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# TWO REMARKS ON AMALGAMS OF LOCALLY FINITE GROUPS

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We construct $2^{\aleph_0}$ non amalgamation bases in the class of locally finite groups, and we present necessary and sufficient conditions for the embeddability of an amalgam into a locally finite group in the case that the common subgroup has finite index in both constituents.

Recall that an amalgam of groups consists of two groups $G, H$ with a common subgroup $U$. Such an amalgam can be realized in a class $\underline{K}$ of groups if there exists a group $F \in \underline{K}$ and embeddings $g: G \to F$ and $h: H \to F$ which coincide on $U$. In the classes of all groups, of abelian groups, or of finite groups any amalgam can be realized. However, this appears to be a rare property for a class of groups. Thus, a group $U \in \underline{K}$ is called an amalgamation base for $\underline{K}$, if all amalgams with constituents $G, H \in \underline{K}$ and common subgroup $U$ can be realized in $\underline{K}$. For example, the amalgamation bases in the class of finite $p$-groups are the cyclic $p$-groups ([2], [3]) and the amalgamation bases in the class of torsion-free nilpotent groups are the subgroups of the additive rationals ([6], Satz 2). See [5], [9] for results in the class of nilpotent groups of class two.

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B.H. Neumann showed that the finite groups are amalgamation bases in the class $\underline{LF}$ of locally finite groups ([7, Theorem 5.2]), but that the countably infinite elementary abelian $2$-group is not an amalgamation base ([7, Section 2]). By the same argument it follows that no infinite elementary abelian $p$-group, $p$ a prime, is an amalgamation base in $\underline{LF}$. In the first part we show that there are $2^{\aleph_0}$ abelian groups of the form $\bigoplus_{i\in\mathbb{N}} C_{p_i}$, $p_i$ a sequence of primes, which are not amalgamation bases in $\underline{LF}$. In contrast to the fact that existentially closed groups in a first-order axiomatizable class are amalgamation bases of the class, Grossberg and Shelah [1] showed that P. Hall's countable universal locally finite group is not an amalgamation base in $\underline{LF}$. In view of these results we ask the following

QUESTION: Are the finite groups the amalgamation bases in $\underline{LF}$? The next interesting test case would be a Prüfer group $C_{p^\infty}$.

In the second part we refine the idea behind the example in the first part to get conditions for the realizability of an amalgam in $\underline{LF}$ if the common subgroup $U$ has countable index in $G$ and $H$ and also locally finite index in $G$ and $H$. By the latter we mean that $|\langle A,U\rangle : U| < \infty$ for any finite subgroup $A$ in $G$ or in $H$. In particular, this yields conditions for the case that $U$ has finite index in both $G$ and $H$. The amalgams will be realized by means of permutational products [8].

1. Consider a sequence $p_i$, $i \in \mathbb{N}$ in $\mathbb{N}$ with $p_i' = k_i p_i + 1$, $1 \le k_i \in \mathbb{N}$, and $p_{i+1} = p_i'$ if $p_i'$ is odd, and $p_{i+1} = p_i'/2$ if $p_i'$ is even. By Dirichlet's theorem there exist $2^{\aleph_0}$ such sequences with all $p_i$ prime.

For any such sequence we construct permutation groups $G$ and $H$ with common subgroup $\oplus C_{p_i}$ such that the amalgam cannot be realized in a locally finite group. Thus no such $\oplus C_{p_i}$ is an amalgamation base in $\underline{LF}$. For $i$ even we choose disjoint sets $\{M_i = m_1^i,\dots,m_{p_i'}^i, n_1^i,\dots,n_{p_i'}^i\}$. We

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consider the following permutations of $M = \cup M_i$ .

