nabla^2\psi=4\pi G\varrho-(1/r^2)(\mathrm{d}/\mathrm{d}r)(r^2\Omega^2)$, thus give for the component of the meridional circulation velocity in the direction of $\nabla\psi$,

$$\mathbf{v}\cdot\mathbf{1}_\psi = -\left\{\frac{\mathrm{d}}{\mathrm{d}\psi}\left(\frac{H}{|\nabla\psi|}\right)[|\nabla\psi|^2 - \langle|\nabla\psi|^2\rangle]\right.$$
$$\left.-\frac{H}{|\nabla\psi|}\left[\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}(r^2\Omega^2) - \left\langle\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}(r^2\Omega^2)\right\rangle\right]\right\}\Big/\varrho T\frac{\mathrm{d}s}{\mathrm{d}\psi}|\nabla\psi|. \tag{15}$$

Note that $\mathbf{H}$ is parallel to $\nabla\psi$ and, from (9), that $\mathbf{H}/|\nabla\psi|$ is a function of ψ only.

When Ω is independent of position within the star, the second term in (15) vanishes, and the first term gives the usual result for the Eddington-Sweet circulation (Mestel, 1965). Our schematic derivation indicates that this result is more general than might have been supposed, however; it appears still to be correct even when nuclear burning and gravitational contraction are included, when electron conduction is considered, and – perhaps most surprising – when neutrino energy losses are taken into account. The latter is an important point since, in the evolutionary phases we are considering, the timescale of the evolution is determined by the neutrino loss rate. Equation (15) indicates that the timescale of the meridional circulations is still determined by the variations of thermal flux, however. This is shorter for neutrino-dominated evolution than for the case in which neutrino emission is neglected, but it is still much longer than the timescale $\tau_\nu \sim E_{\mathrm{th}}/L_\nu$ that characterizes this phase of the evolution.

Equation (15) can be used to obtain a rough approximation for the circulation timescale. If R is again a characteristic dimension of the star, $L \sim HR^2$ is the optical luminosity, $E_{\mathrm{th}} \sim R^3 \varrho T \delta s$ is the thermal energy content of the star, and if $|\nabla\psi| \sim \sim GM/R^2 - R\Omega_0^2$, where Ω_0 is a characteristic rotation frequency and the centrifugal potential energy is still small compared to the gravitational term (it is at most ~ 0.14 for Ostriker and Bodenheimer's secularly stable models), we may roughly estimate

$$\tau \sim \frac{R}{v} \sim \frac{E_{\mathrm{th}}}{L}\Big/\left\{\frac{\Omega_0^2}{GM/R^3}\left[1 + \alpha\frac{\delta\Omega^2}{\Omega_0^2}\right]\right\}, \tag{16}$$

where α is a number of order unity. Apart from the (presumably small) correction term arising from the non-uniformity of the rotation this is just the usual timescale for the Eddington-Sweet circulations,

$$\tau_{\mathrm{ES}} \sim \frac{E_{\mathrm{th}}}{L}\frac{GM/R^3}{\Omega_0^2}. \tag{17}$$

Because Ω_0^2 is always less than GM/R^3 for an object in pressure balance, this is at best comparable to the Kelvin-Helmholtz timescale. Since this is long compared to the timescale τ_ν of the contraction, we may thus conclude that such thermally driven circulations are ineffective in eliminating the differential rotation during this evolutionary phase.

There are other processes which can also drive meridian circulations, however. For example, the instability* discussed by Goldreich and Schubert (1967) will generate

* The existence of this instability is due to the very large heat diffusivity in a stellar interior. This permits the growth of certain perturbations that are stable under an adiabatic displacement, by causing rapid relaxation of the isentropic constraint.

