REAL TOPOLOGICAL HOCHSCHILD HOMOLOGY OF SCHEMES

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Abstract We prove that real topological Hochschild homology THR for schemes with involution satisfies base change and descent for the $\mathbb{Z}/2$-isovariant étale topology. As an application, we provide computations for the projective line (with and without involution) and the higher-dimensional projective spaces.

1. Introduction

Hochschild and cyclic homology and their refinements THH and TC have been extensively studied for many decades, both for their own sake and for their deep connections with algebraic K-theory via traces; see, for example, [11]. Standard textbook references include [40] and [19]. More recently, Bhatt–Morrow–Scholze [7] have introduced a filtration on THH and TC that is strongly related to integral p-adic Hodge theory. Hahn–Raksit–Wilson [25] have provided an alternative construction of the filtration that applies to commutative ring spectra satisfying certain assumptions.

Both topological and algebraic K-theory have real or Hermitian refinements. This is also true for THH and TC, but the serious study of their real variants THR and TCR has just started; see, for example, [17], [16] and [48]. These theories apply to very different branches in algebra and geometry: commutative rings, group rings, schemes, ring spectra, . . . all with or without a nontrivial involution. Concerning the $\mathbb{Z}/2$-equivariant cyclotomic trace, we

understand that there is work in progress by Harpaz, Nikolaus and Shah studying this map in the very general setting of stable Poincaré ∞-categories. (When studying traces, beware of the difference between rings, which is what most algebraic K-theorists and algebraic geometers look at, and ring spectra, which are the input of THH and TC.)

The current article contributes to a better understanding of THR in algebraic geometry, although some of our results also apply to other settings. Our first main result is the base change result in Theorem 3.2.3 for the isovariant étale topology. This then is one of the main ingredients in the proof of the following isovariant étale descent theorem for THR.

Theorem 3.4.3. *The presheaf*

$$\mathrm{THR} \in \mathcal{P}\mathrm{sh}(\mathbf{Aff}_{\mathbb{Z}/2}, \mathrm{Sp}_{\mathbb{Z}/2})$$

satisfies isovariant étale descent, where $\mathbf{Aff}_{\mathbb{Z}/2}$ *denotes the category of affine schemes with involutions, and* $\mathrm{Sp}_{\mathbb{Z}/2}$ *denotes the* ∞-*category of* $\mathbb{Z}/2$-*spectra.*

In the nonequivariant case, similar results have been established for THH and TC by Weibel–Geller [60] and Geisser–Hesselholt [21].

This descent theorem implies in particular that THR satisfies the equivariant Zariski–Mayer–Vietoris property for affine schemes with involutions. This can be used to extend THR to nonaffine schemes with involutions; see Definition 3.4.6, in a way that is compatible with existing definitions of THH, see Proposition 3.4.7.

Using this Zariski–Mayer–Vietoris theorem and various explicit computations of THR of products of monoid rings and maps between them, we are able to compute $\mathrm{THR}(X)$ for $X = \mathbb{P}^1, \mathbb{P}^\sigma$ and more generally $\mathbb{P}^n$ with trivial involution; see Theorems 5.1.2, 5.1.4 and 5.2.6. Here. $\mathbb{P}^\sigma$ denotes the projective line with the involution switching the homogeneous coordinates. We recall that Blumberg and Mandell [10] compute $\mathrm{THH}(\mathbb{P}^n)$. For $\mathbb{P}^n$ with the trivial involution, we obtain the following.

Theorem 5.2.6. *For any separated scheme with involution X and integer $n \geq 0$, there is an equivalence of $\mathbb{Z}/2$-spectra*

$$\mathrm{THR}(X \times \mathbb{P}^n) \simeq \begin{cases} \mathrm{THR}(X) \oplus \bigoplus_{j=1}^{\lfloor n/2 \rfloor} i_* \mathrm{THH}(X) & \text{if } n \text{ is even,} \\ \mathrm{THR}(X) \oplus \bigoplus_{j=1}^{\lfloor n/2 \rfloor} i_* \mathrm{THH}(X) \oplus \Sigma^{n(\sigma-1)} \mathrm{THR}(X) & \text{if } n \text{ is odd.} \end{cases}$$

For a $\mathbb{Z}/2$-spectra E, $i_* E := E \oplus E$ with the obvious involution, see equation (A.15). We refer to Remarks 2.0.1, 5.1.1 and 5.2.7 for a comparison with Hermitian and real algebraic K-theory.

Recall that, unlike algebraic and Hermitian K-theory, THH and THR are not $\mathbb{A}^1$-invariant even on regular schemes. On the other hand, THH and TC do extend to log schemes, and using descent, trivial $\mathbb{P}^1$-bundle formula and computations on the log schemes $(\mathbb{P}^n, \mathbb{P}^{n-1})$, can be shown to be representable in the log-variant of the $\mathbb{P}^1$-stable motivic Morel–Voevodsky homotopy category. This is done in the very recent joint work [8] of the second author with Federico Binda and Paul Arne Østvær. We refer to [9] for an overview of their work. The results of this article are expected to provide most of the necessary ingredients for showing stable representability of THR or at least its fixed points THO in the corresponding equivariant log-homotopy categories. Furthermore, the work

of Quigley–Shah [48] allows us to extend many results of this article including Theorems 3.4.3 and 5.2.6 to TCR. The second author hopes to carry out motivic representability of THR and TCR in forthcoming work.

We conclude with a short overview of the different sections. Section 2 discusses some generalities about commutative ring spectra with involutions and their THR. This includes the study of several functors between $\mathbb{Z}/2$-equivariant and nonequivariant categories, including the norm and equivariant notion of flatness. The necessary background on G-stable equivariant homotopy theory for finite groups G, both ∞- and model categorical, is provided in Appendix A. Section 3 reviews and extends several definitions for Grothendieck topologies on schemes with involutions, notably Thomason's isovariant étale topology. Base change and descent for THR are established in subsections 3.2 and 3.4. Note that the proof of étale base change in subsection 3.2 heavily relies on various results about Green functors established in subsections Appendix A.4 and Appendix A.5. Section 4 recollects and extends material on real and dihedral nerves, which is crucial when computing THR of spherical monoid rings and of monoid rings over Eilenberg MacLane spectra, and where the monoids may have involutions. These monoid ring computations together with isovariant Zariski descent are then used in the computations for projective spaces in section 5.

2. Real topological Hochschild homology of rings with involutions

Remark 2.0.1. THR is to real algebraic K-theory **KR** what THH is to algebraic K-theory. Hence, we recall some recent results on *real algebraic* K-*theory* **KR**. This is a $\mathbb{Z}/2$-equivariant motivic spectrum constructed in [34] and [12]. We recover Hermitian K-theory when restricting **KR** to schemes with trivial $\mathbb{Z}/2$-action. (This is one incarnation of the philosophy 'the fixed points of **KR** are Hermitian K-theory' in the world of presheaves on $\mathbb{Z}/2$-schemes). Forgetting the action, we recover Voevodsky's algebraic K-theory spectrum KGL. As observed in [61, section 7.3], see also [12], Schlichting's techniques generalize to show that **KR** is representable in $\mathcal{SH}^{\mathbb{Z}/2}(k)$. In particular, **KR** satisfies equivariant Nisnevich descent. Here and below, following [12] and [34], we consider **KR** as a motivic spectrum with respect to the circle $T^\rho \simeq \mathbb{P}^1 \wedge \mathbb{P}^\sigma$, where ρ is the regular representation of $\mathbb{Z}/2$, and $\mathbb{P}^\sigma$ denotes $\mathbb{P}^1$ with involution switching the homogeneous coordinates. It might be more consistent with the notations for THH to denote the (motivic) Hermitian K-theory spectrum on schemes with involution by **KO** and to reserve the notation **KR** for a (motivic) spectrum with involution whose fixed points are **KO**. For further results on **KR** and projective spaces, we refer to Remarks 5.1.1 and 5.2.7 below.

Throughout this section, we fix morphisms of finite groupoids

$$\mathrm{pt} \xrightarrow{i} \mathrm{B}(\mathbb{Z}/2) \xrightarrow{p} \mathrm{pt}, \tag{2.1}$$

where BG denotes the finite groupoid consisting of a single set $*$ with $\mathrm{Hom}_{\mathrm{B}G}(*,*) := G$. According to section Appendix A.2, we often use the alternative notation

$$N^{\mathbb{Z}/2} = i_\otimes,\ \Phi^{\mathbb{Z}/2} = p_\otimes,\ \iota = p^*,\ \text{and } (-)^G = p_*$$

instead of the notation in section Appendix A.1. In particular, all functors in the sequel are ∞-functors, which admit lifts to Quillen functors between model categories.

We then have adjoint pairs

$$i^* : \mathrm{Sp}_{\mathbb{Z}/2} \rightleftarrows \mathrm{Sp} : i_* \text{ and } \iota : \mathrm{Sp} \rightleftarrows \mathrm{Sp}_{\mathbb{Z}/2} : (-)^G \tag{2.2}$$

and functors

$$N^{\mathbb{Z}/2} : \mathrm{Sp} \to \mathrm{Sp}_{\mathbb{Z}/2} \text{ and } \Phi^{\mathbb{Z}/2} : \mathrm{Sp}_{\mathbb{Z}/2} \to \mathrm{Sp}.$$

By [53, Theorem 7.12], the pair of functors $(i^*, \Phi^{\mathbb{Z}/2})$ is conservative. We will use this fact frequently. We also have adjoint pairs

$$N^{\mathbb{Z}/2} : \mathrm{CAlg} \rightleftarrows \mathrm{NAlg}_{\mathbb{Z}/2} : i^*, \ i^* : \mathrm{NAlg}_{\mathbb{Z}/2} \rightleftarrows \mathrm{CAlg} : i_*,$$
$$\text{and } \iota : \mathrm{CAlg} \rightleftarrows \mathrm{CAlg}_{\mathbb{Z}/2} : (-)^{\mathbb{Z}/2}$$

and a colimit preserving functor

$$\Phi^{\mathbb{Z}/2} : \mathrm{CAlg}_{\mathbb{Z}/2} \to \mathrm{CAlg}.$$

We refer sections Appendix A.1 and Appendix A.2 for further properties of all these functors.

We now define real topological Hochschild homology for commutative ring spectra with involution.

2.1. Definition of THR

Definition 2.1.1. Suppose $A \in \mathrm{NAlg}_{\mathbb{Z}/2}$. For abbreviation, we set

$$A^{\wedge \mathbb{Z}/2} := N^{\mathbb{Z}/2} i^* A.$$

Since $N^{\mathbb{Z}/2}$ is left adjoint to i^*, we have the counit map $A^{\wedge \mathbb{Z}/2} \to A$. We use this map to define

$$\mathrm{THR}(A) := A \wedge_{A^{\wedge \mathbb{Z}/2}} A \in \mathrm{NAlg}_{\mathbb{Z}/2},$$

which is called the *real topological Hochschild homology of A*. The first $\wedge$ in the formulation of THR is the pushout in $\mathrm{NAlg}_{\mathbb{Z}/2}$. Under a certain flatness condition, this is equivalent to the Bökstedt model of the real topological Hochschild homology; see [16, Theorem, p. 65]. Note also that it is possible to define THR for $\mathbb{Z}/2$-spectra with slightly less structure, for example, for $\mathbb{E}_\sigma$-algebras as in [1].

The map to the second smash factor in $A \wedge_{A^{\wedge \mathbb{Z}/2}} A$ gives a canonical map

$$A \to \mathrm{THR}(A). \tag{2.3}$$

In analogy with Hermitian K-theory **KO** and real algebraic K-theory **KR**, we define

$$\mathrm{THO}(A) = (\mathrm{THR}(A))^{\mathbb{Z}/2} \simeq (A \wedge_{N^{\mathbb{Z}/2} i^* A} A)^{\mathbb{Z}/2} \in \mathrm{CAlg}$$

for $A \in \mathrm{NAlg}_{\mathbb{Z}/2}$. For $B \in \mathrm{NAlg} = \mathrm{CAlg}$, recall that the *topological Hochschild homology of B* is defined to be

$$\mathrm{THH}(B) := B \wedge_{B \wedge B} B \in \mathrm{CAlg},$$

which in turn generalizes the classical [40, Proposition 1.1.13] from rings to ring spectra.

Definition 2.1.2. Let $\mathcal{C}$ be a category. An object X of $\mathrm{Fun}(\mathrm{B}(\mathbb{Z}/2),\mathcal{C})$ is called an *object of $\mathcal{C}$ with involution*. Explicitly, X is an object of $\mathcal{C}$ equipped with an automorphism $w\colon X \to X$ such that $w \circ w = \mathrm{id}$.

In particular, we have the notions of commutative rings with involutions, commutative monoids with involutions and so on.

We do not discuss definitions of $\mathrm{HR}(A)$ refining $\mathrm{HH}(A)$ and comparison results between $\mathrm{THR}(A)$ and $\mathrm{HR}(A)$, similarly to, for example, [46, Proposition IV.4.2], but see Remark 4.2.3 below. The adjoint functors $\underline{\pi}_0$ and H, which are studied in the appendix, preserve many of the adjunctions we study; see, for example, Definition 2.3.2.

Proposition 2.1.3. *For $A \in \mathrm{NAlg}_{\mathbb{Z}/2}$ and $B \in \mathrm{CAlg}$, there exist canonical equivalences*

$$\mathrm{THH}(i^*A) \simeq i^*\mathrm{THR}(A) \text{ and } \mathrm{THR}(N^{\mathbb{Z}/2}B) \simeq N^{\mathbb{Z}/2}\mathrm{THH}(B).$$

Hence, there exists a canonical equivalence

$$\mathrm{THR}(A^{\wedge \mathbb{Z}/2}) \simeq \mathrm{THR}(A)^{\wedge \mathbb{Z}/2}.$$

In particular, the first equivalence implies $i^*\mathrm{THR}(\iota B) \simeq \mathrm{THH}(B)$, using (2) of Proposition Appendix A.2.7.

Proof. Since $N^{\mathbb{Z}/2}$ and i^* preserve colimits, we have equivalences

$$\mathrm{THH}(i^*A) \simeq i^*A \wedge_{i^*A \wedge i^*A} i^*A \simeq i^*A \wedge_{i^* N^{\mathbb{Z}/2} i^*A} i^*A \simeq i^*\mathrm{THR}(A)$$

and

$$\mathrm{THR}(N^{\mathbb{Z}/2}B) \simeq N^{\mathbb{Z}/2}B \wedge_{N^{\mathbb{Z}/2} i^* N^{\mathbb{Z}/2}B} N^{\mathbb{Z}/2}B \simeq N^{\mathbb{Z}/2}(B \wedge_{B \wedge B} B) \simeq N^{\mathbb{Z}/2}\mathrm{THH}(B)$$

by Proposition Appendix A.2.7(6). $\qquad\square$

Proposition 2.1.4. *For $B \in \mathrm{CAlg}$, there exists a canonical equivalence*

$$\mathrm{THR}(i_*B) \xrightarrow{\simeq} i_*\mathrm{THH}(B). \tag{2.4}$$

Proof. The composite

$$\mathrm{THR}(i_*B) \to i_*i^*\mathrm{THR}(i_*B) \xrightarrow{\simeq} i_*\mathrm{THH}(i^*i_*B) \to i_*\mathrm{THH}(B) \tag{2.5}$$

defines equation (2.4), where the first (resp. third) map is induced by the unit (resp. counit). Proposition 2.1.3 shows that the induced map

$$i^*\mathrm{THR}(i_*B) \to i^*i_*\mathrm{THH}(B)$$

is an equivalence. By [53, Theorem 7.12] (note that $i^* = \Phi^e$), it remains to show that the induced map

$$\Phi^{\mathbb{Z}/2}\mathrm{THR}(i_*B) \to \Phi^{\mathbb{Z}/2}i_*\mathrm{THH}(B)$$

is an equivalence. The right-hand side is equivalent to 0 by Proposition Appendix A.2.7(3),(5). On the other hand, we have equivalences

$$\Phi^{\mathbb{Z}/2}\mathrm{THR}(i_*B) \simeq \Phi^{\mathbb{Z}/2}(i_*B) \wedge_{i^*i_*B} \Phi^{\mathbb{Z}/2}(i_*B) \simeq 0 \wedge_{i^*i_*B} 0,$$

which is equivalent to 0 too. □

Applying i^* and Proposition Appendix A.2.7(2), we obtain $\mathrm{THO}(i_*B) \simeq \mathrm{THH}(B)$, which compares nicely with the well-known $\mathbf{KO}(X \amalg X) \simeq K(X)$ for schemes X, where the involution on $X \amalg X$ switches the components.

Proposition 2.1.5. *Let $R \to A, B$ be maps in $\mathrm{NAlg}_{\mathbb{Z}/2}$. Then there exists a canonical equivalence*

$$\mathrm{THR}(A) \wedge_{\mathrm{THR}(R)} \mathrm{THR}(B) \simeq \mathrm{THR}(A \wedge_R B).$$

Proof. Both $N^{\mathbb{Z}/2}$ and i^* preserve colimits. Hence, we obtain a canonical equivalence

$$A^{\wedge \mathbb{Z}/2} \wedge_{R^{\wedge \mathbb{Z}/2}} B^{\wedge \mathbb{Z}/2} \simeq (A \wedge_R B)^{\wedge \mathbb{Z}/2}.$$

On the other hand, there are canonical equivalences

$$\mathrm{THR}(A) \wedge_{\mathrm{THR}(R)} \mathrm{THR}(B) \simeq (A \wedge_{A^{\wedge \mathbb{Z}/2}} A) \wedge_{R \wedge_{R^{\wedge \mathbb{Z}/2}} R} (B \wedge_{B^{\wedge \mathbb{Z}/2}} B)$$

$$\simeq (A \wedge_R B) \wedge_{A^{\wedge \mathbb{Z}/2} \wedge_{R^{\wedge \mathbb{Z}/2}} B^{\wedge \mathbb{Z}/2}} (A \wedge_R B).$$

Combine the two equations to obtain the desired equivalence. □

2.2. Mackey functors for $\mathbb{Z}/2$

We refer to the appendix for a general discussion of Mackey and Green functors for finite groups G. We now restrict to the case $G = \mathbb{Z}/2$.