$\phi = \Pi\{(m^i_j, n^i_j) \mid j = 1,\dots,p'_i, \quad i \text{ even }\}$,

$\psi = \Pi\{(m^i_{kp_i}, m^i_{kp_i+1}) \mid k = 1,\dots,k_i, \quad i \text{ even}\}$,

$a_i = \Pi\{(m^i_{kp_i+1}, m^i_{kp_i+2},\dots,m^i_{(k+1)p_i}) \mid k = 0,\dots,k_i - 1\} \quad i \text{ even}$,

$a_{i+1} = \phi(a_i\psi)^2\phi$ , $i$ even.

It is easy to see that for $i$ even $a_{i+1}$ acts non-trivially only on $\{n^i_1,\dots,n^i_{p'_i}\}$ , and $o(a_{i+1}) = p_{i+1}$ . Furthermore $[a_i,a_j] = 1$ , and $\langle a_i \mid i \in \mathbb{N}\rangle = \oplus\langle a_i\rangle$ with $\langle a_i\rangle \cong C_{p_i}$ . Now consider the group $G = \langle\phi,\psi,a_i \mid i \in \mathbb{N}\rangle$ . We claim that $G \in \underline{LF}$ . To see this it suffices to show that the transitivity systems of any subgroup $G_n = \langle\phi,\psi,a_0,\dots,a_n\rangle$ are bounded ([7, Lemma 5.3]). This holds because $G$ stabilizes each $M_i$, $i$ even, and the transitivity systems in $M_i$, $i > n$ , consist of 2 or 4 elements due to the action of $\phi$ and $\psi$ and the trivial action of $a_0,\dots,a_n$ in these $M_i$ . The crucial property of $G$ is that $a_{2i+1} \in \langle\phi,\psi,a_{2i}\rangle$, $i \in \mathbb{N}$ . Similarly we can construct a group $H = \langle\phi',\psi',b_i \mid i \in \mathbb{N}\rangle \in \underline{LF}$ such that $\langle b_i \mid i \in \mathbb{N}\rangle = \oplus\langle b_i\rangle$ , $\langle b_i\rangle \cong C_{p_i}$ and $b_{2i+2} \in \langle\phi',\psi',b_{2i+1}\rangle$, $i \in \mathbb{N}$ . Thus we have $\oplus C_{p_i}$ as a common subgroup in $G$ and $H$ . However any realization $F$ of this amalgam is not locally finite. For the subgroup $F_0 = \langle\phi,\psi,\phi',\psi',a_0\rangle$ contains the infinite group $\oplus C_{p_i}$ , because $a_{2i} \in F_0$ implies

$$b_{2i+1} = a_{2i+1} \in \langle\phi,\psi,a_{2i}\rangle \le F_0 \text{ , and}$$

$$a_{2i+2} = b_{2i+2} \in \langle\phi',\psi',b_{2i+1}\rangle \le F_0, \quad i \in \mathbb{N} . \qquad \square$$

2. Suppose $U \le G \in \underline{LF}$ . We say that $U$ has locally finite index in $G$ , if $|\langle A,U\rangle : U| < \infty$ for all finite subgroups $A$ of $G$ . Here is

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an example of a locally finite group $G = \langle A,U\rangle$ such that $A$ is finite but $U$ is not of finite index in $G$. Let $M_i$ be disjoint sets of $2i$ elements, $i \geq 2$, and choose permutations $a, a_i$ on $M_i$ of order two that generate a dihedral group $D_{4i}$. The group $G = \langle a, a_i \mid i \geq 2\rangle$ permutes $M = \cup M_i$ and is locally finite as the transitivity systems of the subgroups $\langle a, a_2, \ldots, a_n\rangle$ in the sets $M_i$, $i > n$ consist of two elements according to the action of the involution $a$ and the trivial action of $a_2, \ldots, a_n$. The elementary abelian subgroup $\langle a_i \mid i \geq 2\rangle$ has infinite index as $|\langle a, a_i\rangle : \langle a_i\rangle| = 2i$, $i \geq 2$.