such motions if the conditions $\partial(r^2\Omega)/\partial r \geqslant 0$ and $\partial\Omega/\partial z = 0$ are violated. Since $r^2\Omega = \text{constant}$ in the contraction of an inviscid fluid, the first of these conditions is probably satisfied in the present context, although only marginally so. In cases where this criterion is violated, Goldreich and Schubert have estimated that angular momentum redistribution should occur on a timescale $\sim\Omega^{-1}(\partial \ln\Omega/\partial \ln r)^{-1/2}$. However, Colgate (1968) has pointed out that the growth of this instability is limited by non-linear processes, and he has argued that the relevant timescale for the circulation induced by this thermally driven instability is the Kelvin-Helmholtz timescale. Subsequent calculations by James and Kahn (1970, 1971) have confirmed the non-linear limiting of this instability and have shown that the relevant timescale is actually that of the Eddington-Sweet circulations, τ_{ES}. The Goldreich-Schubert instability thus appears also to be ineffective in producing a significant rearrangement of the angular momentum distribution during the timescale of the initial contraction.

Another possible driving mechanism for meridional circulation is the instability that results from the z-variation of Ω produced by the non-homologous contraction. This is not obviously identical with the Goldreich-Schubert instability in spite of the fact that the stability condition $\partial\Omega/\partial z = 0$ is the same and the timescale estimated by (7) is the same as their original estimate. In particular, it is not clear that the restriction to cases where the thermal diffusivity is large compared to the viscosity is required by the argument leading to (7), while it is essential to Goldreich and Schubert's analysis. Nevertheless, the growth of the $\partial\Omega/\partial z$ instability is probably also limited by non-linear effects. However, the timescale in which the limited-amplitude instability can produce a significant redistribution of angular momentum is probably the timescale τ in which the instability is fed by the gravitational contraction rather than the Kelvin-Helmholtz timescale that applies to the corresponding form of the Goldreich-Schubert instability. If this is correct, the $\partial\Omega/\partial z$ instability provides faster relaxation of the differential rotation than any of the other processes so far considered; but it is not enough to eliminate the differential rotation during this phase of the evolution.

4. Subsequent Evolution: Spin-up with a Crystallizing Core

If angular momentum redistribution is ineffective during the final phase of gravitational contraction immediately preceding the onset of complete degeneracy, the star is left at the end of this phase in a state of strong, almost cylindrically symmetric differential rotation. Such objects may be qualitatively very similar to the models of Ostriker and Bodenheimer. These stars are still very hot, however, and the subsequent evolution consists of gradual cooling, as in the case of non-rotating, degenerate stars. The differential rotation has several effects upon this stage of the evolution, which we consider next.

4.1. Evolution without viscous energy production

Perhaps the easiest case to discuss is one in which the kinetic energy stored in differential motions, $E_{\text{diff}} \sim MR^2(\Delta\Omega)^2$, is sufficiently small that the rate of viscous dissipation

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of rotational kinetic energy contributes negligibly to the total luminosity of the star throughout the cooling phase. Because of the extremely long timescale for viscous processes, this does not severely restrict the angular velocity distribution; even for a maximally rotating object, in which the kinetic energy of rotation is comparable to the gravitational energy, we expect $\dot{E}_{\rm diff} \sim E_{\rm diff}/\tau_{\rm visc} \sim 10^{-4}$ to $10^{0} L_{\odot}$. In a star where this condition is satisfied the cooling phase is thus largely unaffected by the rotation until crystallization begins at the center. With the development of a solid core of appreciable size, however, a new physical process – that of spin-up associated with the viscosity-dominated Ekman boundary layer – becomes important and produces angular momentum redistribution on a very much reduced timescale.