Example 2.2.1. According to [53, Example 4.38], a Mackey functor C for $G := \mathbb{Z}/2$ can be described as a diagram

$$C(G/e) \underset{\mathrm{res}}{\overset{\mathrm{tran}}{\rightleftarrows}} C(G/G),$$
$$\circlearrowleft_w$$

where $C(G/e)$ is an abelian group with an involution w, $C(G/G)$ is an abelian group with the trivial involution, res and tran are homomorphisms of abelian groups with involutions, and the equality

$$\mathrm{res} \circ \mathrm{tran} = \mathrm{id} + w$$

is satisfied (i.e. the *double coset formula* holds). A *morphism of Mackey functors* $C \to D$ is a diagram of abelian groups with involutions

$$
\begin{array}{ccc}
C(G/G) & \longrightarrow & D(G/G) \\
{\scriptstyle\text{res}}\big\downarrow\big\uparrow{\scriptstyle\text{tran}} & & {\scriptstyle\text{res}}\big\downarrow\big\uparrow{\scriptstyle\text{tran}} \\
\circlearrowright C(G/e) & \longrightarrow & D(G/e) \circlearrowleft
\end{array}
$$

such that the horizontal homomorphisms commute with tran and res.

Example 2.2.2. If M is an abelian group with involution, then we can associate the Mackey functor

$$
M \underset{\text{res}}{\overset{\text{tran}}{\rightleftarrows}} M^{\mathbb{Z}/2},
$$

where tran maps $x \in M$ to $x + w(x)$, and res is the inclusion. In this way, we obtain a fully faithful functor from the category of abelian groups with involutions to the category of Mackey functors $\mathrm{Mack}_{\mathbb{Z}/2}$. We often regard an abelian group with involution as a Mackey functor if no confusion seems likely to arise.

Lemma 2.2.3. *Let A be a Green functor for $G := \mathbb{Z}/2$, and let M and L are A-modules. If L is associated (in the sense of the previous example) with an abelian group with involution, then the induced map*

$$
\mathrm{Hom}_{\mathrm{Mod}_A}(M,L) \to \mathrm{Hom}_{\mathrm{Mod}_{A(G/e)}}(M(G/e),L(G/e)) \tag{2.6}
$$

is an isomorphism.

Proof. Let $f\colon M(G/e) \to L(G/e)$ be a homomorphism of $A(G/e)$-modules. Then the image of $f \circ \mathrm{res}$ is in $L(G/e)^{\mathbb{Z}/2}$. Hence, there exists a unique homomorphism of $A(G/G)$-modules $g\colon M(G/G) \to L(G/G)$ such that in the diagram

$$
\begin{array}{ccc}
M(G/G) & \overset{g}{\longrightarrow} & L(G/G) \\
{\scriptstyle\text{res}}\big\downarrow\big\uparrow{\scriptstyle\text{tran}} & & {\scriptstyle\text{res}}\big\downarrow\big\uparrow{\scriptstyle\text{tran}} \\
\circlearrowright M(G/e) & \overset{f}{\longrightarrow} & L(G/e) \circlearrowleft
\end{array}
$$

the pair (f,g) commutes with res. We have

$$
\mathrm{res} \circ g \circ \mathrm{tran} = f \circ \mathrm{res} \circ \mathrm{tran} = f \circ (\mathrm{id} + w) = (\mathrm{id} + w) \circ f = \mathrm{res} \circ \mathrm{tran} \circ f.
$$

Since res for L is injective, we deduce that the pair (f,g) commutes with tran. This constructs an inverse of equation (2.6). □

2.3. Equivariant Eilenberg–MacLane spectra

In this subsection, we explain basic properties of equivariant Eilenberg–MacLane spectra. We also explain how to define THR of commutative rings.

Definition 2.3.1. Let $\mathcal{C}$ be a category (not an ∞-category). We have the functors

$$\mathcal{C} \xrightarrow{\iota} \mathrm{Fun}(\mathrm{B}(\mathbb{Z}/2),\mathcal{C}) \xrightarrow{i^*} \mathcal{C}$$

induced by (2.1). Let $(-)^{\mathbb{Z}/2}$ denote the right adjoint of ι if it exists.

Here, we give some examples. For a commutative ring A, ιA is the commutative ring A with the trivial involution. For an A-module M, ιM is the ιA-module M with the trivial involution. By abuse of notation we sometimes denote the constant Mackey functors by ιA and ιM as well.

For a commutative ring B with involution, $i^* B$ is the commutative ring obtained by forgetting the involution, and $B^{\mathbb{Z}/2}$ is the $\mathbb{Z}/2$-fixed point ring. For a B-module L, $i^* L$ is the $i^* B$-module obtained by forgetting the involution.

Definition 2.3.2. For an abelian group M with involution, we regard M as a Mackey functor, and take the functor (A.25) to obtain the equivariant Eilenberg–MacLane spectrum $\mathrm{H}M$. There are canonical equivalences

$$i^*\mathrm{H}M \simeq \mathrm{H}i^*M \text{ and } (\mathrm{H}M)^{\mathbb{Z}/2} \simeq \mathrm{H}(M^{\mathbb{Z}/2}). \tag{2.7}$$

For a commutative ring A with involution, we can regard $\mathrm{H}A$ as an object of $\mathrm{NAlg}_{\mathbb{Z}/2}$ as explained in [53, Example 11.12].

Note that for a given commutative ring B the commutative ring spectra $\mathrm{H}\iota B$ and $\iota \mathrm{H}B$ are quite different. For example, applying $\underline{\pi}_0$ to the first one yields the constant Mackey functor associated with B whereas for the second one a tensor product over $\mathbb{Z}$ with the Burnside ring Mackey functor of $\mathbb{Z}/2$ appears.

Proposition 2.3.3. *Let M be an abelian group. Then there is a canonical equivalence*

$$\mathrm{H}(M^{\oplus\mathbb{Z}/2}) \simeq i_*\mathrm{H}M, \tag{2.8}$$

where $M^{\oplus\mathbb{Z}/2}$ denotes the abelian group $M \oplus M$ with the involution given by $(x,y) \mapsto (y,x)$. The equivalence (2.8) can be promoted to an equivalence in $\mathrm{NAlg}_{\mathbb{Z}/2}$ if M is a commutative ring.

Proof. There is an equivalence

$$i^*\mathrm{H}(M^{\oplus\mathbb{Z}/2}) \simeq \mathrm{H}(M \oplus M) \tag{2.9}$$

by equation (2.7). Compose this with the map $\mathrm{H}(M \oplus M) \to \mathrm{H}M$ induced by the summation homomorphism $M \oplus M \to M$, and then we construct equation (2.8) by adjunction. We need to show that the induced map

$$\mathrm{Hom}_{\mathrm{Ho}(\mathrm{Sp}_{\mathbb{Z}/2})}(\Sigma^n\Sigma^\infty X_+, \mathrm{H}(M^{\oplus\mathbb{Z}/2})) \to \mathrm{Hom}_{\mathrm{Ho}(\mathrm{Sp}_{\mathbb{Z}/2})}(\Sigma^n\Sigma^\infty X_+, i_*\mathrm{H}M)$$

is an isomorphism for $X = \mathbb{Z}/2, e$ and integer $n \in \mathbb{Z}$. If $n \neq 0$, then both sides are vanishing. Assume $n = 0$. More concretely, it remains to show that the composite of the induced maps

$$\operatorname{Hom}_{\operatorname{Ho}(\operatorname{Sp}_{\mathbb{Z}/2})}(\Sigma^{\infty} X_+, \mathrm{H}(M^{\oplus \mathbb{Z}/2})) \to \operatorname{Hom}_{\operatorname{Ho}(\operatorname{Sp})}(i^* \Sigma^{\infty} X_+, i^* \mathrm{H}(M^{\oplus \mathbb{Z}/2}))$$
$$\to \operatorname{Hom}_{\operatorname{Ho}(\operatorname{Sp})}(i^* \Sigma^{\infty} X_+, \mathrm{H}M) \tag{2.10}$$

is an isomorphism.

If $X = \mathbb{Z}/2$, then equation (2.10) can be written as the homomorphisms

$$M \oplus M \to M \oplus M \oplus M \oplus M \to M \oplus M$$

given by $(x,y) \mapsto (x,0,y,0)$ and $(x,y,z,w) \mapsto (x+y, z+w)$. The composite is an isomorphism. If $X = e$, then equation (2.10) can be written as the homomorphisms

$$M \to M \oplus M \to M$$

given by $x \mapsto (x,0)$ and $(x,y) \mapsto x + y$. The composite is also an isomorphism.

If M is a commutative ring, then equation (2.9) is an equivalence in $\operatorname{NAlg}_{\mathbb{Z}/2}$. Hence, we obtain equation (2.8) as an equivalence in $\operatorname{NAlg}_{\mathbb{Z}/2}$. $\square$

Definition 2.3.4. For a commutative ring A with involution, we set

$$\operatorname{THR}(A) := \operatorname{THR}(\mathrm{H}A).$$

From the map (2.3), we see that $\operatorname{THR}(A)$ is an $\mathrm{H}A$-module.

Recall from equation (A.18) that $\underline{\pi}_0$ of an equivariant spectrum is a Mackey functor.

Proposition 2.3.5. *For every commutative ring A, the morphism of Mackey functors*

$$\iota A \to \underline{\pi}_0(\operatorname{THR}(\iota A)) \tag{2.11}$$

induced by equation (2.3) is an isomorphism.

Proof. By [16, Theorem 5.1], the morphism (2.11) can be described as a diagram

$$\begin{array}{ccc} A & \xrightarrow{\ \alpha\ } & (A \otimes A)/T, \\ {\scriptstyle \text{res}}\big\downarrow\big\uparrow{\scriptstyle \text{tran}} & & {\scriptstyle \text{res}}\big\downarrow\big\uparrow{\scriptstyle \text{tran}} \\ \circlearrowright\ A & \xrightarrow{\ \text{id}\ } & A,\ \circlearrowleft \end{array}$$

where $\alpha(x) = 1 \otimes x$ for $x \in A$, and T is the subgroup generated by $ax \otimes b - a \otimes xb$ for $a, b, x \in A$. The description of T means that α is an isomorphism. It follows that equation (2.11) is an isomorphism. $\square$

2.4. Norm functors and flat modules

The purpose of this subsection is to prove Proposition 2.4.2, which is one ingredient of the proof of Theorem 3.2.3.

Definition 2.4.1. Suppose A is a commutative ring and M is an A-module. Let $N_A^{\mathbb{Z}/2}M$ denote the ιA-module $M \otimes_A M$ whose $\mathbb{Z}/2$-action is given by $a \otimes b \mapsto b \otimes a$. If $A = \mathbb{Z}$, we simply write $N^{\mathbb{Z}/2}M$ instead of $N_{\mathbb{Z}}^{\mathbb{Z}/2}M$.

Observe that there is an isomorphism $N_A^{\mathbb{Z}/2}A \cong \iota A$ of commutative rings with involutions. We prefer to use the notation ιA instead of $N_A^{\mathbb{Z}/2}A$ for brevity.

Proposition 2.4.2. *Let M be an A-module, where A is a commutative ring. Then there exists a canonical map of $\mathrm{H}\iota A$-modules*

$$N^{\mathbb{Z}/2}\mathrm{H}M \wedge_{N^{\mathbb{Z}/2}\mathrm{H}A} \mathrm{H}\iota A \to \mathrm{H}(N_A^{\mathbb{Z}/2}M). \tag{2.12}$$

If M is flat, then this map is an equivalence.

Proof. First, we have a map $N^{\mathbb{Z}/2}\mathrm{H}A \to \mathrm{H}\iota A$ using the adjunction $(N^{\mathbb{Z}/2}, i^*)$, that i^* commutes with H and that $i^*\iota \simeq id$. We have the ιA-module structure on $N_A^{\mathbb{Z}/2}M$ given by $a(x \otimes y) = ax \otimes y$ for all $a \in A$ and $x \otimes y \in N_A^{\mathbb{Z}/2}M$. Hence, we can regard $\mathrm{H}(N_A^{\mathbb{Z}/2}M)$ as an $\mathrm{H}\iota A$-module. By [42, Proposition 4.6.2.17], to construct equation (2.12), it suffices to construct a map of $N^{\mathbb{Z}/2}\mathrm{H}A$-modules

$$N^{\mathbb{Z}/2}\mathrm{H}M \to \mathrm{H}(N_A^{\mathbb{Z}/2}M). \tag{2.13}$$

By Proposition Appendix A.3.7, $N^{\mathbb{Z}/2}\mathrm{H}A$ and $N^{\mathbb{Z}/2}\mathrm{H}M$ are (-1)-connected. From the adjoint pairs (A.27) and (A.29), we obtain a map in $\mathrm{CAlg}_{\mathbb{Z}/2}$

$$N^{\mathbb{Z}/2}\mathrm{H}A \to \mathrm{H}\underline{\pi}_0(N^{\mathbb{Z}/2}\mathrm{H}A)$$

and a map of $N^{\mathbb{Z}/2}\mathrm{H}A$-modules

$$N^{\mathbb{Z}/2}\mathrm{H}M \to \mathrm{H}\underline{\pi}_0(N^{\mathbb{Z}/2}\mathrm{H}M).$$

Hence, to construct equation (2.13), it suffices to construct a map of $\underline{\pi}_0(N^{\mathbb{Z}/2}\mathrm{H}A)$-modules

$$\underline{\pi}_0(N^{\mathbb{Z}/2}\mathrm{H}M) \to N_A^{\mathbb{Z}/2}M.$$

By Lemma 2.2.3 and Definition 2.4.1, this is equivalent to constructing a morphism of $\pi_0(i^*N^{\mathbb{Z}/2}\mathrm{H}A)$-modules

$$\pi_0(i^*N^{\mathbb{Z}/2}\mathrm{H}M) \to M \otimes_A M, \tag{2.14}$$

that is, a morphism of $A \otimes A$-modules $M \otimes M \to M \otimes_A M$ since $i^*N^{\mathbb{Z}/2} \simeq (-)^{\wedge 2}$ by Proposition Appendix A.2.7(6). The canonical assignment $x \otimes y \to x \otimes y$ finishes the construction.

The class of A-modules such that equation (2.12) is an equivalence is closed under filtered colimits. By Lazard's theorem, every flat A-module is a filtered colimit of finitely generated free A-modules. Hence, to show that equation (2.12) is an equivalence if M is flat, we may assume $M = A^n$. In this case, there is an equivalence

$$N^{\mathbb{Z}/2}\mathrm{H}M \simeq V_n \wedge N^{\mathbb{Z}/2}\mathrm{H}A,$$

where V_n is the set $[n] \times [n]$ with the $\mathbb{Z}/2$-action given by $(a,b) \mapsto (b,a)$. Hence, we obtain equivalences

$$N^{\mathbb{Z}/2}\mathrm{H}M \wedge_{N^{\mathbb{Z}/2}\mathrm{H}A} \mathrm{H}\iota A \simeq V_n \wedge \mathrm{H}\iota A \simeq \mathrm{H}(V_n \times \iota A).$$

Combining this with equation (2.12), we obtain a map

$$f \colon \mathrm{H}(V_n \times \iota A) \to \mathrm{H}(N_A^{\mathbb{Z}/2} A^n). \tag{2.15}$$

To show that f is an equivalence, it suffices to show that $\underline{\pi}_0(f)$ is an equivalence. Since $V_n \times \iota A$ and $N_A^{\mathbb{Z}/2} A^n$ are rings with involutions, by Lemma 2.2.3 it suffices to show that $\pi_0(f)$ is an equivalence of modules over rings with involution. Hence, to show that equation (2.12) is an equivalence, it suffices to show that it is an equivalence after applying π_0, that is, the induced map

$$g \colon (M \otimes M) \otimes_{A \otimes A} A \to M \otimes_A M$$

is an isomorphism. From the description of equation (2.14), we see that g is given by

$$g((x \otimes y) \otimes a) = ax \otimes y$$

for $x, y \in M$ and $a \in A$. One can readily check that this g is an isomorphism. $\square$

3. Descent properties of THR

3.1. Some equivariant topologies

We refer to [26] for the definition of the stable $\mathbb{Z}/2$-equivariant motivic homotopy category $\mathcal{SH}^{\mathbb{Z}/2}(k)$, which is compatible with the later work of Hoyois [33], but differs from Hermann's [27]. See his Corollary 2.13 and Example 3.1, as well as [26, Example 2.16 and section 6.1], for a comparison. The following definitions are taken from [26]. *Throughout this section, we assume that G is an abstract finite group, which we will identify with its associated finite group scheme over a fixed base scheme.* We are mostly interested in the case $G = \mathbb{Z}/2 = C_2$.