The hypothesis of locally finite index will be used as follows. Suppose subgroups $A_o, U$ of $G \in \underline{LF}$ are given such that $|\langle A_o, U\rangle : U| < \infty$. Then we find a finite subgroup $A \leq G$ such that $\langle A_o, U\rangle = AU = UA$, and we set $U' = A \cap U$. We define an operation of $A$ on the right cosets $U'\backslash U$ by $U'ua = U'u'$ if $ua = a'u'$, $a' \in A, u' \in U$. To show that this is well defined let $v \in U'$ and $vua = a''u''$, $a'' \in A$, $u'' \in U$. Then $va'u' = a''u''$, $a''^{-1}va' = u''u'^{-1} \in A \cap U = U'$, and hence $U'u' = U'u''$. Now suppose $ua = a'u'$ and $u'b = b'u''$ with $u, u', u'' \in U, a, b, a', b' \in A$. Then $(U'ua)b = U'u'b = U'u''$ and also $U'u(ab) = U'u''$ since $u(ab) = a'u'b = a'b'u''$ with $a'b' \in A$. Thus we have an operation of the group $A$ on $U'\backslash U$. The orbits of the finite group $A$ give rise to a partition of $U'\backslash U$ into finite subsets. Assume that an action of another finite group $B$ on $U'\backslash U$ is given. We may form the finest partition of $U'\backslash U$ which is coarser than both the orbit partitions of $A$ and $B$ on $U'\backslash U$. The sets in this new partition are unions of orbits of $A$ and $B$ with nontrivial intersection. We say that the orbits of $A$ and $B$ generate a bounded partition if this new partition consists of finite subsets of bounded order. B.H. Neumann's example [7, Section 2] shows that in general this new partition may contain infinite subsets, and it is also easy to construct an example where the new partition consists of finite subsets only, but the orders of those subsets are not bounded. Since this example illustrates the necessity of the second condition of our main result we include it here. Consider an elementary abelian $2$-group $U = \oplus U_i$, where

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$U_i = \langle c_{io} \rangle \times \ldots \times \langle c_{i,2i} \rangle$ , and define automorphisms $a$ and $b$ of $U$ by $c_{io}a = c_{io}$, $c_{i,2j-1}a = c_{i,2j}$, $c_{i,2j}a = c_{i,2j-1}$, $c_{i,2j-2}b = c_{i,2j-1}$, $c_{i,2j-1}b = c_{i,2j-2}$, $c_{i,2i}b = c_{i,2i}$, $j = 1,\ldots,i$, $i \geq 1$. Let $G,H$ be the semidirect product of $U$ by $A = \langle a \rangle$ and $B = \langle b \rangle$ , respectively. Then $G = AU = UA$, $H = BU = UB$, and $A \cap U = \langle 1 \rangle = B \cap U$ . As above $A$ and $B$ operate on $U = \langle 1 \rangle \backslash U$ , which is clearly the same as the action of $A$ and $B$ as automorphisms on $U$ . Since $A$ and $B$ stabilize the subgroups $U_i$ of $U$ , the new partition of $U$ consists of finite subsets only. But it contains the subsets $\{c_{i,o},\ldots,c_{i,2i}\}$, $i \geq 1$ and thus is not bounded.

THEOREM. *Suppose $U$ is a common subgroup of countable, locally finite index in the locally finite groups $G$ and $H$ . Then the following are equivalent.*

1. *The amalgam $U \leq G,H$ can be realized in a locally finite group.*

2. *a) For all finite subgroups $A_o \leq G$ and $B_o \leq H$ there exist finite subgroups $A_o \leq A \leq G$ and $B_o \leq B \leq H$ such that $A \cap U = B \cap U$, and $AU = UA$, $BU = UB$.*

*b) For all finite subgroups $A \leq G$ and $B \leq H$ with $U' = A \cap U = B \cap U$, $AU = UA$, $BU = UB$ the orbits of the action of $A$ and $B$ on $U'\backslash U$ generate a bounded partition.*

*In the examples in part 1 condition 2.a) is not satisfied, whereas in the example preceding the theorem 2.a) holds but 2.b) is violated.*