This process has been studied theoretically by Greenspan and Howard (1963) in the context of a simple laboratory experiment. While this case is far too idealized to be directly applicable in the context of stellar interiors, this simplification permits them to give a clear, physical exposition of the basic process, which I shall summarize here. The case which they consider is that of a homogeneous fluid initially constrained to rotate uniformly at angular velocity Ω between two parallel, semi-infinite disks located in the planes $z = -L$ and $z = +L$. At time $t = 0$ the angular velocities of the disks are both instantaneously increased to $\Omega + \Delta\Omega$, where $\Delta\Omega/\Omega \ll 1$, and the problem is to describe the subsequent process of spin-up of the fluid. This occurs in three stages. First, a viscous boundary layer (the Ekman layer) forms at each disk. The thickness δ of this layer is given to order of magnitude by $\varrho\delta^2/\eta \sim \Omega^{-1}$. This boundary layer is no longer in pressure balance, however; the pressure in the layer is that required to balance centrifugal force at the original angular velocity Ω. The excess centrifugal force thus impels fluid radially outward, away from the rotation axis, at a velocity $v_{\rm E} \sim L\Delta\Omega$. In order to conserve mass, however, this material must be replaced by fluid from the approximately inviscid region between the two boundary layers; this 'interior' fluid thus moves toward the axis, at a speed $v_{\rm I} \sim v_{\rm E}\delta/L$. Angular momentum conservation of a contracting ring of fluid spins up this material to angular velocity $\Omega + \Delta\Omega$ when the fluid ring has moved radially inward through a distance $\Delta L \sim \frac{1}{2} L\Delta\Omega/\Omega$. The timescale for spin-up of the fluid to the new angular velocity is thus

$$\tau_{\rm GH} \sim \Delta L/v_{\rm I} \sim \tfrac{1}{2}\left(\varrho L^2/\eta\Omega\right)^{1/2} \sim \left(\tau_{\rm visc}/\Omega\right)^{1/2}. \tag{18}$$

The third and final stage of the spin-up process is the gradual viscous dissipation of small residual motions about the new equilibrium state, and these ultimately decay on the timescale $\sim \tau_{\rm visc}$.

In a rapidly rotating white dwarf, the timescale given by (18) can be very short in spite of the long viscous timescale. For a star in which $\Omega^{-1} \sim 1$ s, for example, $\tau_{\rm GH} \sim 10^1$ to 10^3 yr. This is so much shorter than the cooling time, $\tau_{\rm KH} \sim 10^7$ to 10^9 yr in these phases, that it would imply virtually instantaneous spin-up of the entire star upon commencement of crystallization if this analysis were directly applicable. A star is not a homogeneous fluid, however, and this leads to an important difference from the Greenspan-Howard problem.

In particular, the stratification of density within a star restricts the effect of 'Ekman pumping' to a region of characteristic dimension $\Delta \sim R/S$, where the dimensionless 'stratification parameter' S is given by

$$S^2 \approx \xi R/H_\varrho\,, \quad H_\varrho = (\mathrm{d} \ln \varrho/\mathrm{d}r)^{-1}\,, \quad \xi = G\bar{\varrho}/\Omega^2\,. \tag{19}$$

This effect, first pointed out by Holton (1965), has received extensive discussion by Mestel (1970), and more recently by Sakurai *et al.* (1971), and by Clark (1972), in connection with the solar oblateness problem. In the solar case $S^2 \gg 1$ and thus $\Delta \ll R$; near the center of an object of the type we are presently considering, however, $\xi \sim 10$ to 100, $H_\varrho \sim R$ and consequently $\Delta \sim 0.1$ to $0.3\,R$. Because of the limited extent of the region in which Ekman pumping is effective in this case, spin-up now leads to a quasi-steady state of non-uniform rotation; the region of dimension Δ is brought into uniform rotation in a time $\sim \tau_{\mathrm{GH}}$ appropriate to this size, while the external regions not penetrated by the Ekman currents are essentially unaffected. The residual differential rotation of these outer layers then persists until the Eddington-Sweet circulations eventually complete the redistribution of angular momentum on the timescale τ_{ES} (Sakurai *et al.*, 1971).