Definition 3.1.1. Let x be a point of a G-scheme X. The *set-theoretic stabilizer of X at x* is defined to be

$$S_x := \{g \in G : gx = x\}.$$

The *scheme-theoretic stabilizer of X at x* is defined to be

$$G_x := \ker(S_x \to \mathrm{Aut}(k(x))).$$

Let $f \colon Y \to X$ be an equivariant morphism of G-schemes. We say that

(i) f is *(equivariant) étale* if its underlying morphism of schemes is étale,

(ii) f is an *equivariant étale cover* if f is étale and surjective,

(iii) f is *isovariant* if for every point $y \in Y$, the induced homomorphism $G_y \to G_{f(y)}$ is an isomorphism,

(iv) f is an *isovariant étale cover* if f is isovariant and an equivariant étale cover,

(v) f is a *fixed point étale cover* if it is an étale cover and for every point $x \in X$, there exists a point $y \in f^{-1}(x)$ such that $G_x \simeq G_y$,

(vi) f is a *equivariant Nisnevich cover* if f is an étale cover and for every point $x \in X$, there exists a point $y \in f^{-1}(x)$ such that $k(x) \simeq k(y)$ and $S_x \simeq S_y$.

These covers define *equivariant étale, isovariant étale, fixed point étale* and *equivariant Nisnevich topologies* on the category of G-schemes.

The isovariant étale topology was first studied by Thomason [56]; see, for example, [26, section 6.1]. By [51, Remark 3.1], isovariant is the same as 'fixed-point reflecting', compare Definition 3.3 of loc. cit. The equivariant Nisnevich topology is due to Voevodsky [13].

The discussion in [26, p. 1223] shows that the equivariant Nisnevich topology is coarser than the fixed point étale topology. As observed in the proof of [26, Corollary 6.6], the fixed point étale topology is equivalent to the isovariant étale topology. Hence, we have the following inclusions of topologies:

$$\text{(equivariant Nisnevich)} \subset \text{(fixed point étale)}$$
$$\simeq \text{(isovariant étale)} \subset \text{(equivariant étale)}.$$

For a G-scheme X, let X/G denote the geometric quotient, which is an algebraic space. For the existence, see, for example, [51, Corollary 5.4]. If S is a locally noetherian scheme with the trivial G-action and $X \to S$ is a quasi-projective G-equivariant morphism, then X/G is representable by an S-scheme according to [14, Théorème V.7.1]. If A is a commutative ring with G-action, then there is a canonical isomorphism $\text{Spec}(A)/G \simeq \text{Spec}(A^G)$; see, for example, [51, Theorem 4.1].

Proposition 3.1.2. *Let $f\colon Y \to X$ be a separated isovariant étale morphism of G-schemes. Then the quotient morphism $Y/G \to X/G$ of algebraic spaces is étale, and the induced square*

$$\begin{array}{ccc} Y & \xrightarrow{\ f\ } & X \\ \downarrow & & \downarrow \\ Y/G & \longrightarrow & X/G \end{array}$$

is Cartesian.

Proof. This appears on [26, p. 1225]. □

Definition 3.1.3. For a G-scheme X, let $X_{iso\acute{e}t}$ denote the small isovariant étale site with the isovariant étale coverings $Y \to X$ of G-schemes.

Proposition 3.1.4 (Thomason). *Let X be a G-scheme. Then there exists an equivalence of sites*

$$(X/G)_{\acute{e}t} \xrightarrow{\ \simeq\ } X_{iso\acute{e}t}$$

sending any X/G-scheme Y to $Y \times_{X/G} X$.

Proof. We refer to [26, Proposition 6.11]. □

Definition 3.1.5. Let X be a separated G-scheme. The presheaf X^G on the category of separated schemes **Sch** is defined to be

$$X^G(Y) := \mathrm{Hom}_{\mathbf{Sch}}(Y,X)^G$$

for $Y \in \mathbf{Sch}$. By [14, Proposition 9.2 in Exposé XII], X^G is representable by a closed subscheme of X. Furthermore, the points of the topological space underlying the scheme X^G are in canonical bijection with the points of the topological space of fixed points (recall G is finite).

3.2. Isovariant étale base change

Lemma 3.2.1. *Let $A \to B$ be an étale homomorphism of commutative rings. Then the map $\mathrm{H}(m)\colon \mathrm{H}(N_A^{\mathbb{Z}/2}B) \to \mathrm{H}(\iota B)$ induced by the multiplication map*

$$m\colon N_A^{\mathbb{Z}/2}B \to \iota B,\ x \otimes y \mapsto xy$$

is flat.

Proof. Let C be the kernel of m, which is an ideal of $B \otimes_A B$ with the induced involution. Since $A \to B$ is étale, the diagonal morphism of schemes $\mathrm{Spec}(B) \to \mathrm{Spec}(B) \times_{\mathrm{Spec}(A)} \mathrm{Spec}(B)$ is an open and closed immersion by the implication a)⇒b) in [15, Corollaire IV.17.4.2]. This implies that the ring structure on $B \otimes_A B$ makes a ring structure on C. Hence, we have an isomorphism of commutative rings with involutions $N_A^{\mathbb{Z}/2}B \cong \iota B \times C$. We set $e := (1.0) \in \iota B \times C$. There is an isomorphism $N_A^{\mathbb{Z}/2}B[1/e] \cong \iota B$, which gives an isomorphism

$$\mathrm{colim}(N_A^{\mathbb{Z}/2}B \xrightarrow{\cdot e} N_A^{\mathbb{Z}/2}B \xrightarrow{\cdot e} \cdots) \cong \iota B.$$

Hence, ιB is a filtered colimit of free $N_A^{\mathbb{Z}/2}B$-modules. By Proposition Appendix A.5.11, m is flat. It follows that $\mathrm{H}(m)$ is flat too. $\square$

Let $A \to B$ be a map in $\mathrm{NAlg}_{\mathbb{Z}/2}$. Then we have the commutative square in $\mathrm{NAlg}_{\mathbb{Z}/2}$

$$
\begin{array}{ccc}
A & \longrightarrow & \mathrm{THR}(A) \\
\downarrow & & \downarrow \\
B & \longrightarrow & \mathrm{THR}(B),
\end{array}
$$

where the horizontal maps are given by equation (2.3). Hence, we obtain an induced map

$$\mathrm{THR}(A) \wedge_A B \to \mathrm{THR}(B) \tag{3.1}$$

in $\mathrm{NAlg}_{\mathbb{Z}/2}$. We will now study this map for $\mathrm{H}\iota A \to \mathrm{H}\iota B$.

Note that the following result does not directly follow from the étale base change result [21, Proposition 3.2.1] for THH as neither H nor THR commute with ι. This result is crucial for the descent results further below.

Proposition 3.2.2. *Let $A \to B$ be an étale homomorphism of commutative rings. Then we have an induced equivalence*

$$\mathrm{THR}(\iota A) \wedge_{\mathrm{H}\iota A} \mathrm{H}\iota B \overset{\simeq}{\to} \mathrm{THR}(\iota B). \tag{3.2}$$

Proof. Consider the commutative square

$$\begin{array}{ccc}
\mathrm{H}\iota A \wedge_{N^{\mathbb{Z}/2}\mathrm{H}A} \mathrm{H}\iota B & \longrightarrow & \mathrm{H}\iota B \wedge_{N^{\mathbb{Z}/2}\mathrm{H}B} \mathrm{H}\iota B \\
\simeq \downarrow & & \downarrow \simeq \\
\mathrm{H}(N_A^{\mathbb{Z}/2}B) \wedge_{N^{\mathbb{Z}/2}\mathrm{H}B} \mathrm{H}\iota B & \longrightarrow & \mathrm{H}(N_B^{\mathbb{Z}/2}B) \wedge_{N^{\mathbb{Z}/2}\mathrm{H}B} \mathrm{H}\iota B,
\end{array} \tag{3.3}$$

where the horizontal maps are induced by the map $A \to B$, the vertical equivalences are obtained by Proposition 2.4.2 and the right-hand one even comes from the algebraic isomorphism mentioned after Definition 2.4.1. We have an equivalence $\mathrm{H}\iota B \wedge_{N^{\mathbb{Z}/2}\mathrm{H}B} \mathrm{H}\iota B \simeq \mathrm{THR}(\iota B)$. (This equivalence involves a computation for the index under the $\wedge$, namely $i^* \mathrm{H}\iota B \simeq \mathrm{H}B$, which follows from equation (2.7). The same equivalence is used when identifying the upper left entry in the square.) This implies that equation (3.2) is equivalent to the upper horizontal map of equation (3.3). Proposition Appendix A.5.5 and Lemma 3.2.1 show that the lower horizontal map of equation (3.3) is flat. Putting everything together, we deduce that the map (3.2) is flat.

We will show that this map is an equivalence as claimed by applying Proposition Appendix A.5.6. By Proposition Appendix A.3.7, $N^{\mathbb{Z}/2}\mathrm{H}A$ and $N^{\mathbb{Z}/2}\mathrm{H}B$ are (-1)-connected. Hence, we may use Proposition Appendix A.4.4, and are reduced to showing that the induced morphism

$$\underline{\pi}_0(\mathrm{THR}(\iota A)) \square_{\iota A} \iota B \to \underline{\pi}_0(\mathrm{THR}(\iota B))$$

is an isomorphism. This follows easily from applying Proposition 2.3.5 to A and B. $\qquad\square$

We now establish isovariant étale base change for commutative rings with possibly nontrivial involution.

Theorem 3.2.3. *Let $A \to B$ be an isovariant étale homomorphism of commutative rings with involutions. Then there are canonical equivalences in $\mathrm{NAlg}_{\mathbb{Z}/2}$*

$$\mathrm{THR}(B) \simeq \mathrm{THR}(A) \wedge_{\mathrm{H}\iota(A^{\mathbb{Z}/2})} \mathrm{H}\iota(B^{\mathbb{Z}/2}) \simeq \mathrm{THR}(A) \wedge_{\mathrm{H}A} \mathrm{H}B.$$

Proof. We have isomorphisms $\mathrm{Spec}(A^{\mathbb{Z}/2}) \simeq \mathrm{Spec}(A)/(\mathbb{Z}/2)$ and $\mathrm{Spec}(B^{\mathbb{Z}/2}) \simeq \mathrm{Spec}(B)/(\mathbb{Z}/2)$. Hence, by Proposition 3.1.2, the induced homomorphism $A^{\mathbb{Z}/2} \to B^{\mathbb{Z}/2}$ is étale, and there is an isomorphism of commutative rings with involution

$$B \cong A \otimes_{\iota(A^{\mathbb{Z}/2})} \iota(B^{\mathbb{Z}/2}). \tag{3.4}$$

Since $A^{\mathbb{Z}/2} \to B^{\mathbb{Z}/2}$ is flat, $B^{\mathbb{Z}/2}$ is a filtered colimit of finite free $A^{\mathbb{Z}/2}$-modules. It follows that $\iota A^{\mathbb{Z}/2} \to \iota B^{\mathbb{Z}/2}$ is flat by Proposition Appendix A.5.11. Hence, the isomorphism (3.4) induces an equivalence $\mathrm{H}B \simeq \mathrm{H}A \wedge_{\mathrm{H}(\iota A^{\mathbb{Z}/2})} \mathrm{H}(\iota B^{\mathbb{Z}/2})$ by Proposition Appendix A.5.7. Then Proposition 2.1.5 yields an equivalence

$$\mathrm{THR}(B) \simeq \mathrm{THR}(A) \wedge_{\mathrm{THR}(\iota(A^{\mathbb{Z}/2}))} \mathrm{THR}(\iota(B^{\mathbb{Z}/2})).$$

Now, Proposition 3.2.2 implies the left equivalence of the Proposition after canceling out one smash factor $\mathrm{THR}(\iota(A^{\mathbb{Z}/2}))$. Applying the isovariance condition once more gives the right-hand side equivalence. $\square$

3.3. Presheaves of equivariant spectra

Zariski and other sheaves are completely determined by their behaviors on *affine* schemes. This is known to be true in some homotopical settings as well. The purpose of this section is to establish a rather general result, namely Proposition 3.3.13, which applies to our setting, that is the isovariant étale site and $\mathbb{Z}/2$-equivariant spectra. For this, a result of [3] will be very useful.

Definition 3.3.1. Let $\mathcal{C}$ be a category with a Grothendieck topology t, and let $\mathcal{V}$ be a presentable ∞-category. Let

$$\mathcal{P}\mathrm{sh}(\mathcal{C},\mathcal{V}) := \mathrm{Fun}(\mathrm{N}(\mathcal{C})^{op},\mathcal{V})$$

denote the ∞-category of presheaves on $\mathcal{C}$ with values in $\mathcal{V}$. We say that a presheaf $\mathcal{F} \in \mathcal{P}\mathrm{sh}(\mathcal{C},\mathcal{V})$ satisfies *t-descent* if the induced map

$$\mathcal{F}(X) \to \lim_{i \in \Delta} \mathcal{F}(\mathscr{X}_i) \tag{3.5}$$

is an equivalence for every t-hypercover $\mathscr{X} \to X$. Let $\mathcal{S}\mathrm{hv}_t(\mathcal{C},\mathcal{V})$ denote the full subcategory of $\mathcal{P}\mathrm{sh}(\mathcal{C},\mathcal{V})$ consisting of presheaves satisfying t-descent. We often omit t in the above notation if it is clear from the context.

The above condition is sometimes called *t-hyperdescent*, in order to distinguish it from the weaker descent condition only for covering sieves rather than all hypercovers. We refer to [41, section 6.5] and [18] for a careful comparison.

If $\mathcal{V} \to \mathcal{V}'$ is a functor of ∞-categories, then this induces a functor $\mathcal{P}\mathrm{sh}(\mathcal{C},\mathcal{V}) \to \mathcal{P}\mathrm{sh}(\mathcal{C},\mathcal{V}')$. In this way, we obtain functors i^*, $(-)^{\mathbb{Z}/2}$ and so on for the presheaf categories.

Proposition 3.3.2. *Suppose* $\mathcal{F} \in \mathcal{P}\mathrm{sh}(\mathcal{C},\mathrm{Sp}_{\mathbb{Z}/2})$, *where* $\mathcal{C}$ *is a site. Then* $\mathcal{F} \in \mathcal{S}\mathrm{hv}(\mathcal{C},\mathrm{Sp}_{\mathbb{Z}/2})$ *if and only if* $i^*\mathcal{F}, \mathcal{F}^{\mathbb{Z}/2} \in \mathcal{S}\mathrm{hv}(\mathcal{C},\mathrm{Sp})$.

Proof. Let $\mathscr{X} \to X$ be a hypercover. Since i^* and $(-)^{\mathbb{Z}/2}$ preserve limits, equation (3.5) is an equivalence if and only if

$$i^*\mathcal{F}(X) \to \lim_{i \in \Delta} i^*\mathcal{F}(\mathscr{X}_i) \text{ and } \mathcal{F}(X)^{\mathbb{Z}/2} \to \lim_{i \in \Delta} \mathcal{F}(\mathscr{X}_i)^{\mathbb{Z}/2}$$

are equivalences. Equivalently, $i^*\mathcal{F}, \mathcal{F}^{\mathbb{Z}/2} \in \mathcal{S}\mathrm{hv}(\mathcal{C},\mathrm{Sp})$. $\square$

Definition 3.3.3. Let $\mathcal{C}$ be a site. As a consequence of Proposition 3.3.2, we see that $\mathcal{S}\mathrm{hv}(\mathcal{C},\mathrm{Sp}_{\mathbb{Z}/2})$ is the full subcategory of local objects of [41, Definition 5.5.4.1] with respect to the class of maps consisting of

$$\iota\Sigma^n\Sigma^\infty\mathscr{X}_+ \to \iota\Sigma^n\Sigma^\infty X_+ \text{ and } i_\sharp\Sigma^n\Sigma^\infty\mathscr{X}_+ \to i_\sharp\Sigma^n\Sigma^\infty X_+$$

for all hypercovers $\mathscr{X} \to X$ and integers n. In particular, there is an adjoint pair

$$L : \mathcal{P}\mathrm{sh}(\mathcal{C},\mathrm{Sp}_{\mathbb{Z}/2}) \rightleftarrows \mathcal{S}\mathrm{hv}(\mathcal{C},\mathrm{Sp}_{\mathbb{Z}/2}) : \eta \tag{3.6}$$

by [41, Proposition 5.5.4.15(3)], where η is the inclusion functor. A map $\mathcal{F} \to \mathcal{G}$ in $\mathcal{P}\mathrm{sh}(\mathcal{C},\mathrm{Sp}_{\mathbb{Z}/2})$ is called a *local equivalence* if $L\mathcal{F} \to L\mathcal{G}$ is an equivalence.

Proposition 3.3.4. *Let $\mathcal{C}$ be a site. Then the functor*

$$i^* : \mathcal{P}\mathrm{sh}(\mathcal{C},\mathrm{Sp}_{\mathbb{Z}/2}) \to \mathcal{P}\mathrm{sh}(\mathcal{C},\mathrm{Sp})$$

preserves local equivalences.

