AN INTEGRAL OVER THE INTERIOR OF A SIMPLEX

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1. Introduction

If $f(z)$ is analytic in a suitable domain, it is shown how the integral of $f(\alpha_1 x_1 + \ldots + \alpha_n x_n)$ over the interior of a simplex may be reduced to the evaluation of a contour integral, in fact to an exercise in partial fractions.

The contour integral is expressed in two ways, according as the simplex is given in terms of its vertices or faces.

2. Notation

It will be convenient to use a matrix notation. Let $x = \{x_1, \ldots, x_n\}$ be a column vector representing a point in real n-dimensional Euclidean space, where the volume element is $dx = \prod_{r=1}^{n} dx_r$. Using a dash for the transpose let $\alpha' = (\alpha_1, \ldots, \alpha_n)$ be a row vector of real or complex constants.

Suppose that the co-ordinate vectors of the vertices of a simplex are

$$v_r = \{x_{r1}, \ldots, x_{rs}, \ldots, x_{rn}\} \, (1 \leqq r \leqq n+1), \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

and define $n+1$ numbers by

$$z_r = \alpha' v_r = \sum_{s=1}^{n} \alpha_s x_{rs} (1 \leqq r \leqq n+1). \quad \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

3. Two Lemmas

Lemma 1. *Let S^* be the simplex $x_1 \geqq 0, x_2 \geqq 0, \ldots, x_n \geqq 0, x_1 + x_2 + \ldots + x_n \leqq k$; then*

$$\int_{S^*} \exp(\alpha' x) dx = \frac{1}{2\pi i} \int_\Gamma \frac{e^{kz} dz}{z \prod_{r=1}^{n} (z - \alpha_r)},$$

where Γ is a contour enclosing $z = 0, \alpha_1, \ldots, \alpha_n$.

Proof. See Lemma 1 of (**1**).

Lemma 2. *If V_n is the volume of S, the simplex with vertices* (1), *and C_1 is a contour enclosing all the z_r of* (2), *then*

$$\int_S \exp(\alpha' x) dx = \frac{n! V_n}{2\pi i} \int_{C_1} \frac{e^z dz}{\prod_{r=1}^{n+1} (z - z_r)} \quad \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

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Proof. Make the change of variables $y = x - v_{n+1}$; then S becomes a simplex S_0 with one vertex at the origin and the remaining vertices at

$$w_r = v_r - v_{n+1}\,(1 \leqq r \leqq n) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

and the left-hand side of (3) becomes

$$e^{z_{n+1}} \int_{S_0} \exp(\alpha' y) dy. \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

If now W is the $n \times n$ matrix whose rth column is w_r, then the determinant $|W|$ of W is just $\pm n! V_n$ (see (2), page 124). Apply the transformation $y = Wx$ to (5); then S_0 becomes the simplex S^* of Lemma 1, with $k = 1$, and (5) becomes

$$n! V_n e^{z_{n+1}} \int_{S^*} \exp(\alpha' W x) dx. \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6)$$

Let β_r be the rth component of the row vector $\alpha' W$. Then, by Lemma 1, with $k = 1$, (6) becomes, for a contour Γ enclosing $\zeta = 0, \beta_1, \ldots, \beta_n$,

$$\frac{n! V_n}{2\pi i} \int_\Gamma \frac{\exp(\zeta + z_{n+1}) d\zeta}{\zeta \prod_{r=1}^{n} (\zeta - \beta_r)}.$$

The lemma now follows on letting $z = \zeta + z_{n+1}$, since

$$\beta_r + z_{n+1} = \alpha' w_r + \alpha' v_{n+1} = \alpha' v_r = z_r \quad (1 \leqq r \leqq n).$$

4. The Main Theorem

Theorem 1. *Suppose that* $f(z) = \sum_{r=0}^{\infty} c_r z^r$ *for* $|z| < R$, *and that*

$$F_s(z) = \frac{1}{(s-1)!} \int_0^z f(t)(z-t)^{s-1} dt \quad (s = 1, 2, \ldots).$$

Let S be the simplex with vertices (1). *Suppose further that the numbers z_r of* (2) *satisfy $|z_r| < R_1$ for some $R_1 < R$, and let C be the circle $|z| = \rho$, where $R_1 < \rho < R$. Then*

$$\int_S f(\alpha' x) dx = \int_C F_n(z) K_n(z) dz, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(7)$$

where

$$K_n(z) = \frac{n! V_n}{2\pi i} \frac{1}{\prod_{r=1}^{n+1} (z - z_r)}.$$

Proof. In (3) of Lemma 2, replace for $\lambda > 1$, α_r by $\lambda \alpha_r$, z by λz, and let the contour C_1 become a contour C_2.