It is important to note that this final redistribution may lead to ejection of matter from the star, especially for objects of larger mass. The reason is that the total angular momentum of a *uniformly* rotating, degenerate star is severely constrained by the requirement that the velocity of rotation at the equator must be less than the local velocity of escape. In a differentially rotating star, however, the outer layers can be in almost Keplerian orbits, as pointed out by Ostriker and Bodenheimer, and this restriction does not apply. This leads to a particularly interesting situation for objects more massive than the Chandrasekhar limit. As Roxburgh (1965) has shown, the maximum mass, M_U, for uniformly rotating white dwarfs is only slightly larger than for non-rotating, degenerate stars. Thus if a star of mass $M > M_U$ reaches a stable, quasi-stationary state of differential rotation at the end of the pre-degenerate contraction phase, there is no corresponding, uniformly rotating object into which it can evolve. Consequently, when the mass of the uniformly rotating core grows to exceed M_U, a catastrophic collapse must result, perhaps leading to a supernova explosion. This is essentially the mechanism suggested by Schwartz and Africk (1970). However, the time between the onset of crystallization and the collapse of the core will be considerably shorter than they found if Ekman pumping causes spin-up on the Greenspan-Howard timescale.

4.2. Evolution with Viscous Energy Production

If the energy of differential rotation is so large that $\dot{E}_{\mathrm{diff}}$ becomes comparable to the luminosity at some point during the radiative cooling phase, the evolution becomes qualitatively different. This is the case that Durisen (1972) has considered. Because the store of rotational energy in such objects is comparable to the total gravitational energy of the star, and because the viscous timescale that characterizes this mode of evolution is so large, such a star represents as truly a 'final' stage of evolution as does an ordinary white dwarf; the store of energy is inexhaustible in cosmological time. However, further changes in the mechanical structure of the star do occur in this case. As the core is gradually brought into uniform rotation by the viscous forces, angular momentum conservation demands an outward transfer of angular momentum to the envelope of the star. In Durisen's calculations this is implicitly assumed to take place on a timescale $\sim \tau_{\rm visc}$; the possibility of Ekman pumping is ignored. This is probably not completely justified, however, as the following argument indicates:

It is conceivable that viscous-dominated evolution will lead to a structure consisting of a uniformly rotating core and a differentially rotating envelope containing most of the angular momentum, as found by Durisen. However, in this case there must be a layer of high shear between the core and the envelope. This may act as an Ekman layer and bring about uniform rotation on a timescale $\sim \tau_{\rm GH}$. On the other hand, since a fluid does not have the shear resistance of a solid, it appears equally plausible *a priori* that an initial shear layer may have the opposite effect of generating differential motions within the initially uniformly rotating core. In either case Durisen's calculations – while an important first attack on this problem – needs to be extended to include an improved treatment of the fluid dynamics.

5. Summary and Conclusions

I have discussed in some detail the fluid dynamic processes associated with the final contraction and cooling phases of rotating stars near the end-points of their evolution. The strongly non-homologous nature of the neutrino-dominated, pre-degenerate contraction phase almost certainly leads to the development of strong differential rotation by the time degeneracy brings the contraction to a halt. The instability of a configuration in which $\partial\Omega/\partial z \neq 0$ further suggests that the contracting star will maintain an almost cylindrically symmetric angular velocity distribution through the action of the rapid circulations driven by this instability.* At the end of this phase the star may thus resemble the models of Ostriker and Bodenheimer.

The behavior of the star during the subsequent cooling phase depends sensitively upon the total angular momentum and the viscosity. It is furthermore uncertain because the possible effect of Ekman pumping upon the spin-up in this phase is not clear. *If* Ekman pumping is important, one can distinguish two situations, depending upon the ratio of the timescale $\tau_{\rm ev}$ of the evolution (defined by (3)) to $\tau_{\rm GH}$.

In the case where $\tau_{\rm GH} \gtrsim \tau_{\rm ev}$, Ekman pumping is ineffective anyway, even if it must in principle be taken into account. This inequality is satisfied both for slow rotators and for stars in very rapid contraction. Defined this way, a slow rotator is one for which $\Omega^{-1} \gtrsim \tau_{\rm ev}^2/\tau_{\rm visc}$. With the parameters of the non-rotating star models of Savedoff *et al.* (1969), this gives $\Omega^{-1} \gtrsim 1$ h (for a 1 $M_\odot$ white dwarf), $\gtrsim 4$ days (for 0.631 $M_\odot$), and

* Except perhaps near the stellar surface, where the requirement of continuity of the gravitational potential in spite of a discontinuous change in the total potential may result in the establishment of a surface boundary layer. This problem has recently been investigated by Marks and Clement (1971) and references therein.