Proof. We need to show that the maps

$$i^*\iota\Sigma^n\Sigma^\infty \mathscr{X}_+ \to i^*\iota\Sigma^n\Sigma^\infty X_+ \text{ and } i^*i_\sharp\Sigma^n\Sigma^\infty \mathscr{X}_+ \to i^*i_\sharp\Sigma^n\Sigma^\infty X_+$$

are local equivalences for all hypercovers $\mathscr{X} \to X$ and integers n, which follow from Proposition Appendix A.2.7(2),(7). $\qquad\square$

Proposition 3.3.5. *Let $\mathcal{C}$ be a site. Then the functor*

$$(-)^{\mathbb{Z}/2} : \mathcal{P}\mathrm{sh}(\mathcal{C},\mathrm{Sp}_{\mathbb{Z}/2}) \to \mathcal{P}\mathrm{sh}(\mathcal{C},\mathrm{Sp})$$

preserves local equivalences.

Proof. We need to show that the maps

$$(\iota\Sigma^\infty \mathscr{X}_+)^{\mathbb{Z}/2} \to (\iota\Sigma^\infty X_+)^{\mathbb{Z}/2} \text{ and } (i_\sharp\Sigma^\infty \mathscr{X}_+)^{\mathbb{Z}/2} \to (i_\sharp\Sigma^\infty X_+)^{\mathbb{Z}/2} \qquad (3.7)$$

are local equivalences. Since $(-)^{\mathbb{Z}/2}i_\sharp \simeq (-)^{\mathbb{Z}/2}i_* \simeq \mathrm{id}$ by Proposition Appendix A.2.7(2),(3), the second map in equation (3.7) is an equivalence.

The formulation (A.4) means that ι commutes with Σ^∞, and there is an equivalence $(\iota X)^{\mathbb{Z}/2} \simeq X$. Together with the tom Dieck splitting [23, Theorem 3.10], we have an equivalence

$$(\iota\Sigma^\infty X_+)^{\mathbb{Z}/2} \simeq \Sigma^\infty X_+ \oplus \Sigma^\infty(\mathrm{B}(\mathbb{Z}/2) \times X)_+.$$

We have a similar equivalence for $\mathscr{X}$ too. Since

$$\Sigma^\infty \mathscr{X}_+ \to \Sigma^\infty X_+ \text{ and } \Sigma^\infty(\mathrm{B}(\mathbb{Z}/2) \times \mathscr{X})_+ \to \Sigma^\infty(\mathrm{B}(\mathbb{Z}/2) \times X)_+$$

are local equivalences in $\mathcal{P}\mathrm{sh}(\mathcal{C},\mathrm{Sp})$, the first map in equation (3.7) is a local equivalence.
$\qquad\square$

Remark 3.3.6. Some of the results in this subsection including Propositions 3.3.2 and 3.3.4 have obvious generalizations to Sp_G for finite groups G. We also expect that Proposition 3.3.5 can be generalized too, but this should require extra effort since the tom Dieck splitting becomes more complicated.

Proposition 3.3.7. *Let $f : \mathcal{F} \to \mathcal{G}$ be a morphism in $\mathcal{P}\mathrm{sh}(\mathcal{C},\mathrm{Sp}_{\mathbb{Z}/2})$, where $\mathcal{C}$ is a site. If i^*f and $f^{\mathbb{Z}/2}$ are local equivalences, then f is a local equivalence.*

Proof. By considering the fiber of f, we reduce to the case when $\mathcal{G} = 0$. This means that $i^*\mathcal{F}$ and $\mathcal{F}^{\mathbb{Z}/2}$ are local equivalent to 0. The adjoint pair (2.4) gives a local equivalence $\mathcal{F} \to \mathcal{F}'$ with $\mathcal{F}' \in \mathcal{S}\mathrm{hv}(\mathcal{C},\mathrm{Sp}_{\mathbb{Z}/2})$. By Propositions 3.3.4 and 3.3.5, $i^*\mathcal{F}'$ and $\mathcal{F}'^{\mathbb{Z}/2}$ are local

equivalent to 0. Since $i^*\mathcal{F}', \mathcal{F}'^{\mathbb{Z}/2} \in \mathcal{S}\mathrm{hv}(\mathcal{C}, \mathrm{Sp})$ by Proposition 3.3.2, it follows that $i^*\mathcal{F}'$ and $\mathcal{F}'^{\mathbb{Z}/2}$ are equivalent to 0. Hence, $\mathcal{F}'$ is equivalent to 0, that is, $\mathcal{F}$ is local equivalent to 0. $\qquad\square$

Definition 3.3.8. Let $\mathfrak{M}$ be a combinatorial model category. For a category $\mathcal{C}$, we set

$$\mathbf{Psh}(\mathcal{C}, \mathfrak{M}) := \mathrm{Fun}(\mathcal{C}^{op}, \mathfrak{M}).$$

A morphism $\mathcal{F} \to \mathcal{G}$ in $\mathbf{Psh}(\mathcal{C}, \mathfrak{M})$ is a weak equivalence (resp. fibration) if $\mathcal{F}(X) \to \mathcal{G}(X)$ is a weak equivalence (resp. fibration) for all $X \in \mathcal{C}$. One can form a projective model structure based on these; see [3, Definition 4.4.18].

If $\mathcal{V}$ is the underlying ∞-category of $\mathfrak{M}$, then the underlying ∞-category of $\mathbf{Psh}(\mathcal{C}, \mathfrak{M})$ with respect to the projective model structure is equivalent to $\mathcal{P}\mathrm{sh}(\mathcal{C}, \mathcal{V})$ by [42, Proposition 1.3.4.25].

Definition 3.3.9 ([3, Definition 4.4.23]). A *category of coefficients* is a left proper cofibrantly generated stable model category $\mathfrak{M}$ satisfying the following conditions:

(i) Finite coproducts of weak equivalences are weak equivalences.

(ii) There exists a set $\mathcal{E}$ of compact objects of $\mathfrak{M}$ that generates the homotopy category $\mathrm{Ho}(\mathfrak{M})$.

The set $\mathcal{E}$ is considered as a part of the data.

Example 3.3.10. According to [29, Proposition B.63], Sp_G^O is a cofibrantly generated stable model category. By [29, Example B.10, Remark B.64], Sp_G^O is left proper. A consequence of [29, Corollary B.43] is that finite coproducts of weak equivalences are weak equivalences. Let $\mathcal{E}$ be the set of $\Sigma^n \Sigma^\infty (G/H)_+$ for all subgroups H of G and integers n, which consists of compact objects and generates $\mathrm{Ho}(\mathrm{Sp}_G^O)$ by Proposition Appendix A.1.6. In conclusion, Sp_G^O is a category of coefficients.

We also note that Sp_G^O is a combinatorial model category.

Definition 3.3.11. Let $\mathcal{C}$ be a site. According to [3, Definition 4.4.28], a morphism $\mathcal{F} \to \mathcal{G}$ in $\mathbf{Psh}(\mathcal{C}, \mathrm{Sp}_G^O)$ is called a *local weak equivalence* if the induced morphism of presheaves

$$(X \in \mathcal{C} \mapsto \mathrm{Hom}_{\mathrm{Ho}(\mathbf{Psh}(\mathcal{C}, \mathrm{Sp}_G^O))}(\Sigma^n \Sigma^\infty (G/H \times X)_+, \mathcal{F})$$

$$\to (X \in \mathcal{C} \mapsto \mathrm{Hom}_{\mathrm{Ho}(\mathbf{Psh}(\mathcal{C}, \mathrm{Sp}_G^O))}(\Sigma^n \Sigma^\infty (G/H \times X)_+, \mathcal{G})$$

becomes an isomorphism after sheafification for every subgroup H of G and integer n. There is a local projective model structure; see [3, Definition 4.4.34]. This is a Bousfield localization of the projective model structure with respect to local weak equivalences.

Proposition 3.3.12. *Let $\mathcal{C}$ be a site. The underlying ∞-category of $\mathbf{Psh}(\mathcal{C}, \mathrm{Sp}_{\mathbb{Z}/2}^O)$ with respect to the local projective model structure is equivalent to $\mathcal{S}\mathrm{hv}(\mathcal{C}, \mathrm{Sp}_{\mathbb{Z}/2})$.*

Proof. Let $f: \mathcal{F} \to \mathcal{G}$ be a morphism of fibrant objects in $\mathbf{Psh}(\mathcal{C}, \mathrm{Sp}_G^O)$ with respect to the projective model structure. We need to show that f is a local weak equivalence if and only if the corresponding map g in $\mathcal{P}\mathrm{sh}(\mathcal{C}, \mathrm{Sp}_G)$ is a local equivalence.

By adjunction, f is a local weak equivalence if and only if the induced morphism of presheaves

$$(X \in \mathcal{C} \mapsto \mathrm{Hom}_{\mathrm{Ho}(\mathrm{Sp}^O)}(\Sigma^n, \mathcal{F}(X)^H)) \to (X \in \mathcal{C} \mapsto \mathrm{Hom}_{\mathrm{Ho}(\mathrm{Sp}^O)}(\Sigma^n, \mathcal{G}(X)^H))$$

becomes an isomorphism after sheafification for $H = e, \mathbb{Z}/2$ and integer n. By [18, Theorem 1.3], this is equivalent to saying that i^*g and $g^{\mathbb{Z}/2}$ are local equivalences. Proposition 3.3.7 finishes the proof. $\qquad \square$

Proposition 3.3.13. *Let $\mathcal{C}$ and $\mathcal{C}'$ be sites. If there is an equivalence of topoi $\mathbf{Shv}(\mathcal{C}) \simeq \mathbf{Shv}(\mathcal{C}')$, then there is an equivalence of ∞-categories*

$$\mathcal{S}\mathrm{hv}(\mathcal{C}, \mathrm{Sp}_{\mathbb{Z}/2}) \simeq \mathcal{S}\mathrm{hv}(\mathcal{C}', \mathrm{Sp}_{\mathbb{Z}/2}).$$

Proof. Apply [3, Proposition 4.4.56] to the canonical (compose Yoneda and sheafification) functors $\mathcal{C} \to \mathbf{Shv}(\mathcal{C})$ and $\mathcal{C}' \to \mathbf{Shv}(\mathcal{C}')$, where the right-hand side categories are equipped with the topology described in [3, after Théorème 4.4.51]. Hence, we obtain left Quillen equivalences

$$\mathbf{Psh}(\mathcal{C}, \mathrm{Sp}^O_{\mathbb{Z}/2}) \to \mathbf{Psh}(\mathbf{Shv}(\mathcal{C}), \mathrm{Sp}^O_{\mathbb{Z}/2}) \text{ and } \mathbf{Psh}(\mathcal{C}', \mathrm{Sp}^O_{\mathbb{Z}/2}) \to \mathbf{Psh}(\mathbf{Shv}(\mathcal{C}'), \mathrm{Sp}^O_{\mathbb{Z}/2})$$

with respect to the local model structures of Definition 3.3.11. Owing to Proposition 3.3.12 and [42, Lemma 1.3.4.21], we obtain equivalences of ∞-categories

$$\mathcal{S}\mathrm{hv}(\mathcal{C}, \mathrm{Sp}_{\mathbb{Z}/2}) \simeq \mathcal{S}\mathrm{hv}(\mathbf{Shv}(\mathcal{C}), \mathrm{Sp}_{\mathbb{Z}/2}) \text{ and } \mathcal{S}\mathrm{hv}(\mathcal{C}', \mathrm{Sp}_{\mathbb{Z}/2}) \simeq \mathcal{S}\mathrm{hv}(\mathbf{Shv}(\mathcal{C}'), \mathrm{Sp}_{\mathbb{Z}/2}).$$

We obtain the desired equivalence of ∞-categories thanks to the equivalence of topoi (and hence sites) $\mathbf{Shv}(\mathcal{C}) \simeq \mathbf{Shv}(\mathcal{C}')$. $\qquad \square$

3.4. Isovariant étale descent

For an abelian group M, equation (2.7) gives a canonical equivalence

$$\mathrm{H}M \xrightarrow{\simeq} (\mathrm{H}\iota M)^{\mathbb{Z}/2}.$$

The last adjunction in equation (2.2) then yields a maps $\iota \mathrm{H}M \to \mathrm{H}\iota M$. We have a similar map for a commutative ring A.

Lemma 3.4.1. *Let A be a commutative ring and M a flat A-module. Then there is an equivalence*

$$\mathrm{H}\iota M \simeq \mathrm{H}\iota A \wedge_{\iota \mathrm{H}A} \iota \mathrm{H}M.$$

Proof. By Proposition Appendix A.3.6, H preserves filtered colimits. The two functors ι also preserve filtered colimits since they are left adjoints. Every flat A-module is a filtered colimit of finitely generated free A-modules by Lazard's theorem, so we reduce to the case when M is a finitely generated free A-module. In this case, the claim is clear. $\qquad \square$

For a scheme X, let $\mathbf{\acute{E}tAff}/X$ denote the category of affine schemes étale over X. We start by considering the affine case $X = \mathrm{Spec}(A)$.

Lemma 3.4.2. *Let A be a commutative ring, and let M be an $\mathrm{H}A$-module. The presheaf $\mathcal{F}$ of spectra on $\mathbf{\acute{E}tAff}/\operatorname{Spec}(A)$ given by*

$$\mathcal{F}(\operatorname{Spec}(B)) := M \wedge_{\mathrm{H}A} \mathrm{H}B$$

for étale homomorphisms $A \to B$ satisfies étale descent.

Proof. Since $\mathrm{H}A \to \mathrm{H}B$ is flat in the sense of [42, Definition 7.2.2.10], we have an isomorphism

$$\pi_q\mathcal{F}(\operatorname{Spec}(B))) \simeq \pi_q(M) \otimes_A B$$

for all integers q. In particular, $\pi_q\mathcal{F}$ is a quasi-coherent sheaf on $\mathbf{\acute{E}tAff}/\operatorname{Spec}(A)$.

According to the paragraph preceding [21, Proposition 3.1.2], there is a conditionally convergent spectral sequence

$$E_2^{st} := H^s_{\acute{e}t}(X, a_{\acute{e}t}\pi_{-t}\mathcal{F}) \Rightarrow \pi_{-s-t}L_{\acute{e}t}\mathcal{F}(X),$$

where $a_{\acute{e}t}$ denotes the étale sheafification functor. Since $\pi_{-t}\mathcal{F}$ is a quasi-coherent sheaf for every integer t, the cohomology $H^s(X, a_{\acute{e}t}\pi_{-t}\mathcal{F})$ vanishes for every integer $s \neq 0$. It follows that $\pi_{-t}\mathcal{F} \to \pi_{-t}L_{\acute{e}t}\mathcal{F}$ is an isomorphism for every integer t, that is, $\mathcal{F}$ satisfies étale descent. $\square$

Let A be a commutative ring with involution. Let $\mathbf{iso\acute{E}tAff}/\operatorname{Spec}(A)$ denote the category of affine schemes with involutions isovariant étale over $\operatorname{Spec}(A)$.

Theorem 3.4.3. *The presheaf*

$$\mathrm{THR} \in \mathcal{P}\mathrm{sh}(\mathbf{Aff}_{\mathbb{Z}/2}, \mathrm{Sp}_{\mathbb{Z}/2})$$

satisfies isovariant étale descent, where $\mathbf{Aff}_{\mathbb{Z}/2}$ denotes the category of affine schemes with involutions.

Proof. We only need to show that the restriction of THR to $\mathbf{iso\acute{E}tAff}/\operatorname{Spec}(A)$ satisfies isovariant étale descent. The functor

$$\mathbf{\acute{E}tAff}/\operatorname{Spec}(A^{\mathbb{Z}/2}) \to \mathbf{iso\acute{E}tAff}/\operatorname{Spec}(A) \tag{3.8}$$

sending $\operatorname{Spec}(B)$ to $\operatorname{Spec}(A \otimes_{\iota A^{\mathbb{Z}/2}} \iota B)$ is an equivalence of sites by Proposition 3.1.4. Let

$$\mathcal{F} \in \mathcal{P}\mathrm{sh}(\mathbf{\acute{E}tAff}/\operatorname{Spec}(A^{\mathbb{Z}/2}), \mathrm{Sp}_{\mathbb{Z}/2})$$

be the presheaf given by $\mathcal{F}(\operatorname{Spec}(B)) := \mathrm{THR}(A) \wedge_{\mathrm{H}\iota A^{\mathbb{Z}/2}} \mathrm{H}\iota B$ for étale homomorphisms $A^{\mathbb{Z}/2} \to B$. If we show that $\mathcal{F}$ satisfies étale descent, then the presheaf

$$\mathcal{G} \in \mathcal{P}\mathrm{sh}(\mathbf{iso\acute{E}tAff}/\operatorname{Spec}(A), \mathrm{Sp}_{\mathbb{Z}/2})$$

obtained from $\mathcal{F}$ and errant (3.8) satisfies isovariant étale descent. Theorem 3.2.3 gives the third equivalence in

$$\mathcal{G}(\operatorname{Spec}(\iota B \otimes_{\iota A^{\mathbb{Z}/2}} A)) = \mathcal{F}(\operatorname{Spec}(B)) = \mathrm{THR}(A) \wedge_{\mathrm{H}\iota A^{\mathbb{Z}/2}} \mathrm{H}\iota B \simeq \mathrm{THR}(A \otimes_{\iota A^{\mathbb{Z}/2}} \iota B),$$

that is, $\mathcal{G} \simeq \mathrm{THR}$. This means that THR satisfies isovariant étale descent.