Then since

$$\int_{C_2} z^m K_n(z) dz = 0 \text{ for } m \leqq n-1, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(8)$$

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on expanding both exponentials and equating corresponding powers of λ, it follows that

$$\int_S (\alpha' x)^r dx = \frac{r!}{(n+r)!} \int_{C_2} z^{n+r} K_n(z) dz. \qquad \text{.....................(9)}$$

Now choose C_2 to be the circle C, multiply both sides of (9) by the c_r and sum the resulting series. Then

$$\int_S f(\alpha' x) dx = \int_C F_n(z) K_n(z) dz,$$

where

$$F_n(z) = \sum_{r=0}^{\infty} \frac{r! c_r z^{n+r}}{(n+r)!} = \sum_{r=0}^{\infty} c_r \int_0^z \frac{t^r (z-t)^{n-1}}{(n-1)!} dt$$

$$= \frac{1}{(n-1)!} \int_0^z f(t)(z-t)^{n-1} dt,$$

as in the enunciation. The circle C may now be expanded into any contour in the domain of regularity of $F_n(z)$. Because of remark (8) $F_n(z)$ need only be evaluated to within a polynomial of degree $n-1$.

In the case when the row vector α' is real and the z_r are all distinct there is a geometrical interpretation of the contour integral. Project the vertices $V_1, \ldots, V_{n+1}$ of S orthogonally onto a set of points $P_1, \ldots, P_{n+1}$ along the vector α'. Let D_r be the product of the signed distances from the remaining points P to P_r, then the multiple integral of (7) is

$$n! V_n \sum_{r=1}^{n+1} \frac{F_n(P_r)}{D_r}.$$

When the z_r are not distinct, a result of Turnbull (3) shows how the contour integral of (7) may be expressed in terms of a determinant which involves the $F_1(z), \ldots, F_n(z)$.

5. Alternative Form of the Main Theorem

Suppose that the simplex S of Theorem 1 is defined as the region bounded by the $n+1$ hyperplanes

$$\begin{aligned} a_{11}x_1 + \ldots + \quad a_{1n}x_n + a_{1,n+1} &= 0, \\ \cdots\cdots\cdots\cdots\cdots\cdots & \\ a_{n+1,1}x_1 + \ldots + a_{n+1,n}x_n + a_{n+1,n+1} &= 0. \end{aligned} \qquad \text{..................(10)}$$

We show in this paragraph how the $K_n(z)$ of Theorem 1 can be expressed in terms of $z, \alpha_1, \ldots, \alpha_n$ and the a_{rs} of (10). To do this it will be convenient to denote by $A = (a_{rs})$ the $(n+1)\times(n+1)$ matrix of coefficients of (10), and by A_{rs}, the $n \times n$ determinant left when the row r and column s of $|A|$ are deleted.

Lemma 3. *The volume V_n of the simplex* (10) *is given by*

$$n!V_n = \pm \frac{|A|^n}{\prod_{r=1}^{n+1} A_{r,n+1}}. \qquad \text{(11)}$$

Proof. Let the co-ordinates of the vertex $v_r = (x_{r1}, \ldots, x_{rs}, \ldots, x_{rn})$ be the solution of (10) when the rth equation is omitted, then by Cramer's rule, allowing for an interchange of columns in the numerator, we have

$$x_{rs} = (-)^{n+s+1} \frac{A_{rs}}{A_{r,n+1}}. \qquad \text{(12)}$$

Denote by Δ the $(n+1)\times(n+1)$ matrix whose rth row is $(x_{r1}, \ldots, x_{rn}, 1)$ and consider the product $\Delta A'$. Since v_r is the solution of (10) with the rth equation omitted, every element in the rth row of $\Delta A'$ is zero, except that in the rth place, where the entry is

$$\sum_{s=1}^{n} x_{rs}a_{rs} + a_{r,n+1};$$

but, by (12), this is

$$\frac{(-)^{n+r+1}}{A_{r,n+1}} \left\{ \sum_{s=1}^{n} (-)^{r+s} a_{rs}A_{rs} + (-)^{n+r+1} a_{r,n+1}A_{r,n+1} \right\} = \frac{(-)^{n+r+1}}{A_{r,n+1}} |A|.$$

Thus

$$|\Delta|\,|A| = \pm \frac{|A|^{n+1}}{\prod_{r=1}^{n+1} A_{r,n+1}}.$$

The lemma follows, since $n!V_n = \pm|\Delta|$ (see (2) p. 124).

Theorem 2. *If $A_{(r)}(z)$ is the determinant of the matrix formed from A by replacing the rth row by $(\alpha_1, \ldots, \alpha_n, -z)$, then*

$$2\pi i K_n(z) = \pm \frac{|A|^n}{\prod_{r=1}^{n+1} A_{(r)}(z)}, \qquad \text{(13)}$$

the sign $\pm$ being chosen to make $\pm(-)^{\frac{1}{2}(n+1)(n+2)} \dfrac{|A|^n}{\prod_{r=1}^{n+1} A_{r,n+1}}$ *positive.*

Proof. Expanding $A_{(r)}(z)$ by its rth row, we get

$$A_{(r)}(z) = \sum_{s=1}^{n} (-)^{r+s}\alpha_s A_{rs} + (-)^{r+n+1}(-z)A_{r,n+1}$$

and so, by (12),

$$A_{(r)}(z) = (-)^{r+n}A_{r,n+1}\left(z - \sum_{s=1}^{n} \alpha_s x_{rs}\right) = (-)^{r+n}A_{r,n+1}(z-z_r),$$

The theorem follows from this and Lemma 3, apart from the choice of sign. This is determined by putting $\alpha_1 = \alpha_2 = \ldots = \alpha_n = 0$ and $z = 1$ in (13).

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