Proof. $1 \Rightarrow 2$. Let the amalgam be realized in the locally finite group $F$ . We may assume that $G,H \leq F$ and $F = \langle G,H \rangle$ . We first show that $U$ has locally finite index in $F$ . Let $C_o$ be a finite subgroup of $F$ . Thus $C_o \leq \langle A,B \rangle$ with $A \leq G$, $B \leq H$ finite. We may assume that $AU = UA$ and $BU = UB$ , as $U$ has locally finite index in $G$ and $H$ . This implies that $\langle A,B \rangle U = U\langle A,B \rangle 1$, and hence the index $|\langle C_o,U \rangle : U| \leq |\langle A,B,U \rangle : U| \leq |\langle A,B \rangle|$ is finite as $\langle A,B \rangle \leq F \in \underline{LF}$ .

a) Let $A_o \leq G$, $B_o \leq H$ be finite subgroups. Now $\langle A_o,B_o \rangle \in F$ is finite and there exists a finite group $C$ such that $\langle A_o, B_o\rangle \leq C \leq F$ and $CU = UC$. We set $A = C \cap G$ and $B = C \cap H$. Then $U \leq G,H$ yields $A \cap U = C \cap U = B \cap U$ and $AU = (C \cap G)U = CU \cap G = UC \cap G = U(C \cap G)$ $= UA$ by Dedekind's modular law, and similarly $BU = UB$.

b) Suppose $A \leq G$, $B \leq H$ finite and $U' = A \cap U = B \cap U$, and $AU = UA$, $BU = UB$. Then $C = \langle A,B\rangle \leq F$ is finite and satisfies $CU = UC$. Set $U'' = C \cap U \geq U'$, and we have an action of $C$ on $U''\backslash U$, whose orbits are bounded by $|C|$. Now the sets of the new partition generated by the orbits of $A$ and $B$ on $U'\backslash U$ are bounded by $|C|\cdot|U'' : U'|$, since the projections of any two orbits of $A$ or $B$ in $U'\backslash U$ which intersect nontrivially to $U''\backslash U$ are contained in a single orbit of $C$ and hence the projections of the sets of the new partition of $U'\backslash U$ to $U''\backslash U$ are contained in single orbits of $C$, and the bound for those sets follows.

$2 \Rightarrow 1$. As $U$ has countable, locally finite index in $G$, we find an ascending chain of finite subgroups $A_n$, $n \in \mathbb{N}$, in $G$ such that $G = \cup A_n U$, $A_n U = UA_n$, $A_o = \langle 1\rangle$. We choose a left transversal of $U$ in $G$ as follows. Set $U_n = A_n \cap U$. Then $A_n U_{n+1} = A_n U \cap A_{n+1} = UA_n \cap A_{n+1}$ $= U_{n+1} A_n$, where the first and last equality are instances of Dedekind's modular law. Choose a left transversal $S_n$ of $A_n U_{n+1}$ in $A_{n+1}$, $n \in \mathbb{N}$, with $1 \in S_n$. Then $S_n$ is also a left transversal of $A_n U$ in $A_{n+1} U$, as $A_{n+1} U \cap A_{n+1} = A_{n+1}$ and $A_n U \cap A_{n+1} = A_n U_{n+1}$. Therefore $S_n S_{n-1} \ldots S_1 S_o$ is a left transversal of $U = A_o U$ in $A_{n+1} U$, and hence $S = \cup S_n \ldots S_o$ is a left transversal of $U$ in $G$. Similarly, we have $H = \cup B_n U$ and a left transversal $T = \cup T_n \ldots T_o$ of $U$ in $H$, where $T_n$ is a left transversal of $B_n(B_{n+1} \cap U)$ in $B_{n+1}$. Now let $F$ be the permutational product [8] of $G$ and $H$ with respect to the transversals $S$ and $T$ of $U$ in $G$ and $H$, respectively. $F$ is the subgroup of permutations on $M = S \times T \times U$, generated by the following actions of $G$ and $H$ on $M$. For $g \in G$ we have $(s,t,u)g = (s',t,u')$ if $sug = s'u'$ with $s' \in S$, $u' \in U$, and for $h \in H$ we have $(s,t,u)\, h = (s,t',u')$ if $tuh = t'u'$ with $t' \in T$, $u' \in U$. These two actions coincide on the common subgroup $U$, and therefore $F$ is a realization of the amalgam of $G$ with $H$ over