Hence, it remains to check that $\mathcal{F}$ satisfies étale descent. There is an equivalence

$$i^*\mathcal{F}(\mathrm{Spec}(B)) \simeq i^*\mathrm{THR}(A) \wedge_{\mathrm{H}A^{\mathbb{Z}/2}} \mathrm{H}B.$$

By Lemmas Appendix A.2.9 and 3.4.1, there are equivalences

$$\mathcal{F}(\mathrm{Spec}(B))^{\mathbb{Z}/2} \simeq (\mathrm{THR}(A) \wedge_{\iota \mathrm{H}A^{\mathbb{Z}/2}} \iota \mathrm{H}B)^{\mathbb{Z}/2} \simeq \mathrm{THR}(A)^{\mathbb{Z}/2} \wedge_{\mathrm{H}A^{\mathbb{Z}/2}} \mathrm{H}B.$$

Lemma 3.4.2 implies that $i^*\mathcal{F}$ and $\mathcal{F}^{\mathbb{Z}/2}$ satisfy étale descent, which means that $\mathcal{F}$ satisfies étale descent. $\qquad\square$

Proposition 3.4.4. *For every separated scheme X with involution, there exists an isovariant étale covering $\{U_i \to X\}_{i \in I}$ such that each U_i is an affine scheme with involution.*

Proof. Let $X^{\mathbb{Z}/2}$ denote the closed subscheme of X obtained by Definition 3.1.5. We set $Y := X - X^{\mathbb{Z}/2}$, which is an open subscheme of X.

If x is a point of $X^{\mathbb{Z}/2}$, choose an affine open neighborhood U_x of x in X. Then $V_x := U_x \cap w(U_x)$ is again an affine open neighborhood of x since X is separated, and w can be restricted to V_x. It follows that

$$\{V_x \to X\}_{x \in X} \cup \{Y \to X\}$$

is a Zariski covering.

The $\mathbb{Z}/2$-action on Y is free, so the quotient morphism $Y \to Y/(\mathbb{Z}/2)$ is étale since the algebraic space $Y/(\mathbb{Z}/2)$ is formed by the étale equivalence relation $Y \times \mathbb{Z}/2 \rightrightarrows Y$, and there is an isomorphism $Y \times_{Y/(\mathbb{Z}/2)} Y \cong Y \times \mathbb{Z}/2$. It follows that the morphism $Y \times \mathbb{Z}/2 \to Y$ obtained by the first projection $Y \times_{Y/(\mathbb{Z}/2)} Y \to Y$ is isovariant étale. Choose a Zariski covering $\{W_j \to Y\}_{j \in J}$ after forgetting involution such that each W_j is an affine scheme, and then

$$\{V_x \to X\}_{x \in X} \cup \{W_j \times \mathbb{Z}/2 \to X\}_{j \in J} \tag{3.9}$$

is an isovariant étale covering. $\qquad\square$

Let $\mathbf{Sch}_{\mathbb{Z}/2}$ denote the category of separated schemes with involutions.

Proposition 3.4.5. *There is an equivalence of topoi*

$$\mathbf{Shv}_{iso\acute{e}t}(\mathbf{Aff}_{\mathbb{Z}/2}) \simeq \mathbf{Shv}_{iso\acute{e}t}(\mathbf{Sch}_{\mathbb{Z}/2}). \tag{3.10}$$

Hence, there is an equivalence of ∞-categories

$$\mathcal{S}\mathrm{hv}_{iso\acute{e}t}(\mathbf{Aff}_{\mathbb{Z}/2}, \mathrm{Sp}_{\mathbb{Z}/2}) \simeq \mathcal{S}\mathrm{hv}_{iso\acute{e}t}(\mathbf{Sch}_{\mathbb{Z}/2}, \mathrm{Sp}_{\mathbb{Z}/2}). \tag{3.11}$$

Proof. Combine [2, Théorème III.4.1] and Proposition 3.4.4 to obtain equation (3.10). Use Proposition 3.3.13 for equation (3.11). $\qquad\square$

Definition 3.4.6. By Theorem 3.4.3 and Proposition 3.4.5, we obtain

$$\mathrm{THR} \in \mathcal{S}\mathrm{hv}_{iso\acute{e}t}(\mathbf{Sch}_{\mathbb{Z}/2}, \mathrm{Sp}_{\mathbb{Z}/2}).$$

This definition immediately implies that $\mathrm{THR}(X)$ that satisfies isovariant étale descent for all $X \in \mathbf{Sch}_{\mathbb{Z}/2}$.

Below, we carry out computations of $\mathrm{THR}(X)$ for projective spaces $X = \mathbb{P}^n$, even with involution for $n = 1$. Let us first check that extending Proposition 2.1.3 to schemes yields a definition of THH for schemes equivalent to the one of [21] as follows. See, for example, [10, p. 1055 and chapter 3] for a discussion of other equivalent definitions of $\mathrm{THH}(X)$.

Proposition 3.4.7. *For $X \in \mathbf{Sch}_{\mathbb{Z}/2}$ and $Y \in \mathbf{Sch}$, there are canonical equivalences*

$$\mathrm{THH}(i^*X) \simeq i^*\mathrm{THR}(X) \ \textit{and} \ \mathrm{THR}(Y \amalg Y) \simeq i_*\mathrm{THH}(Y),$$

where the involution on $Y \amalg Y$ switches the components.

Proof. The question is isovariant étale local on X and étale local on Y, so we reduce to the case when $X = \mathrm{Spec}(A)$ for some commutative ring A with involution and $Y = \mathrm{Spec}(B)$ for some commutative ring B. We obtain equivalences

$$\mathrm{THR}(i^*A) = \mathrm{THR}(\mathrm{H}(i^*A)) \simeq \mathrm{THR}(i^*\mathrm{H}A) \simeq i^*\mathrm{THR}(\mathrm{H}A) = i^*\mathrm{THR}(A)$$

using Proposition 2.1.3 and equation (2.7). We similarly obtain the remaining desired equivalence using Propositions 2.1.4 and 2.3.3. $\qquad \square$

Definition 3.4.8. Recall from [26, section 2.1] that an *equivariant Nisnevich distinguished square* is a Cartesian square in $\mathbf{Sch}_{\mathbb{Z}/2}$ of the form

$$Q := \begin{array}{ccc} V & \longrightarrow & Y \\ \downarrow & & \downarrow f \\ U & \stackrel{j}{\longrightarrow} & X \end{array} \qquad (3.12)$$

such that j is an open immersion, f is equivariant étale and the morphism of schemes $(Y - V)_{\mathrm{red}} \to (X - U)_{\mathrm{red}}$ is an isomorphism. The collection of such squares forms a cd-structure, which is bounded, complete and regular in the sense of [58] by [26, Theorem 2.3]. Furthermore, the associated topology is the equivariant Nisnevich topology in Definition 3.1.1 by [26, Proposition 2.17].

Corollary 3.4.9. *For every equivariant Nisnevich distinguished square Q of the form (3.12), the induced square*

$$\mathrm{THR}(Q) = \begin{array}{ccc} \mathrm{THR}(X) & \longrightarrow & \mathrm{THR}(U) \\ \downarrow & & \downarrow \\ \mathrm{THR}(Y) & \longrightarrow & \mathrm{THR}(V) \end{array}$$

is Cartesian.

Proof. Let P be the collection of equivariant Nisnevich distinguished squares. Consider the class G_P of morphisms between simplicial presheaves in [58, p. 1392]. By [58, Proposition 3.8(2)], every G_P-local equivalence is an equivariant Nisnevich local equivalence.

This implies that if $\mathcal{F}$ is a simplicial presheaf satisfying equivariant Nisnevich descent, then $\mathcal{F}(Q)$ is Cartesian for all $Q \in P$.

Since THR satisfies isovariant étale descent, it satisfies equivariant Nisnevich descent. Hence, we can apply the argument in the above paragraph to $\Omega^\infty \Sigma^n i^*\text{THR}$ and $\Omega^\infty \Sigma^n \Phi^{\mathbb{Z}/2}\text{THR}$ for all integers n to see that $i^*\text{THR}(Q)$ and $\Phi^{\mathbb{Z}/2}\text{THR}(Q)$ are Cartesian, which implies the claim. □

For the computations about $\text{THR}(\mathbb{P}^n)$ we provide in the next sections, we may simply use appropriate equivariant Nisnevich covers by affine schemes and Corollary 3.4.9, which, for example, exhibits $\text{THR}(\mathbb{P}^1)$ as part of a homotopy co-Cartesian square in which all other entries are of the form $\text{THR}(Y)$ for $Y = \text{Spec}(A[M])$ corresponding to some commutative monoid rings for $M = \mathbb{N}$ or $M = \mathbb{Z}$ over the commutative base ring A. We could even go one step further and *define* $\text{THR}(\mathbb{P}^n_{\mathbb{S}}, \sigma)$ for projective spaces, possibly with nontrivial involution σ, over the sphere spectrum $\mathbb{S}$ rather than over A or HA, using the homotopy pushout of the appropriate diagram of THR of the corresponding spherical monoid rings $\mathbb{S}[M]$, compare the proofs in section 5.1.

4. The dihedral bar construction

4.1. Crossed simplicial groups

Hochschild and cyclic homology and their real refinements are closely related to cyclic, real and dihedral nerves. We now present a uniform treatment of these constructions. The original references are [20] and [40]; parts of this are also explained, for example, in [17] and [16].

Definition 4.1.1. (See [20, Definition 1.1],[40, chapter 6.3]) A *crossed simplicial group* G is a sequence of groups $\{G_n\}_{n \geq 0}$ together with a category ΔG satisfying the following conditions:

(i) ΔG contains Δ as a subcategory with the same objects.

(ii) $\text{Aut}([n])$ is the opposite group G_n^{op}.

(iii) A morphism in ΔG can be uniquely written as the composite $\alpha \circ g$ for some $\alpha \in \text{Hom}_\Delta([m],[n])$ and $g \in G_m^{op}$.

Observe that every crossed simplicial group is a (or rather has an underlying) simplicial set; see [20, Lemma 1.3]. A *G-set* is a functor $\Delta G^{op} \to \mathbf{Set}$, which by restriction has an underlying simplicial set.

Construction 4.1.2. Let X be a simplicial set. For a crossed simplicial group G, the *G-set* $F_G(X)$ is defined in [20, Definition 4.3]. In simplicial degree q, we have

$$F_G(X)_q := G_q \times X_q.$$

The G_q action on $F_G(X)_q$ is the left multiplication on G_q. The faces and degeneracy maps are given by

$$d_i(g,x) := (d_i(g), d_{g^{-1}(i)}(x)) \text{ and } s_i(g,x) := (s_i(g), s_{g^{-1}(i)}(x)). \tag{4.1}$$

According to [20, Proposition 5.1], we have the projection $p_1 \colon F_G(X) \to G$ given by $p_1(g,x) := g$. We also have the projection $p_2 \colon |F_G(X)| \to |X|$ given by

$$p_2[(g,x),u] := [x,gu]$$

for $u \in \Delta_{top}^q$. Furthermore, the two projections define a homeomorphism

$$|F_G(X)| \cong |G| \times |X|.$$

If X is a G-set, then we have the evaluation map $ev \colon F_G(X) \to X$ given by

$$ev(g,x) := gx.$$

According to [20, Theorem 5.3], the composite map

$$|G| \times |X| \xrightarrow{\cong} |F_G(X)| \xrightarrow{ev} |X|$$

defines a $|G|$-action on $|X|$. The *G-geometric realization of X* is $|X|$ with this $|G|$-action.

Example 4.1.3. We have the following three fundamental examples of crossed simplicial groups.

(1) [20, Example 2, section 1.5] introduces a crossed simplicial group G with $G_n := \mathbb{Z}/2$. For this G, a G-set is called a *real simplicial set*. The G-geometric realization is called the *real geometric realization*.

Explicitly, a real simplicial set X is a simplicial set equipped with isomorphisms $w_n \colon X_n \to X_n$ with $w_n^2 = \mathrm{id}$ for all integers $n \geq 1$ satisfying the relations

$$d_i w_n = w_{n-1} d_{n-i} \text{ and } s_i w_n = w_{n+1} s_{n-i}$$

for $0 \leq i \leq n$.

(2) [20, Example 4, section 1.5] introduces a crossed simplicial group G with $G_n := C_{n+1}$. For this G, a G-set is called a *cyclic set*, which was defined by Connes. The G-geometric realization is called the *cyclic geometric realization* and comes with an action of $S^1 = SO(2)$ (see also [40, Theorem 7.1.4]).

Explicitly, a cyclic set X is a simplicial set equipped with isomorphisms $t_n \colon X_n \to X_n$ with $(t_n)^{n+1} = \mathrm{id}$ for all integers $n \geq 1$ satisfying the relations

$$d_i t_n = t_{n-1} d_{i-1} \text{ and } s_i t_n = t_{n+1} s_{i-1}$$

for $1 \leq i \leq n$.

(3) [20, Example 5, section 1.5] introduces a crossed simplicial group G with $G_n := D_{n+1}$, where D_n denotes the dihedral group of order $2n$. For this G, a G-set is called a *dihedral set*. The G-geometric realization is called the *dihedral geometric realization*.

Explicitly, a dihedral set X is a simplicial set equipped with t_n and w_n for all integers $n \geq 1$ satisfying $w_n t_n = t_n^{-1} w_n$ and all the above conditions for t_n and w_n.

There are obvious forgetful functors from dihedral to real and to cyclic sets.

For $|G|$, we obtain $\mathbb{Z}/2$, $S^1 = U(1) = SO(2)$ and $O(2)$ in the three examples above.

Example 4.1.4. Let G be the crossed simplicial set in Example 4.1.3(1). For a simplicial set X, the bijection $G \times X_q = X_q \amalg X_q$ induces a canonical isomorphism of G-sets

$$F_G(X) \cong X \amalg X^{op},$$

where $w_n \colon X_n \amalg X_n^{op} \to X_n \amalg X_n^{op}$ is the switching map. If X is a real simplicial set, then the evaluation map $ev \colon F_G(X) \to X$ sends $x \in X_n^{op}$ to $w_n(x)$. It follows that the $\mathbb{Z}/2$-action on $|X|$ is given by the composite map

$$|X| \xrightarrow{\cong} |X^{op}| \xrightarrow{w} |X|,$$

where the first map is the canonical identity.

Example 4.1.5. Let G be the crossed simplicial set in Example 4.1.3(2). For every integer $n \geq 0$, we set

$$\Lambda^n := F_G(\Delta^n).$$

This is Connes' cyclic n-complex. We warn the reader that $F_G(X)$ for a simplicial set is different from the cyclic bar construction.

4.2. Real and dihedral nerves

We now recall how commutative monoids with involution give rise to real and dihedral simplicial sets, refining nerves and cyclic nerves for commutative monoids without involutions.

Definition 4.2.1. Let M be a commutative monoid with involution σ. The *real nerve of M*, denoted $\mathrm{N}^\sigma M$, is the real simplicial set whose underlying simplicial set is the nerve of M with $(\mathrm{N}^\sigma M)_q := M^{\times q}$ and w is given by

$$w(x_1, \ldots, x_q) := (\sigma(x_q), \ldots, \sigma(x_1))$$

in simplicial degree q. The *real bar construction*, denoted $\mathrm{B}^\sigma M$, is the real geometric realization of $\mathrm{N}^\sigma M$. We refer to [17, Example 2.1.1] for a similar account.

Definition 4.2.2. Let M be a commutative monoid with involution σ. The *dihedral nerve of M*, denoted $\mathrm{N}^{\mathrm{di}} M$, is the dihedral set defined as follows. In simplicial degree q, we have $(\mathrm{N}^{\mathrm{di}} M)_q := M^{\times (q+1)}$. The face maps are

$$d_i(x_0, \ldots, x_q) := \begin{cases} (x_0, \ldots, x_{i-1}, x_i + x_{i+1}, x_{i+2}, \ldots, x_q) & \text{if } i = 0, \ldots, q-1 \\ (x_q + x_0, x_1, \ldots, x_{q-1}) & \text{if } i = q. \end{cases}$$

The degeneracy maps are

$$s_i(x_0, \ldots, x_q) := (x_0, \ldots, x_i, 0, x_{i+1}, \ldots, x_q) \text{ for } i = 0, \ldots, q.$$

The rotation maps are

$$t_q(x_0, \ldots, x_q) := (x_q, x_0, \ldots, x_{q-1}).$$

The involution is

$$w(x_0, \ldots, x_q) := (\sigma(x_0), \sigma(x_q), \ldots, \sigma(x_1)),$$

compare also with [17, Example 2.1.2].

Remark 4.2.3. This involution looks similar to the involution on the Hochschild complex as studied in [40, 5.2.1]. In [40, section 5.2] Loday investigates several 'real' versions of Hochschild and cyclic homology for rings. The decompositions of [40, Proposition 5.2.3, (5.2.7.1)] should be compared to the Bott sequence [52, section 6] relating algebraic to Hermitian K-theory, which also splits after inverting 2 or rather $1 - \epsilon$.

Note that we obtain the *cyclic nerve of M* [59, section 2.3], denoted $\mathrm{N}^{\mathrm{cy}} M$, if we forget the involution structure.

The *dihedral bar construction*, denoted $\mathrm{B}^{\mathrm{di}} M$, is the dihedral geometric realization of $\mathrm{N}^{\mathrm{di}} M$. By Construction 4.1.2, $\mathrm{B}^{\mathrm{di}} M$ admits a canonical $O(2)$-action. The underlying real simplicial set of $\mathrm{N}^{\mathrm{di}} M$ is different from $\mathrm{N}^{\sigma} M$.

