PLANE COLLINEATIONS

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1. If in a complex projective plane a point P, with coordinate vector x, corresponds to a point P^*, with coordinate vector x^*, under a non-singular collineation, then

$$x^* = Ax$$

where A is a non-singular 3×3 matrix, the coordinates and the elements of A being complex numbers. It is well known that there are six types of plane collineation, corresponding to the Segre characteristics of A, which are

$$[111],\ [(11)1],\ [(111)],\ [21],\ [(21)],\ [3];$$

(Todd, *Projective and Analytical Geometry*, p. 168). To obtain the six types in this way, however, requires considerable preliminary algebra, and more time than a usual student course allows.

An alternative method depends on using an algebraic theorem that if A is an $n \times n$ matrix and α is a root of the equation $|A-\rho I| = 0$ with multiplicity p, then $n-1 \geqq r(A-\alpha I) \geqq n-p$, where $r(T)$ is the rank of the matrix T. (Mirsky, *Linear Algebra*, p. 214; Semple and Kneebone, *Algebraic Projective Geometry*, p. 211). United points of the collineation correspond to characteristic roots of A, and from this theorem when $n = 3$ we are led to expect, either 3 distinct roots α, β, γ giving 3 isolated united points; or roots α, α, β, giving either an isolated united point and a line of united points, or 2 united points; or roots α, α, α, giving either all the points of the plane united, or a line of united points, or an isolated united point.

Unfortunately this method cannot be relied on to give all the possible cases, for when $n = 4$ it confuses two. The collineations

$$x^* = \begin{pmatrix} \lambda & 1 & . & . \\ . & \lambda & . & . \\ . & . & \lambda & 1 \\ . & . & . & \lambda \end{pmatrix} x \quad \text{and} \quad x^* = \begin{pmatrix} \lambda & 1 & . & . \\ . & \lambda & 1 & . \\ . & . & \lambda & . \\ . & . & . & \lambda \end{pmatrix} x$$

are not distinguishable from the point of view of the preceding algebra, for although in each case the characteristic roots are λ, four times, and $r(A-\lambda I) = 2$, the collineations are geometrically different; the first leaves fixed all the points on a line, XZ, and all the planes through the same line, while the second leaves fixed all the points on XT and all the planes through a different line XY. Their Segre characteristics are $[(22)]$ and $[(31)]$.

Once this failure of the method has been noticed, one feels less confident when expounding it in the case $n = 3$. We give an alternative treatment which is essentially elementary, and which provides some geometrical insight into what is happening.

2. We consider the collineation $x^* = Ax$, where $|A| \neq 0$ and

$$A = \begin{pmatrix} \lambda & a_2 & a_3 \\ b_1 & \mu & b_3 \\ c_1 & c_2 & \nu \end{pmatrix}.$$

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If u is the coordinate vector of a line whose equation is $u'x = 0$, the line transforms to a line u^* such that $u = A'u^*$, where A' is the transpose of A. (Strictly, we should express this as $u^* = (A')^{-1}u$, but as we are simply concerned to find united points and lines, distinctions between the directions of the transformations need not concern us.) United points occur when $x^* = \rho x$, and so correspond to roots of

$$| A-\rho I | = 0;$$

dually, united lines correspond to roots of

$$| A'-\rho I | = 0.$$

Since A and A' have the same characteristic equation, there are thus the same number of united points and united lines in the collineation; if there is a line of united points there is also a pencil of united lines; and in any collineation there is at least one united point and one united line.

We use repeatedly the basic property of one-dimensional homographies, that either every element is united, or there are two united elements, or just one.

3. The collineation has at least one united point; take this as $X(1, 0, 0)$, so that $b_1 = c_1 = 0$, $\lambda \neq 0$. Consider the pencil of lines through X; three cases may arise.

