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TOTALLY DISCONNECTED, NILPOTENT, LOCALLY COMPACT GROUPS

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It is shown that, if G is a totally disconnected, compactly generated and nilpotent locally compact group, then it has a base of neighbourhoods of the identity consisting of compact, open, normal subgroups. An example is given showing that the hypothesis that G be compactly generated is necessary.

It is known that every totally disconnected, locally compact group, G, has a compact, open subgroup, see [**2**, **4**] or [**5**]. The study of such groups would be greatly simplified if there always existed a compact, open, *normal* subgroup in G. However there are many totally disconnected groups for which that is not the case; the first example was found by van Danzig, [**2**]. Work in [**6**] shows that this example is fundamental in the sense that, if G has an element x which does not normalise any compact, open subgroup, then G has a closed subgroup, containing x, which is very much like van Dantzig's example.

If G is nilpotent and compactly generated in addition to being totally disconnected, then it does have a compact, open, normal subgroup, [**3**], Théorème 2, and this fact has proved useful for the study of the representation theory of nilpotent groups, [**1**]. A stronger statement is proved below, where it is shown that such groups have sufficiently many compact, open, normal subgroups to form a base of neighbourhoods of the identity. This answers a question of Alan Carey. Also, an example is given of a totally disconnected, nilpotent (not compactly generated) group which does not have a compact, open normal subgroup. In the example every element, x, normalises some compact, open subgroup but the subgroup depends on x.

The proof of the main theorem requires the following result from [**6**].

Lemma 1. *Let G be a locally compact group, V be a compact, open subgroup of G and x be in G. Then there is an integer, k, such that, putting $U = \bigcap_{0 \leqslant n \leqslant k} x^n V x^{-n}$, we have*

$$U = K_+ K_- = K_- K_+,$$

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where $K_+ = \bigcap_{n\geqslant 0} x^n U x^{-n}$ *and* $K_- = \bigcap_{n\geqslant 0} x^{-n} U x^n$.

This lemma is the first step in the structure theory for totally disconnected groups described in [6]. Note that, if it should happen that $xUx^{-1} = U$, then $K_+ = U = K_-$. In general, $xK_+x^{-1} \supset K_+$ and $x^{-1}K_-x \supset K_-$.

Lemma 2. *Let G be a nilpotent locally compact group and x be in G. Then for each compact, open subgroup V of G there is an open subgroup $U \subset V$ such that $xUx^{-1} = U$.*

Proof: Choose U as in lemma 1. As in lemma 1, let $K_+ = \bigcap_{n\geqslant 0} x^n U x^{-n}$. Then $U = K_+K_-$ and $xK_+x^{-1} \supset K_+$.

Suppose that $xK_+x^{-1} \setminus K_+ \neq \emptyset$ and choose $z_1 \in xK_+x^{-1} \setminus K_+$. Then $x^{-1}z_1x \in K_+$ and so $z_2 = z_1x^{-1}z_1^{-1}x \in xK_+x^{-1} \setminus K_+$. Now define elements z_j inductively by $z_{j+1} = z_jx^{-1}z_j^{-1}x$, $j = 1, 2, 3, \ldots$. Then $z_j \in xK_+x^{-1} \setminus K_+$ for every j. In particular, $z_j \neq 1$ for every j, which contradicts the fact that G is nilpotent. Therefore $xK_+x^{-1} = K_+$.

By a similar argument we also have that $xK_-x^{-1} = K_-$ and so

$$xUx^{-1} = \left(xK_+x^{-1}\right)\left(xK_-x^{-1}\right) = K_+K_- = U.$$

□

We have then that each element x of a totally disconnected, nilpotent group G normalises some compact, open subgroup but that subgroup may depend on x. However, if G is compactly generated, then we can find a subgroup normalised by all elements of G.

Theorem. *Let G be a compactly generated, totally disconnected, nilpotent group and let V be a neighbourhood of 1 in G. Then V contains a compact, open, normal subgroup of G. In other words, G has a base of neighbourhoods of 1 consisting of compact, open, normal subgroups.*

Proof: Let

$$G = G^0 \supset G^1 \supset \cdots \supset G^n \supset G^{n+1} = \{1\}$$

be the lower central series of G. Let U_n be a compact, open subgroup of G contained in V. Since $G^n \subset Z(G)$, we have

$$xU_nx^{-1} = U_n, \qquad (x \in U_n \cup G^n).$$

We shall construct compact, open subgroups U_i, $i = n, n-1, \ldots, 1, 0$, such that $U_{i-1} \subset U_i$ and

$$xU_ix^{-1} = U_i, \qquad \left(x \in U_n \cup G^i\right). \tag{1}$$

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Then U_0 will be the desired compact, open subgroup of G contained in V.