$\gtrsim 60$ days (for 0.398 $M_\odot$). For comparison, the observational data give $\Omega^{-1} \sim 1.34$ days for the white dwarf G195-19 (Angel and Landstreet, 1971; but the mass of this object is unknown) and $\Omega^{-1} \sim 1$ h for the 0.45 $M_\odot$ white dwarf 40 Eri B (Greenstein and Trimble, 1972). Thus, if it is important at all, Ekman pumping has apparently played a role in 40 Eri B, which violates this 'slow rotator' condition.

The case of rapid contraction is illustrated by the final evolution of the 2.5 $M_\odot$ iron star model of Savedoff *et al.* This model is so much more massive than the Chandrasekhar limit that degeneracy is unable to stop the contraction, and τ_{ev} ultimately becomes orders of magnitude shorter than τ_{GH} even for maximal rotation. Strong differential rotation is therefore clearly expected for this case. Theoretical confirmation of this is provided by the calculations of LeBlanc and Wilson (1970) who studied the collapse phase of an initially rapidly and uniformly rotating iron star of 7 $M_\odot$; their calculation represents a continuation of the evolution of such a star subsequent to the phase considered by Savedoff *et al.* In LeBlanc and Wilson's fully hydrodynamic, two-dimensional (but inviscid) calculation, a strong differential rotation did indeed develop, and the angular velocity distribution was found to be approximately given by $\Omega \propto r^{-1.85}$ (except, of course, near the axis, where their finite difference equations give too coarse a representation of the physics).

In the case where $\tau_{GH} \ll \tau_{ev} \ll \tau_{ES}$, Ekman pumping may be important for the evolution. For stars in which crystallization begins at high enough luminosity so that viscous energy production can still be neglected, this process will lead to the growth of an appreciable, uniformly rotating core in the rather short timescale τ_{GH} (appropriate to a region of dimension Δ). If the stellar mass is $M \gg M_U$, this will lead to core collapse and perhaps to explosion as a supernova.

If viscous energy production cannot be ignored, growth of the crystallizing core proceeds on a much longer timescale $\sim \tau_{visc}$, and the evolution is dominated by angular momentum transfer from the core to the envelope, as Durisen has shown. The role of Ekman pumping in this case is problematical at present and is an important area for further investigation.

There are a substantial number of other problems that are also of interest in connection with the evolution of highly rotating, degenerate stars. From an observational point of view, for example, it is evidently important to establish whether such objects do in fact exist. In this regard, a determination of the rotation velocities of stars which are thought to be the progenitors of the white dwarfs – e.g. the nuclei of planetary nebulae or the hot subdwarfs – would be of considerable importance. In addition, the incontrovertible detection of a rotationally distorted object would be of still more interest. (Wegner (1972) has recently suggested that the white dwarf CD-42°14462 may be just such a star; however this remains to be confirmed.)

For the theorist, an even greater variety of problems are available: How does the instability resulting from $\partial\Omega/\partial z \neq 0$ manifest itself? How does the existence of rotation affect the evolutionary development of a star? Can the development of differential rotation halt a collapse? Is a viscosity-dominated core sufficiently similar to a solid core to permit Ekman pumping to be effective? Quantitative answers to questions of this sort will evidently require detailed numerical computations. However, the gross dissimilarity of scales of the important fluid dynamical processes (e.g., $\tau_{GH} \sim 10$ to 10^3 yr vs. $\tau_{visc} \sim 10^{10}$ to 10^{14} yr; similarly, the thickness of the Ekman layer is of order centimeters to meters, compared to a stellar radius $\sim 10^3$ km) makes it necessary also to carry out analytical studies of this problem in simpler situations. In addition, calculations of, e.g., the viscosity of relativistically degenerate electrons must be carried out, and a better understanding of the phenomenon of crystallization under conditions of high shear is also needed.

Hopefully, an attack upon some of these problems will soon be forthcoming; for the potential for increasing our understanding of the evolution of rotating, degenerate stars is truly enormous.

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