We have the map of dihedral sets

$$\mathrm{N}^{\mathrm{di}} M \to M \tag{4.2}$$

sending $(x_0, \ldots, x_q)$ to $x_0 + \cdots + x_q$ in simplicial degree q, where we regard M as the constant dihedral set whose rotation maps are the identities and whose involutions are given by σ. For every σ-orbit I in M, we set

$$\mathrm{N}^{\mathrm{di}}(M; I) := \mathrm{N}^{\mathrm{di}} M \times_M I. \tag{4.3}$$

Let $\mathrm{B}^{\mathrm{di}}(M; I)$ be its dihedral geometric realization. We have an isomorphism of dihedral sets

$$\mathrm{N}^{\mathrm{di}} M \cong \coprod_I \mathrm{N}^{\mathrm{di}}(M; I), \tag{4.4}$$

where the coproduct runs over the σ-orbits in M.

Proposition 4.2.4. *Let M and L be commutative monoids with involutions. Then there is an isomorphism of dihedral sets*

$$\mathrm{N}^{\mathrm{di}}(M \times L) \cong \mathrm{N}^{\mathrm{di}} M \times \mathrm{N}^{\mathrm{di}} L. \tag{4.5}$$

Proof. The two projections $M \times L \rightrightarrows M, L$ induce equation (4.5). To show that it is an isomorphism, observe that it is given by the shuffle homomorphism

$$(M \times L)^{\times (q+1)} \to M^{\times (q+1)} \times L^{\times (q+1)}$$

in simplicial degree q. $\qquad \square$

Proposition 4.2.5. *Let M and L be commutative monoids with the involutions. Then there is an isomorphism of dihedral sets*

$$\mathrm{N}^{\mathrm{di}}(M \times L; (x, y)) \cong \mathrm{N}^{\mathrm{di}}(M; x) \times \mathrm{N}^{\mathrm{di}}(L; y)$$

for all $x \in M$ and $y \in L$.

Proof. This follows from the commutativity of the square

$$\begin{array}{ccc} (M \times L)^{\times(q+1)} & \longrightarrow & M^{\times(q+1)} \times L^{\times(q+1)} \\ \downarrow & & \downarrow \\ M \times L & \xrightarrow{\ \mathrm{id}\ } & M \times L, \end{array}$$

where the upper horizontal homomorphism is the shuffle homomorphism, and the vertical homomorphisms are the summation homomorphisms. $\qquad\square$

Proposition 4.2.6. *Let G be an abelian group with an involution w, and let j be an element of G. If $j = w(j)$, then there is an isomorphism of real simplicial sets*

$$\mathrm{N}^{\mathrm{di}}(G; j) \xrightarrow{\cong} \mathrm{N}^{\sigma} G. \tag{4.6}$$

If $j \neq w(j)$, then there is an isomorphism of real simplicial sets

$$\mathrm{N}^{\mathrm{di}}(G; \{j, w(j)\}) \xrightarrow{\cong} \mathbb{Z}/2 \times \mathrm{N}^{\sigma} G. \tag{4.7}$$

As a consequence, there is an isomorphism of real simplicial sets

$$\mathrm{N}^{\mathrm{di}} G \simeq G \times \mathrm{N}^{\sigma} G. \tag{4.8}$$

Proof. If $j = w(j)$, the assignment

$$(x_0, \dots, x_q) \mapsto (x_1, \dots, x_q)$$

in simplicial degree q constructs the isomorphism (4.6). If $j \neq w(j)$, the assignment

$$(x_0, \dots, x_q) \mapsto (a, x_1, \dots, x_q)$$

in simplicial degree q constructs the isomorphism (4.7), where $a := 0$ (resp. $a := 1$) if $x_0 + \dots + x_q = j$ (resp. $x_0 + \dots + x_q = -j$). $\qquad\square$

Definition 4.2.7. Let S^{σ} be the $\mathbb{Z}/2$-space whose underlying space is S^1 and whose $\mathbb{Z}/2$-action is given by the reflection $(x, y) \in S^1 \subset \mathbb{R}^2 \mapsto (x, -y)$. This is the real geometric realization of the real simplicial set whose underlying simplicial set is the simplicial circle $\Delta^1/\partial\Delta^1$ and whose involution on the nondegenerate simplices is the identity. One easily checks that the relations in Example 4.1.3(1) uniquely determines the higher w_n. To understand the involution on the real realization, the reader is advised to look at the explanations in Example 4.1.4.

Proposition 4.2.8. *For the additive monoid $\mathbb{Z}$ with the trivial involution, there is a $\mathbb{Z}/2$-homotopy equivalence*

$$\mathrm{B}^{\sigma} \mathbb{Z} \simeq S^{\sigma}.$$

Proof. Consider the element $1 \in (\mathrm{N}^{\sigma}\mathbb{Z})_1 \simeq \mathbb{Z}$ in simplicial degree 1. This is fixed by the involution on $\mathrm{N}^{\sigma}\mathbb{Z}$, so the real geometric realization of the real simplicial subset of $\mathrm{N}^{\sigma}\mathbb{Z}$ whose only nondegenerate simplices are 1 and the base point is S^{σ}. Hence, we obtain a

map of $\mathbb{Z}/2$-spaces $S^\sigma \to B^\sigma\mathbb{Z}$. This is the usual homotopy equivalence after forgetting the involutions. It remains to check that the induced map

$$(S^\sigma)^{\mathbb{Z}/2} \to (B^\sigma\mathbb{Z})^{\mathbb{Z}/2} \tag{4.9}$$

is a homotopy equivalence as well.

By [17, Proposition 2.1.6], $(B^\sigma\mathbb{Z})^{\mathbb{Z}/2}$ is the classifying space of the category $\mathrm{Sym}\,\mathbb{Z}$, whose objects are integers and whose morphisms are given by

$$\mathrm{Hom}_{\mathrm{Sym}\,\mathbb{Z}}(a,b) := \left\{ \begin{array}{ll} * & \text{if } 2\mid a-b, \\ \emptyset & \text{if } 2\nmid a-b. \end{array} \right.$$

The classifying space of $\mathrm{Sym}\,\mathbb{Z}$ is obviously homotopy equivalent to S^0, and hence we deduce a homotopy equivalence

$$S^0 \simeq (B^\sigma\mathbb{Z})^{\mathbb{Z}/2}. \tag{4.10}$$

This implies that $\pi_n((B^\sigma\mathbb{Z})^{\mathbb{Z}/2})$ is trivial for every integer $n>0$ and $\pi_0((B^\sigma\mathbb{Z})^{\mathbb{Z}/2})$ consists of two points.

Hence, it suffices to show that the map

$$\pi_0((S^\sigma)^{\mathbb{Z}/2}) \to \pi_0((B^\sigma\mathbb{Z})^{\mathbb{Z}/2}) \tag{4.11}$$

induced by equation (4.9) is injective. Let sd_σ denote Segal's edgewise subdivision functor in [54]. In simplicial degree 1, the two degeneracy maps in $\mathrm{sd}_\sigma(N^\sigma\mathbb{Z})$ are given by

$$(x_1,x_2,x_3) \mapsto x_2, x_1+x_2+x_3.$$

Then edges (x_1,x_2,x_1) in the $\mathbb{Z}/2$-fixed point space connects x_2 and $2x_1+x_2$. In simplicial degree 0, $\mathrm{sd}_\sigma(S^\sigma)$ consists of two vertices 0 and 1, whose images in $B^\sigma\mathbb{Z}$ are not connected. Hence, equation (4.11) is injective as claimed. $\qquad\square$

Together with equation (4.6), we obtain an equivalence

$$B^{\mathrm{di}}(\mathbb{Z};j) \simeq S^\sigma \tag{4.12}$$

for all $j \in \mathbb{Z}$.

Definition 4.2.9. Let $\mathbb{Z}^\sigma$ denote the monoid $\mathbb{Z}$ with the involution $x \mapsto -x$ for $x \in \mathbb{Z}$.

Proposition 4.2.10. *There is a $\mathbb{Z}/2$-homotopy equivalence*

$$B^\sigma\mathbb{Z}^\sigma \simeq S^1.$$

Proof. See [16, Example 5.13]. $\qquad\square$

Combine this with equation (4.8) to obtain a $\mathbb{Z}/2$-homotopy equivalence

$$B^{\mathrm{di}}\mathbb{Z}^\sigma \simeq S^1 \amalg \coprod_{j>0} (\mathbb{Z}/2 \times S^1).$$

Together with the description of i_* in equation (A.15), we obtain an equivalence in $\mathrm{Sp}_{\mathbb{Z}/2}$

$$\mathbb{S}[\mathrm{B}^{\mathrm{di}}\mathbb{Z}^\sigma] \simeq \mathbb{S}[S^1] \oplus \bigoplus_{j>0} i_* i^* \mathbb{S}[S^1]. \tag{4.13}$$

For every integer j, let $S^\sigma(j)$ denote the space S^1 with the $O(2)$-action, whose $SO(2)$-action is given by $(t,x) \mapsto t^j x$ for $t \in SO(2)$ and $x \in S^1$, and whose $\mathbb{Z}/2$-action is given by the complex conjugate $x \mapsto \overline{x}$. Here, we regard S^1 as the unit circle in $\mathbb{C}$. For $j \geq 0$, let Δ_σ^j be the $\mathbb{Z}/2$-space whose involution is the reflection mapping the vertex i to $j-i$ for all $0 \leq i \leq j$.

The following computations refine [49, Proposition 3.21].

Proposition 4.2.11. *We have* $\mathrm{B}^{\mathrm{di}}(\mathbb{N};0) \cong *$. *For every integer* $j \geq 1$, *there is an* $O(2)$-*equivariant homeomorphism*

$$\mathrm{B}^{\mathrm{di}}(\mathbb{N};j) \simeq (S^\sigma \times \Delta_\sigma^{j-1})/C_j, \tag{4.14}$$

where the C_j-*action on* Δ_σ^{j-1} *is the cyclic permutation. Hence, there is an* $O(2)$-*equivariant deformation retract*

$$\mathrm{B}^{\mathrm{di}}\mathbb{N} \simeq * \amalg \coprod_{j \geq 1} S^\sigma(j). \tag{4.15}$$

Proof. The $(j-1)$-simplex $(1,\ldots,1)$ generates $\mathrm{N}^{\mathrm{cy}}(\mathbb{N};j)$ as a cyclic set. Use this to construct a surjective map of cyclic sets

$$\Lambda^{j-1} \to \mathrm{N}^{\mathrm{cy}}(\mathbb{N};j). \tag{4.16}$$

In simplicial degree q, this can be written as the map

$$C_{q+1} \times \mathrm{Hom}([q],[j-1]) \to \{(x_0,\ldots,x_q) \in \mathbb{N}^{q+1} : x_0 + \cdots + x_q = j\}.$$

sending (t,f) to

$$(f(t(0)) - f(t(q)), f(t(1)) - f(t(0)), \ldots, f(t(q)) - f(t(q-1))).$$

In this formulation, the values are calculated modulo j. Let Λ_σ^{j-1} be the dihedral set whose underlying cyclic set is Λ^{j-1} and whose involution is given by

$$(t,f) \mapsto (q+1-t, \rho \circ f),$$

where $\rho\colon [j-1] \to [j-1]$ is the map sending $x \in [j-1]$ to $j-1-x$. Then equation (4.16) becomes a morphism of dihedral sets

$$\Lambda_\sigma^{j-1} \to \mathrm{N}^{\mathrm{di}}(\mathbb{N};j). \tag{4.17}$$

The cyclic geometric realization of equation (4.16) is S^1-homeomorphic to the quotient map

$$S^1 \times \Delta^{j-1} \to (S^1 \times \Delta^{j-1})/C_j;$$

see the proof of [28, Lemma 2.2.3] or [49, Proposition 3.20]. Use the observation in Example 4.1.4 to show that the real geometric realization of $|\Lambda_\sigma^{j-1}|$ is $\mathbb{Z}/2$-homeomorphic

to $S^\sigma \times \Delta^{j-1}_\sigma$. Combine these two facts to deduce that the dihedral geometric realization of equation (4.17) is $O(2)$-homeomorphic to the quotient map

$$S^\sigma \times \Delta^{j-1}_\sigma \to (S^\sigma \times \Delta^{j-1}_\sigma)/C_j$$

In particular, we obtain equation (4.14). Since Δ^{j-1}_σ is D_j-contractible to its barycenter, we obtain equation (4.15). □

Let us review what the proof of [49, Proposition 3.21] contains. For every integer $r \geq 1$, let sd_r denote the r-fold edgewise subdivision functor in [11]. For every integer j, the r-fold power map $\mathrm{N}^{\mathrm{di}}(\mathbb{Z};j) \to \mathrm{sd}_r \mathrm{N}^{\mathrm{di}}(\mathbb{Z};j)$ given by

$$(m_0,\ldots,m_q) \mapsto (m_0,\ldots,m_q,\ldots,m_0,\ldots,m_q) \tag{4.18}$$

induces a homeomorphism

$$\mathrm{B}^{\mathrm{di}}(\mathbb{Z};j) \xrightarrow{\cong} \mathrm{B}^{\mathrm{di}}(\mathbb{Z};rj)^{C_r}. \tag{4.19}$$

If $r \nmid j$, then we have

$$\mathrm{B}^{\mathrm{di}}(\mathbb{Z};j)^{C_r} = \emptyset. \tag{4.20}$$

Suppose $j \geq 0$. We similarly have a homeomorphism

$$\mathrm{B}^{\mathrm{di}}(\mathbb{N};j) \xrightarrow{\cong} \mathrm{B}^{\mathrm{di}}(\mathbb{N};rj)^{C_r} \tag{4.21}$$

If $r \nmid j$, then we have

$$\mathrm{B}^{\mathrm{di}}(\mathbb{N};j)^{C_r} = \emptyset. \tag{4.22}$$

If H is a closed subgroup of $SO(2)$, then the induced map

$$\mathrm{B}^{\mathrm{di}}(\mathbb{N};j)^H \to \mathrm{B}^{\mathrm{di}}(\mathbb{Z};j)^H \tag{4.23}$$

is a homotopy equivalence.

Proposition 4.2.12. *For every integer $j \geq 1$, the induced map*

$$\mathrm{B}^{\mathrm{di}}(\mathbb{N};j) \to \mathrm{B}^{\mathrm{di}}(\mathbb{Z};j)$$

is an $O(2)$-equivariant homotopy equivalence.

Proof. We need to show that equation (4.23) is a homotopy equivalence for all closed subgroups H of $O(2)$.

If $H = O(2)$, then $\mathrm{B}^{\mathrm{di}}(\mathbb{N};j)^{O(2)} = \mathrm{B}^{\mathrm{di}}(\mathbb{Z};j)^{O(2)} = \emptyset$ by equations (4.20) and (4.22). Hence, the remaining case is $H = D_r$ for every integer $r \geq 1$. Since equation (4.18) commutes with w, equation (4.21) is $\mathbb{Z}/2$-equivariant. Similarly, equation (4.19) is $\mathbb{Z}/2$-equivariant. Hence, we have the induced homeomorphisms

$$\mathrm{B}^{\mathrm{di}}(\mathbb{N};j)^{\mathbb{Z}/2} \xrightarrow{\cong} \mathrm{B}^{\mathrm{di}}(\mathbb{N};rj)^{D_r} \text{ and } \mathrm{B}^{\mathrm{di}}(\mathbb{Z};j)^{\mathbb{Z}/2} \xrightarrow{\cong} \mathrm{B}^{\mathrm{di}}(\mathbb{Z};rj)^{D_r}.$$

Combine with equation (4.22) to reduce to the case when $H = \mathbb{Z}/2$.