Now suppose that U_i satisfying (1) has been found for some $i \geqslant 1$. Since G is compactly generated, G^{i-1} is compactly generated, see [**3**], and so we may choose z_1, $z_2, \ldots, z_p$ in G^{i-1} such that $\{U_n z_j\}_{j=1}^{p}$ covers a generating subset of G^{i-1}. In the next paragraph a recursive construction is given which yields compact, open subgroups $W_0 = U_i$, W_1, $W_2, \ldots W_p$ such that for $j = 1, \ldots, p$,

$$(2) \qquad xW_jx^{-1} = W_j, \qquad \left(x \in U_n \cup G^i \cup \{z_1, \ldots, z_j\}\right).$$

Then, since W_p is normalised by every x in U_n and by $z_1, z_2, \ldots, z_p$ and since $\{U_n z_j\}_{j=1}^{p}$ covers a generating subset of G^{i-1}, W_p satisfies

$$xW_px^{-1} = W_p, \qquad \left(x \in U_n \cup G^{i-1}\right).$$

Hence we may take $U_{i-1} = W_p$.

We now give the construction of the W_j's. Suppose that, for some j, W_{j-1} satisfying (2) has been constructed. Since z_j belongs to G^{i-1}, we have, for every x in G and every k, that $xz_j^k = z_j^k xw$, where w is in G^i. Hence

$$xz_j^kW_{j-1}z_j^{-k}x^{-1} = z_j^kxwW_{j-1}w^{-1}x^{-1}z_j^{-k} = z_j^kxW_{j-1}x^{-1}z_j^{-k},$$

because w belongs to G^i, and this last is equal to $z_j^kW_{j-1}z_j^{-k}$ for every x in $U_n \cup G^i \cup \{z_1, \ldots, z_{j-1}\}$. Hence for every m we have

$$x\left(\bigcap_{k=0}^{m} z_j^kW_{j-1}z_j^{-k}\right)x^{-1} = \bigcap_{k=0}^{m} z_j^kW_{j-1}z_j^{-k}$$

for every x in $U_n \cup G^i \cup \{z_1, \ldots, z_{j-1}\}$. By the argument of lemma 2 there is an $m \geqslant 0$ such that

$$z_j\left(\bigcap_{k=0}^{m} z_j^kW_{j-1}z_j^{-k}\right)z_j^{-1} = \bigcap_{k=0}^{m} z_j^kW_{j-1}z_j^{-k}$$

as well. Therefore there is a group $W_j = \bigcap_{k=0}^{m} z_j^kW_{j-1}z_j^{-k}$ satisfying (2). □

We now give an example of a nilpotent, totally disconnected, locally compact group which does not have a compact, open, normal subgroup. This example shows that the hypothesis that G should be compactly generated is necessary.

Let $F = \langle a, b, h : a^2 = 1 = b^2;\ ab = ba;\ h^2 = 1;\ \text{and}\ ha = bh\rangle$. Then F is an eight element group such that $[F, F] = \{1, ab\} \equiv Z$, which is the centre of F. Hence F is a nilpotent group.

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Now let G be the set of all functions $g : \mathbf{N} \to F$ such that $g(n) \in \{1, a\}$ for all but finitely many values of n. Then G becomes a nilpotent group when equipped with the coordinatewise product and the sets

$$U(n, g) \equiv \{f \in G : f(m) = g(m),\ m \leqslant n;\ f(m) \in \{1, a\},\ m > n\}$$

form a base of a locally compact, totally disconnected topology on G.

Let H be an open, normal subgroup of G. Then there is an integer n such that H contains the open subgroup $U_n \equiv \{g \in G : g(m) = 1,\ m \leqslant n;\ g(m) \in \{1, a\},\ m > n\}$. Since H is normal, it follows that for each $m > n$ H contains a function g such that $g(m) = b$. Hence H cannot be compact and so G contains no compact, open, normal subgroup. ∎

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