(A) *Suppose that all the lines through X are united.*

Then the matrices A, A' are

$$\begin{pmatrix} \lambda & a_2 & a_3 \\ 0 & \mu & b_3 \\ 0 & c_2 & \nu \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \lambda & 0 & 0 \\ a_2 & \mu & c_2 \\ a_3 & b_3 & \nu \end{pmatrix} \quad (\lambda \neq 0).$$

The lines $y = 0$, $z = 0$, $y+z = 0$ are all united, so $b_3 = c_2 = 0$, $\mu = \nu \neq 0$. Since there is a pencil of united lines, there is also a line of united points.

I. If this line of united points does not pass through X, take it as YZ, where Y is $(0, 1, 0)$, Z is $(0, 0, 1)$. Then Y, Z and $(0, 1, 1)$ are all united, so $a_2 = a_3 = 0$, and the canonical form of the matrix is

$$\begin{pmatrix} \alpha & . & . \\ . & \beta & . \\ . & . & \beta \end{pmatrix} \quad (\alpha\beta \neq 0).$$

This collineation is the plane homology, or plane perspective.

II. If the line of united points passes through X, take it as XZ. Then Z, and $(1, 0, 1)$, are united, so $a_3 = 0$, $\lambda = \mu$. The matrix reduces to

$$\begin{pmatrix} \alpha & 1 & . \\ . & \alpha & . \\ . & . & \alpha \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \alpha & . & . \\ . & \alpha & . \\ . & . & \alpha \end{pmatrix} \quad (\alpha \neq 0)$$

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depending on whether $a_2 \neq 0$ or $a_2 = 0$. In the first case the collineation is a special homology, or elation; in the second case it is the identity.

We have now dealt with all the cases in which there are a line of united points and a pencil of united lines.

(B) *Suppose that just two of the lines through X are united.*

Take these as XY and XZ. Then $y = 0$ and $z = 0$ are united lines, and $y+z = 0$ is not united, so $b_3 = c_2 = 0$, $\mu \neq \nu \neq 0$. If on either of the lines all the points are united, these collineations have already been discussed. Two further cases therefore arise.

I. If each line contains one united point distinct from X, take these as Y and Z. Then $a_2 = a_3 = 0$, and λ, μ, ν are all distinct. We thus have the matrix of the general collineation in the canonical form

$$\begin{pmatrix} \alpha & . & . \\ . & \beta & . \\ . & . & \gamma \end{pmatrix} \quad (\alpha\beta\gamma \neq 0).$$

II. If one line, say XY, has X as the only united point, then, for all θ, $(\theta, 1, 0)$ transforms to $(\theta+k, 1, 0)$, with $k \neq 0$, and so $\lambda = \mu$, $a_2 \neq 0$. There are at least two united lines, so there is at least one further united point, T say, and XT is a united line, so that there is a second united point on the other united line through X. Take this as Z. Then $a_3 = 0$ and dividing by a_2 we obtain for the matrix the canonical form

$$\begin{pmatrix} \alpha & 1 & . \\ . & \alpha & . \\ . & . & \beta \end{pmatrix} \quad (\alpha\beta \neq 0).$$

(C) *Suppose that only one line through X is united.*

We have discussed the cases when through a united point there is more than one united line, and dually when on a line there is more than one united point. It follows that the only remaining case is the one where there is just one united point X and just one united line, XY say, through it. Then for all θ, $(\theta, 1, 0)$ transforms to $(\theta+k, 1, 0)$, $(k \neq 0)$, and $y+\theta z = 0$ transforms to $y+(\theta+h)z = 0$, $(h \neq 0)$. As a result, $c_2 = 0$, $\lambda = \mu$, $a_2 = k\lambda \neq 0$, and $\mu = \nu$, $b_3 = -\nu h \neq 0$. Choosing Y, Z as the points which transform to $(1, \alpha, 0)$, $(0, 1, \alpha)$ respectively, we find that the matrix of this last collineation reduces to

$$\begin{pmatrix} \alpha & 1 & . \\ . & \alpha & 1 \\ . & . & \alpha \end{pmatrix} \quad (\alpha \neq 0).$$

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