Propositions 4.2.6 and 4.2.8 give a homotopy equivalence

$$\mathrm{B}^{\mathrm{di}}(\mathbb{Z};j)^{\mathbb{Z}/2} \simeq S^0. \tag{4.24}$$

By Proposition 4.2.11, we also have a homotopy equivalence $\mathrm{B}^{\mathrm{di}}(\mathbb{N};j)^{\mathbb{Z}/2} \simeq S^0$. Hence, it remains to check that the induced map

$$\pi_0(\mathrm{B}^{\mathrm{di}}(\mathbb{N};j)^{\mathbb{Z}/2}) \to \pi_0(\mathrm{B}^{\mathrm{di}}(\mathbb{Z};j)^{\mathbb{Z}/2}) \tag{4.25}$$

is a bijection. Recall that sd_σ denotes Segal's edgewise subdivision functor. In simplicial degree 1, the two degeneracy maps in $\mathrm{sd}_\sigma(\mathrm{N}^{\mathrm{di}}(\mathbb{N};j))$ and $\mathrm{sd}_\sigma(\mathrm{N}^{\mathrm{di}}(\mathbb{Z};j))$ are given by

$$(x_0, x_1, x_2, x_3) \mapsto (x_3 + x_0 + x_1, x_2), (x_0, x_1 + x_2 + x_3).$$

The edge (x_0, x_1, x_2, x_1) in the $\mathbb{Z}/2$-fixed point spaces connects $(x_0, 2x_1 + x_2)$ and $(x_0 + 2x_1, x_2)$. Hence, we have

$$\pi_0(\mathrm{B}^{\mathrm{di}}(\mathbb{N};j)) \cong \{(x_0, x_1) \in \mathbb{N}^2 : x_0 + x_1 = j\}/ \sim$$

and

$$\pi_0(\mathrm{B}^{\mathrm{di}}(\mathbb{Z};j)) \cong \{(x_0, x_1) \in \mathbb{Z}^2 : x_0 + x_1 = j\}/ \sim,$$

where $(x_0, x_1) \sim (x_0', x_1')$ if and only if $x_0 - x_0'$ is even. This shows that equation (4.25) is a bijection. $\square$

Proposition 4.2.13. *For a commutative monoid M with involution, there is an equivalence in $\mathrm{Sp}_{\mathbb{Z}/2}$*

$$\mathrm{THR}(\mathbb{S}[M]) \simeq \mathbb{S}[\mathrm{B}^{\mathrm{di}}M].$$

Proof. See [35], and also [16, Proposition 5.9]. $\square$

Proposition 4.2.14. *Let A be a commutative ring with involution, and let M be a commutative monoid with involution. Then there is an equivalence in $\mathrm{Sp}_{\mathbb{Z}/2}$*

$$\mathrm{H}(A[M]) \simeq \mathrm{H}A \wedge \mathbb{S}[M].$$

Proof. If we regard M as a $\mathbb{Z}/2$-set, then M is a coproduct of copies $\mathbb{Z}/2$ and e. If $M = e$, then the claim is clear. Hence, it suffices to show that there is an equivalence

$$\mathrm{H}((i^*A)^{\oplus \mathbb{Z}/2}) \simeq \mathrm{H}A \wedge \Sigma^\infty(\mathbb{Z}/2)_+, \tag{4.26}$$

where $(i^*A)^{\oplus \mathbb{Z}/2}$ is the commutative monoid $A \oplus A$ with the involution given by $(x,y) \mapsto (y,x)$. By equation (2.7) and Propositions Appendix A.1.10, Appendix A.2.7(3) and 2.3.3, we obtain equivalences

$$\mathrm{H}A \wedge i_\sharp i^*\mathbb{S} \simeq i_\sharp(i^*\mathrm{H}A \wedge i^*\mathbb{S}) \simeq i_\sharp \mathrm{H}i^*A \simeq i_* \mathrm{H}i^*A \simeq \mathrm{H}((i^*A)^{\oplus \mathbb{Z}/2}).$$

Together with the equivalence $i_\sharp i^*\mathbb{S} \simeq \Sigma^\infty(\mathbb{Z}/2)_+$, we obtain equation (4.26). $\square$

Proposition 4.2.15. *Let A be a commutative ring with involution, and let M be a commutative monoid with involution. Then there is a canonical equivalence in $\mathrm{Sp}_{\mathbb{Z}/2}$*

$$\mathrm{THR}(A[M]) \simeq \mathrm{THR}(A) \wedge \mathbb{S}[\mathrm{B}^{\mathrm{di}}M].$$

Proof. By Propositions 2.1.5 and 4.2.13, we have equivalences

$$\mathrm{THR}(\mathrm{H}A \wedge \mathbb{S}[M]) \simeq \mathrm{THR}(\mathrm{H}A) \wedge \mathrm{THR}(\mathbb{S}[M]) \simeq \mathrm{THR}(A) \wedge \mathbb{S}[\mathrm{B}^{\mathrm{di}}M].$$

Proposition 4.2.14 finishes the proof. □

5. Properties of real topological Hochschild homology

5.1. THR of the projective line

We now establish the first computations of THR for nonaffine schemes, namely $\mathbb{P}^1$ and $\mathbb{P}^\sigma$.

Remark 5.1.1. Hermitian K-theory **KO** resp. **KR** is not an orientable theory, that is the usual projective bundle formula as for Chow groups and algebraic K-theory does not hold. The computation of $\mathbb{P}^1$ over regular rings is [31, Proposition 6.1], where a (8,4)-motivic periodic spectrum **KO** is constructed. This computation is extended by [52, Theorem 9.10] to rather general base schemes (still with 2 invertible). For the projective line with involution, a variation of Schlichting's proof leads to the computation of $\mathbf{KR}(\mathbb{P}^\sigma)$, see [12, Theorem 5.1] and compare [61, Theorem 7.1] and [34] for different proofs. In particular, this leads to an equivariant motivic spectrum **KR** which is $\mathbb{P}^1 \wedge \mathbb{P}^\sigma$-periodic. For further periodicities of **KR**, see [34, Theorem 10]. In their notation, we have $\mathbb{P}^1 \simeq S^1 \wedge S^\alpha$ and $\mathbb{P}^\sigma \simeq S^\gamma \wedge S^{\gamma\alpha} \simeq \mathbb{P}^1_-$. For the definition of $S^\gamma = S^\sigma$ in the motivic setting and a proof of the last equivalence, we refer to [12, section 2.5]. Although THR is not $\mathbb{A}^1$-invariant, it seems reasonable to expect that the formulas for $\mathrm{THR}(\mathbb{P}^n)$ are similar to those for **KR**. In the cases considered below this is indeed the case: The following Proposition implies that $\Omega^{1+\alpha}\mathrm{THR} \simeq \Sigma^{\gamma-1}\mathrm{THR}$. Smashing with S^1, we obtain the same periodicity as (35) in [34]. Similarly, the next proposition corresponds to (36) of loc. cit.

The following computations rely on Proposition 4.2.13 and the computations for dihedral nerves in the previous section.

Theorem 5.1.2. *For any $X \in \mathbf{Sch}_{\mathbb{Z}/2}$, there is an equivalence of $\mathbb{Z}/2$-spectra*

$$\mathrm{THR}(X \times \mathbb{P}^1) \simeq \mathrm{THR}(X) \oplus \Sigma^{\sigma-1}\mathrm{THR}(X).$$

Proof. For notational convenience, we will write the proof as if everything takes place over $\mathbb{S}$ rather than X. Using the description of THR for spherical groups rings from Proposition 4.2.13 and its extension to log schemes with involution from Proposition 4.2.15, we are reduced to consider the following homotopy co-Cartesian square, corresponding to the standard Zariski cover of $\mathbb{P}^1$ by two copies of $\mathbb{A}^1$ and using Corollary 3.4.9 and the description of $\mathrm{THR}(\mathbb{S}[M])$ from Proposition 4.2.13:

$$
\begin{array}{ccc}
\mathrm{THR}(\mathbb{P}^1_{\mathbb{S}}) & \longrightarrow & \mathbb{S}[\mathrm{B}^{\mathrm{di}}\mathbb{N}] \\
\downarrow & & \downarrow \\
\mathbb{S}[\mathrm{B}^{\mathrm{di}}(-\mathbb{N})] & \longrightarrow & \mathbb{S}[\mathrm{B}^{\mathrm{di}}\mathbb{Z}].
\end{array}
$$

Here, the notation $\mathbb{N}$ and $-\mathbb{N}$ indicates the two different embeddings of $\mathbb{A}^1$ in $\mathbb{P}^1$. It is crucial to notice that even as $\mathbb{G}_m$ has trivial involution, the involution given by the dihedral nerve (see Definition 4.2.2 above) yields nontrivial involutions. Using the decomposition of equation (4.4) and Propositions 4.2.6, 4.2.8 and 4.2.11, we obtain the following $\mathbb{Z}/2$-equivariant (homotopy) co-Cartesian square:

$$
\begin{array}{ccc}
\mathrm{THR}(\mathbb{P}^1_{\mathbb{S}}) & \longrightarrow & \mathbb{S}[* \amalg \coprod_{j \geq 1} S^\sigma] \\
\downarrow & & \downarrow \\
\mathbb{S}[* \amalg \coprod_{j \leq -1} S^\sigma] & \longrightarrow & \mathbb{S}[\coprod_{j \in \mathbb{Z}} S^\sigma].
\end{array}
$$

An obvious cancellation, using Proposition 4.2.12 on the relevant maps, yields the following $\mathbb{Z}/2$-equivariant (homotopy) co-Cartesian square

$$
\begin{array}{ccc}
\mathrm{THR}(\mathbb{P}^1_{\mathbb{S}}) & \longrightarrow & \mathbb{S}[*] \\
\downarrow & & \downarrow \\
\mathbb{S}[*] & \longrightarrow & \mathbb{S}[S^\sigma]
\end{array}
$$

and the result follows. $\square$

We now turn to the slightly more subtle computation of the projective line $\mathbb{P}^\sigma$ with involution. Recall that $(i_\sharp, i^*, i_*)$ denotes the free-forgetful-cofree adjunction for the map $i \colon \mathrm{pt} \to \mathrm{B}(\mathbb{Z}/2)$. Let $\Sigma^\sigma \colon \mathrm{Sp}_{\mathbb{Z}/2} \to \mathrm{Sp}_{\mathbb{Z}/2}$ be the functor $\Sigma^\infty S^\sigma \wedge (-)$, which has an inverse functor $\Sigma^{-\sigma}$ since the sphere $\Sigma^\infty S^\sigma$ is $\wedge$-invertible in $\mathrm{Sp}_{\mathbb{Z}/2}$. For integers m and n, we set $\Sigma^{m+n\sigma} := \Sigma^m (\Sigma^\sigma)^{\wedge n}$, which is a functor $\mathrm{Sp}_{\mathbb{Z}/2} \to \mathrm{Sp}_{\mathbb{Z}/2}$. We also set $\Sigma^{m+n\sigma} := \Sigma^{m+n\sigma} \mathbb{S} \in \mathrm{Sp}_{\mathbb{Z}/2}$ for abbreviation.

For an adjoint pair of ∞-categories $F \colon \mathcal{C} \rightleftarrows \mathcal{D} \colon G$, let $ad \colon \mathrm{id} \to GF$ (resp. $ad' \colon FG \to \mathrm{id}$) denote the unit (resp. counit).

Lemma 5.1.3. *There is a natural equivalence of functors*

$$
\Sigma^{-\sigma} \simeq \mathrm{fib}(\mathrm{id} \xrightarrow{ad} i_* i^*).
$$

Proof. Consider the co-Cartesian square of $\mathbb{Z}/2$-spaces

$$
\begin{array}{ccc}
S^\sigma - \{(1,0),(-1,0)\} & \longrightarrow & S^\sigma - \{(1,0)\} \\
\downarrow & & \downarrow \\
S^\sigma - \{(-1,0)\} & \longrightarrow & S^\sigma.
\end{array}
$$

There are $\mathbb{Z}/2$-homotopy equivalences $S^\sigma - \{(1,0),(-1,0)\} \simeq \mathbb{Z}/2$ and $S^\sigma - \{(1,0)\} \simeq S^\sigma - \{(-1,0)\} \simeq \mathrm{pt}$. Together with the explicit descriptions of $i_\sharp$ and i^* in Construction Appendix A.2.4, we obtain a natural equivalence

$$
\Sigma^\sigma \simeq \mathrm{cofib}(i_\sharp i^* \xrightarrow{ad'} \mathrm{id}). \tag{5.1}
$$

By adjunction, we obtain the desired natural equivalence. $\square$

Theorem 5.1.4. *For any $X \in \mathbf{Sch}_{\mathbb{Z}/2}$, there is an equivalence of $\mathbb{Z}/2$-spectra*

$$\mathrm{THR}(X \times \mathbb{P}^\sigma) \simeq \mathrm{THR}(X) \oplus \Sigma^{1-\sigma} \mathrm{THR}(X).$$

Proof. As before, we will write the proof as if everything takes place over $\mathbb{S}$. Consider the Cartesian equivariant Nisnevich square

$$
\begin{array}{ccc}
\mathbb{Z}/2 \times \mathbb{G}_{m,\mathbb{S}}^\sigma & \longrightarrow & (\mathbb{P}_{\mathbb{S}}^1 - \infty) \amalg (\mathbb{P}_{\mathbb{S}}^1 - 0) \\
\downarrow & & \downarrow \\
\mathbb{G}_{m,\mathbb{S}}^\sigma & \longrightarrow & \mathbb{P}_{\mathbb{S}}^\sigma
\end{array}
\qquad (5.2)
$$

as in [12, Lemma 2.23], where the $\mathbb{Z}/2$-action on the upper right corner is induced by the action on $\mathbb{P}_{\mathbb{S}}^\sigma$.

Propositions 2.1.3 and 3.4.7 give equivalences

$$\mathrm{THR}((\mathbb{P}_{\mathbb{S}}^1 - \infty) \amalg (\mathbb{P}_{\mathbb{S}}^1 - 0)) \simeq i_* \mathrm{THH}(i^* \mathbb{A}_{\mathbb{S}}^1) \simeq i_* i^* \mathrm{THR}(\mathbb{A}_{\mathbb{S}}^1).$$

We similarly have an equivalence $\mathrm{THR}(\mathbb{Z}/2 \times \mathbb{G}_{m,\mathbb{S}}^\sigma) \simeq i_* i^* \mathrm{THR}(\mathbb{G}_{m,\mathbb{S}}^\sigma)$ since there are isomorphisms $\mathbb{Z}/2 \times \mathbb{G}_{m,\mathbb{S}}^\sigma \cong \mathbb{Z}/2 \times \mathbb{G}_{m,\mathbb{S}}$ and $i^* \mathbb{G}_{m,\mathbb{S}}^\sigma \cong i^* \mathbb{G}_{m,\mathbb{S}}$. The induced map $\mathrm{THR}(\mathbb{G}_{m,\mathbb{S}}^\sigma) \to \mathrm{THR}(\mathbb{Z}/2 \times \mathbb{G}_{m,\mathbb{S}}^\sigma)$ can be identified with the map

$$\mathrm{THR}(\mathbb{G}_{m,\mathbb{S}}^\sigma) \to i_* i^* \mathrm{THR}(\mathbb{G}_{m,\mathbb{S}}^\sigma)$$

obtained by the unit of the adjunction pair (i^*, i_*). As in the proof of Theorem 5.1.2, use Propositions 4.2.6, 4.2.8, 4.2.11 and 4.2.12 to see that the induced map $\mathrm{THR}(\mathbb{A}_{\mathbb{S}}^1) \to \mathrm{THR}(\mathbb{G}_{m,\mathbb{S}})$ can be identified with the map

$$\mathbb{S} \oplus \bigoplus_{j>0} \Sigma_+^\infty S^\sigma \to \mathbb{S} \oplus \bigoplus_{j>0} i_* i^* \Sigma_+^\infty S^\sigma$$

obtained by the unit of the adjunction pair (i^*, i_*). By equation (4.13), we obtain an equivalence

$$\mathrm{THR}(\mathbb{G}_{m,\mathbb{S}}^\sigma) \simeq \mathbb{S} \oplus \bigoplus_{j>0} i_* i^* \Sigma_+^\infty S^1.$$

Applying THR to equation (5.2) and combining with the above discussion yield the following homotopy co-Cartesian square:

$$
\begin{array}{ccc}
\mathrm{THR}(\mathbb{P}_{\mathbb{S}}^\sigma) & \longrightarrow & i_* i^*(\mathbb{S} \oplus \bigoplus_{j>0} \Sigma_+^\infty S^\sigma) \\
& & \downarrow \simeq \\
& & i_* i^*(\mathbb{S} \oplus \bigoplus_{j>0} \Sigma_+^\infty S^1) \\
\downarrow & & \downarrow \\
\Sigma_+^\infty S^1 \oplus \bigoplus_{j>0} i_* i^* \Sigma_+^\infty S^1 & \longrightarrow & i_* i^*(\Sigma_+^\infty S^1 \oplus \bigoplus_{j>0} i_* i^* \Sigma_+^\infty S^1).
\end{array}
\qquad (5.3)
$$

Consider the commutative square

$$Q := \begin{array}{ccc} 0 & \longrightarrow & \bigoplus_{j>0} i_* i^* \Sigma^\infty_+ S^1 \\ \downarrow & & \downarrow \\ \bigoplus_{j>0} i_* i^* \Sigma^\infty_+ S^1 & \longrightarrow & \bigoplus_{j>0} i_* i^* i_* i^* \Sigma^\infty_+ S^1 \end{array} \tag{5.4}$$

extracted from equation (5.3), where the right vertical map (resp. lower horizontal map) is obtained by applying $i_* i^*$ to the left (resp. right) of the unit map $\mathrm{id} \to i_* i^*$. Since $\Phi^{\mathbb{Z}/2} i_* \simeq 0$ by Proposition Appendix A.2.7(3),(5), $\Phi^{\mathbb{Z}/2} Q$ is Cartesian. The objects in the square $i^* Q$ are direct sums of $\bigoplus_{j>0} \Sigma^\infty_+ S^1$, and the right vertical and lower horizontal maps are the matrix multiplications given by

$$\begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}$$

for certain choices of bases. From this, one can check that $i^* Q$ is Cartesian too. It follows that Q is Cartesian.

Hence, the other direct summand of the square (5.3)

$$\begin{array}{ccc} \mathrm{THR}(\mathbb{P}^\sigma_{\mathbb{S}}) & \longrightarrow & i_* i^* \Sigma^0 \\ \downarrow & & \downarrow \\ \Sigma^0 \oplus \Sigma^1 & \longrightarrow & i_* i^* \Sigma^0 \oplus i_* i^* \Sigma^1 \end{array}$$

is also Cartesian. It follows that $\mathrm{THR}(\mathbb{P}^\sigma_{\mathbb{S}})$ is equivalent to the direct sum

$$\lim(\Sigma^0 \to i_* i^* \Sigma^0 \leftarrow i_* i^* \Sigma^0) \oplus \lim(\Sigma^1 \to i_* i^* \Sigma^1 \leftarrow 0).$$

Together with Lemma 5.1.3, we obtain the desired equivalence.   □

Proposition 5.1.5. *There is an equivalence of $\mathbb{Z}/2$-spectra*

$$\mathrm{THR}(X) \simeq \Omega_{\mathbb{P}^1 \wedge \mathbb{P}^\sigma} \mathrm{THR}(X).$$

Proof. Combine Theorems 5.1.2 and 5.1.4.   □

5.2. THR of projective spaces

Definition 5.2.1. As usual, for any integer $n \geq 1$, we consider the n-cube $(\Delta^1)^n$ as a partially ordered set, and use the same symbol for the associated category. For an ∞-category $\mathcal{C}$, an *n-cube in $\mathcal{C}$* is a functor

$$Q \colon \mathrm{N}(\Delta^1)^n \to \mathcal{C},$$

compare [42, Definition 6.1.1.2]. If $\mathcal{C}$ admits limits, the *total fiber of Q* is defined to be

$$\mathrm{tfib}(Q) := \mathrm{fib}(Q(0,\ldots,0) \to \lim Q|_{(\Delta^1)^n - \{(0,\ldots,0)\}}).$$

The following ∞-categorical result can be shown by dualizing the arguments from [8, Proposition A.6.5].