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$U$. We now show that $F$ is locally finite. It suffices to show that the transitivity systems of any finitely generated subgroup of $F$ on $M$ are finite of bounded order. A finitely generated subgroup of $F$ is contained in a subgroup $\langle G_o, H_o\rangle$ with finite subgroups $G_o \le G$ and $H_o \le H$. By 2.a) there exist finite subgroups $G_o \le A \le G$ and $H_o \le B \le H$ such that $U' = A \cap U = B \cap U$, and $AU = UA$, $BU = UB$. Let $A \le A_n U$ and $B \le B_n U$ then the orbit of $(s,t,u)$ under $\langle A,B\rangle$ is contained in $sA_n \times tB_n \times U''$ where $U''$ is the union of the cosets in the orbit of $U'u$ under the action of $A$ and $B$ on $U'\backslash U$. But the number of cosets in $U'\backslash U$ reached from any $U'u'$, $u' \in U$ is uniformly bounded, by $m$ say, by 2.b) and hence the transitivity systems of $\langle A,B\rangle$ on $M$ are uniformly bounded by $|A_n|\cdot|B_n|\cdot m\cdot|U'|$. □

COROLLARY. *Suppose $U$ is a common subgroup of finite index in the locally finite groups $G$ and $H$. Then the following are equivalent.*

1. *The amalgam of $G$ with $H$ over $U$ can be realized in a locally finite group.*

2. *Suppose $G = AU$, $H = BU$ with finite subgroups $A$ and $B$.*

   a) *For any finite subgroups $A \le A_o \le G$, $B \le B_o \le H$ there exist finite subgroups $A_o \le A_1 \le G$ and $B_o \le B_1 \le H$ such that $A_1 \cap U = B_1 \cap U$.*

   b) *For any finite subgroups $A \le A_1 \le G$, $B \le B_1 \le H$ with $U' = A_1 \cap U = B_1 \cap U$ the orbits of the actions of $A_1$ and $B_1$ on $U'\backslash U$ generate a bounded partition.*

*This follows immediately from the theorem as $A \le A_o \le G$ and $G = AU$ imply $G = A_oU$.*

## References

[1] R. Grossberg, S. Shelah, "On universal locally finite groups", *Israel J. Math.* 44 (1983), 289-302.

https://doi.org/10.1017/S0004972700003750 Published online by Cambridge University Press

[2] G. Higman, "Amalgams of p-groups", *J. Algebra* 1 (1964), 301-305.

[3] F. Leinen, "A uniform way to control chief series in finite p-groups and to construct the countable algebraically closed locally finite p-groups", *J. London Math. Soc.* (2) 33 (1986), 260-270.

[4] F. Leinen, "Concerning amalgamation over $C_{p^\infty}$", (preprint, 1985).

[5] B.J. Maier, "Amalgame nilpotenter Gruppen der Klasse zwei", *Publ. Math. Debrecen*, I 31 (1984), 57-70, II 33 (1986), 43-52.

[6] B.J. Maier, "Amalgame torsionsfreier nilpotenter Gruppen", *J. Algebra* 99 (1986), 520-547.

[7] B.H. Neumann, "Amalgams of periodic groups", *Proc. Roy. Soc. London Ser. A* 255 (1960), 477-489.

[8] B.H. Neumann, "Permutational products of groups", *J. Austral. Math. Soc.* 1 (1960), 299-310.

[9] D. Saracino, "Amalgamation bases for nil-2 groups", *Algebra Universalis* 16 (1983), 47-62.

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