Proposition 5.2.2. *Let Q be an n-cube in an ∞-category $\mathcal{C}$ with small limits, where n is a nonnegative integer. Then for every integer $1 \le i \le n$, there exists a fiber sequence*

$$\mathrm{tfib}(Q) \to \mathrm{tfib}(Q|_{(\Delta^1)^{i-1} \times \{0\} \times (\Delta^1)^{n-i-1}}) \to \mathrm{tfib}(Q|_{(\Delta^1)^{i-1} \times \{1\} \times (\Delta^1)^{n-i-1}}).$$

Proposition 5.2.3. *Let $\mathcal{C}$ be a symmetric monoidal stable ∞-category with small limits such that the tensor product operation on $\mathcal{C}$ preserves fiber sequences in each variable. If Q is an n-cube and $f\colon X_0 \to X_1$ is a map in $\mathcal{C}$, then there is a canonical equivalence*

$$\mathrm{tfib}(Q) \otimes \mathrm{fib}(f) \simeq \mathrm{tfib}(Q \otimes f),$$

where $Q \otimes f$ is the associated $(n+1)$-cube sending $(a_1,\ldots,a_{n+1}) \in (\Delta^1)^{n+1}$ to $Q(a_1,\ldots,a_n) \otimes X_{a_{n+1}}$.

Proof. This is again obtained by dualizing the arguments for the corresponding one [8, Proposition A.6.7]. An intermediate step is to show $\mathrm{tfib}(Q) \otimes X \simeq \mathrm{tfib}(Q \otimes X)$ for any $X \in \mathcal{C}$, where $Q \otimes X$ is the associated n-cube sending $(a_1,\ldots,a_n) \in (\Delta^1)^n$ to $Q(a_1,\ldots,a_n) \otimes X$. Then one can use Proposition 5.2.2. $\square$

Proposition 5.2.4. *Let $\mathcal{C}$ be a symmetric monoidal stable ∞-category with small limits such that the tensor product operation on $\mathcal{C}$ preserves fiber sequences in each variable. If $i_1\colon X_{0,1} \to X_{1,1}, \ldots, i_n\colon X_{0,n} \to X_{1,n}$ are maps in $\mathcal{C}$, then there is a canonical equivalence*

$$\mathrm{fib}(i_1) \otimes \cdots \otimes \mathrm{fib}(i_n) \simeq \mathrm{tfib}(i_1 \otimes \cdots \otimes i_n),$$

where $i_1 \otimes \cdots \otimes i_n$ is the associated n-cube sending $(a_1,\ldots,a_n) \in (\Delta^1)^n$ to $X_{a_1,1} \otimes \cdots \otimes X_{a_n,n}$ and arrows given by tensor products of i_js and identities.

Proof. Use Proposition 5.2.3 repeatedly. $\square$

Proposition 5.2.5. *Suppose $X \in \mathbf{Sch}_{\mathbb{Z}/2}$, and let $\{U_1,\ldots,U_n\}$ be a Zariski covering of X with the induced involutions. Let Q be the S-cube given by*

$$Q(\{i_1,\ldots,i_r\}) := U_{i_1} \cap \cdots \cap U_{i_r}$$

for nonempty $\{i_1,\ldots,i_r\} \subset S := \{1,\ldots,n\}$ and $Q(\emptyset) := X$. Then there is an equivalence

$$\mathrm{tfib}(\mathrm{THR}(Q)) \simeq 0.$$

Proof. Let us use the equivalence $\mathrm{N}(\mathbf{P}(S)) \simeq (\Delta^1)^n$. We include the description of this equivalence if $S = \{1\}$. We regard the partially ordered set $\mathbf{P}(\{1\})$ as the category associated with the diagram $\emptyset \to \{1\}$. The nerve of this category is Δ^1.

We proceed by induction on n. The claim is clear if $n = 1$. Assume $n > 1$. By induction, we fiber sequences

$$\mathrm{tfib}(\mathrm{THR}(Q|_{\{0\} \times (\Delta^1)^{n-1}})) \to \mathrm{THR}(X) \to \mathrm{THR}(U_2 \cup \cdots \cup U_n)$$

and

$$\mathrm{tfib}(\mathrm{THR}(Q|_{\{1\} \times (\Delta^1)^{n-1}})) \to \mathrm{THR}(U_1) \to \mathrm{THR}(U_1 \cap (U_2 \cup \cdots \cup U_n)).$$

Together with Proposition 5.2.2, we reduce to showing that the induced square

$$
\begin{array}{ccc}
\mathrm{THR}(X) & \longrightarrow & \mathrm{THR}(U_2 \cup \cdots \cup U_n) \\
\downarrow & & \downarrow \\
\mathrm{THR}(U_1) & \longrightarrow & \mathrm{THR}(U_1 \cap (U_2 \cup \cdots \cup U_n))
\end{array}
$$

is Cartesian. This follows from Corollary 3.4.9. $\qquad\square$

Theorem 5.2.6. *For any $X \in \mathbf{Sch}_{\mathbb{Z}/2}$ and integer $n \geq 0$, there is an equivalence of $\mathbb{Z}/2$-spectra*

$$
\mathrm{THR}(X \times \mathbb{P}^n) \simeq \left\{
\begin{array}{ll}
\mathrm{THR}(X) \oplus \bigoplus_{j=1}^{\lfloor n/2 \rfloor} i_* \mathrm{THH}(X) & \text{if } n \text{ is even,} \\
\mathrm{THR}(X) \oplus \bigoplus_{j=1}^{\lfloor n/2 \rfloor} i_* \mathrm{THH}(X) \oplus \Sigma^{n(\sigma-1)} \mathrm{THR}(X) & \text{if } n \text{ is odd.}
\end{array}
\right.
$$

Proof. Proposition Appendix A.2.7(3) allows us to replace i_* by $i_\sharp$ in the claim. We set

$$
M_j := \{(x_1, \ldots, x_n) \in \mathbb{Z}^n : x_j \geq 0\}
$$

for $j = 1, \ldots, n$ and

$$
M_{n+1} := \{(x_1, \ldots, x_n) \in \mathbb{Z}^n : x_1 + \cdots + x_n \leq 0\}.
$$

Observe that there is an isomorphism of commutative monoids $M_j \cong \mathbb{Z}^{n-1} \oplus \mathbb{N}$ for all $j = 1, \ldots, n+1$. We set $M_I := M_{i_1} \cap \cdots \cap M_{i_q}$ for all nonempty subsets $I := \{i_1, \ldots, i_q\}$ of $\{1, \ldots, n+1\}$, and consequently $M_\emptyset := \mathbb{Z}^n$. Together with the obvious maps, we obtain an $(n+1)$-cube M in commutative monoids associated with M_I for any subset $I \subset \{1, \ldots, n+1\}$. Here, we set $M(b_1, \ldots, b_{n+1}) = M_I$, where I is the set of indices i such that $b_i = 0$. By equation (4.4), we have a canonical decomposition

$$
\mathrm{B}^{\mathrm{di}} M_I \simeq \coprod_{v \in \mathbb{Z}^n} \mathrm{B}^{\mathrm{di}}(M_I; v),
$$

where $\mathrm{B}^{\mathrm{di}}(M_I; v) := \emptyset$ if $v \notin M_I$. Hence, for every v the above $\mathrm{B}^{\mathrm{di}}(M_I; v)$ assemble to an $(n+1)$-cube in $\mathbb{Z}/2$-spaces, and a decomposition of the $(n+1)$-cube $\mathrm{B}^{\mathrm{di}} M$ into these smaller cubes as v varies in $\mathbb{Z}^n$. Combine Propositions 4.2.4 and 4.2.12 to show that the induced map in this cube

$$
\mathrm{B}^{\mathrm{di}}(\mathbb{N} \oplus \mathbb{N}^s \oplus \mathbb{Z}^{n-s-1}; v) \to \mathrm{B}^{\mathrm{di}}(\mathbb{Z} \oplus \mathbb{N}^s \oplus \mathbb{Z}^{n-s-1}; v)
$$

is an equivalence for every integer $0 \leq s \leq n-1$ if the first coordinate of v is greater than 0. By a change of coordinates in the target, we see that the induced map

$$
\mathrm{B}^{\mathrm{di}}(M_{I \cup \{j\}}; v) \to \mathrm{B}^{\mathrm{di}}(M_I; v)
$$

is an equivalence for all $j = 1, \ldots, n+1$, subsets I of $\{1, \ldots, n+1\} - \{j\}$ and $v \in M_j^+$, where M_j^+ denotes the set of nonunits of M_j. Use Proposition 5.2.2 repeatedly to show

$$
\mathrm{tfib}(\mathbb{S}[\mathrm{B}^{\mathrm{di}}(M; v)]) \simeq 0 \tag{5.5}
$$

for all $v \in M_j^+$. This means $\mathrm{tfib}(\mathbb{S}[\mathrm{B}^{\mathrm{di}}(M; v)]) \simeq 0$ whenever $v \neq O$ since $M_\emptyset - (M_1^+ \cup \cdots \cup M_{n+1}^+) = \{O\}$, where O denotes the origin in $\mathbb{Z}^n$.

Now consider the standard cover of $\mathbb{P}^n_{\mathbb{S}}$ by $(n+1)$ copies of $\mathbb{A}^n_{\mathbb{S}}$, and recall that $\mathbb{A}^n_{\mathbb{S}}$ is the spherical monoid ring of $\mathbb{N}^n$. (We continue switching between spherical monoid rings and honest schemes over rings as before. Also, note that by Proposition 4.2.15 products of affine schemes correspond to products of monoids when computing THR.) Choosing suitable coordinates, the intersections of s elements of the cover with $0 < s \leq n+1$ are given by (the spherical monoid rings of) M_I above such that $|I| = n+1-s$. By Proposition 5.2.5, we have an equivalence

$$\mathrm{THR}(\mathbb{P}^n_{\mathbb{S}}) \simeq \lim \mathbb{S}[\mathrm{B}^{\mathrm{di}} M|_{(\Delta^1)^n - \{(0,\ldots,0)\}}].$$

Together with equation (5.5), we have an equivalence

$$\mathrm{THR}(\mathbb{P}^n_{\mathbb{S}}) \simeq \lim \mathbb{S}[\mathrm{B}^{\mathrm{di}}(M;O)|_{(\Delta^1)^n - \{(0,\ldots,0)\}}]$$

since $\mathrm{B}^{\mathrm{di}}(M_{\{1,\ldots,n\}};O) \cong \mathrm{B}^{\mathrm{di}}(M_{\{1,\ldots,n\}})$. For every subset I of $\{1,\ldots,n\}$, we have the canonical decomposition

$$\mathrm{B}^{\mathrm{di}}(M_I;O) \cong V_I \amalg *,$$

where V_I is obtained by removing the base point $*$ of $\mathrm{B}^{\mathrm{di}}(M_I;O)$ corresponding to the element $0 \in M_I$ in simplicial degree 0. This yields the canonical decomposition

$$\mathbb{S}[\mathrm{B}^{\mathrm{di}}(M;O)] \simeq Q \oplus Q',$$

where every entry of Q' is $\Sigma^\infty_+ * \simeq \mathbb{S}$. Use Proposition 5.2.2 repeatedly to have an equivalence $\mathrm{tfib}(Q') \simeq 0$. This implies that we have an equivalence

$$\lim Q'|_{(\Delta^1)^n - \{(0,\ldots,0)\}} \simeq \mathbb{S}.$$

On the other hand, we have $Q(0,\ldots,0) = 0$ since $M_{\{1,\ldots,n\}} = 0$. This implies that we have an equivalence

$$\mathrm{tfib}(Q) \simeq \Sigma^{-1} \lim Q|_{(\Delta^1)^n - \{(0,\ldots,0)\}}.$$

Combine what we have discussed above to have an equivalence

$$\mathrm{THR}(\mathbb{P}^n_{\mathbb{S}}) \simeq \mathbb{S} \oplus \Sigma^1 \mathrm{tfib}(Q). \tag{5.6}$$

We claim that

$$\mathrm{tfib}(Q|_{(\Delta^1)^{d+1} \times \{0\}^{n-d}}) \simeq \begin{cases} \bigoplus_{j=1}^{\lfloor d/2 \rfloor} i_\sharp \Sigma^{-1} & \text{if } d \text{ is even,} \\ \bigoplus_{j=1}^{\lfloor d/2 \rfloor} i_\sharp \Sigma^{-1} \oplus \Sigma^{d\sigma - d - 1} & \text{if } d \text{ is odd.} \end{cases}$$

Let us proceed by induction on d. The claim is clear if $d = 0$ by the definition of the total fiber. Assume $0 < d \leq n$. Let $\{e_1,\ldots,e_n\}$ be the standard basis in $\mathbb{Z}^n$. The d-cube $M|_{(\Delta^1)^d \times \{1\} \times \{0\}^{n-d}}$ is isomorphic to the naturally associated d-cube sending $(a_1,\ldots,a_d) \in (\Delta^1)^d$ to

$$P_{a_1}(e_1 - e_{d+1}) \oplus \cdots \oplus P_{a_d}(e_d - e_{d+1}) \oplus \mathbb{N}(-e_{d+1}) \oplus \mathbb{N}(e_{d+2} - e_{d+1}) \oplus \cdots \oplus \mathbb{N}(e_n - e_{d+1}),$$

where $P_0 := \mathbb{N}$, $P_1 := \mathbb{Z}$ and $e_{n+1} := 0$. Combine Propositions 4.2.5 and 5.2.4 to obtain an equivalence

$$\mathrm{tfib}(\mathbb{S}[\mathrm{B}^{\mathrm{di}}(M|_{(\Delta^1)^d \times \{1\} \times \{0\}^{n-d}}; O)])$$
$$\simeq \mathrm{fib}(\mathbb{S}[\mathrm{B}^{\mathrm{di}}(\mathbb{N}; 0)] \to \mathbb{S}[\mathrm{B}^{\mathrm{di}}(\mathbb{Z}; 0)])^{\wedge d} \wedge \mathbb{S}[\mathrm{B}^{\mathrm{di}}(\mathbb{N}; 0)]^{\wedge n-d}.$$

Use Proposition 5.2.2 repeatedly to deduce an equivalence

$$\mathrm{tfib}(Q'|_{(\Delta^1)^d \times \{1\} \times \{0\}^{n-d}}) \simeq 0.$$

Together with Proposition 4.2.11 and equation (4.12), we obtain equivalences

$$\mathrm{tfib}(Q|_{(\Delta^1)^d \times \{1\} \times \{0\}^{n-d}}) \simeq \mathrm{fib}(\mathbb{S} \to \mathbb{S}[S^\sigma])^{\wedge d} \wedge \mathbb{S}^{\wedge n-d}$$
$$\simeq \Sigma^{-d}(\Sigma^\infty S^\sigma / (1,0))^{\wedge d} \simeq \Sigma^{d\sigma - d},$$

where $(1,0)$ is the base point of S^σ. Proposition 5.2.2 gives a fiber sequence

$$\mathrm{tfib}(Q|_{(\Delta^1)^{d+1} \times \{0\}^{n-d}}) \to \mathrm{tfib}(Q|_{(\Delta^1)^d \times \{0\} \times \{0\}^{n-d}}) \to \mathrm{tfib}(Q|_{(\Delta^1)^d \times \{1\} \times \{0\}^{n-d}}).$$

If d is odd, then we obtain a fiber sequence

$$\mathrm{tfib}(Q|_{(\Delta^1)^{d+1} \times \{0\}^{n-d}}) \to \bigoplus_{j=1}^{\lfloor d/2 \rfloor} i_\sharp \Sigma^{-1} \xrightarrow{f} \Sigma^{d\sigma - d} \tag{5.7}$$

by induction. Since $\pi_{-1}(\Sigma^0) = 0$, the map $\Sigma^{-1} \to i^* \Sigma^{d\sigma - d} \simeq \Sigma^0$ obtained by adjunction is equivalent to 0. It follows that f is equivalent to 0, that is, equation (5.7) splits. This completes the induction argument for odd d.

If d is even, we obtain a fiber sequence

$$\mathrm{tfib}(Q|_{(\Delta^1)^{d+1} \times \{0\}^{n-d}}) \to \bigoplus_{j=1}^{\lfloor d/2 \rfloor - 1} i_\sharp \Sigma^{-1} \oplus \Sigma^{(d-1)\sigma - d} \to \Sigma^{d\sigma - d}$$

by induction. As above, the induced map $\bigoplus_{j=1}^{\lfloor d/2 \rfloor - 1} i_\sharp \Sigma^{-1} \to \Sigma^{d\sigma - d}$ is equivalent to 0. It follows that we have an equivalence

$$\mathrm{tfib}(Q|_{(\Delta^1)^{d+1} \times \{0\}^{n-d}}) \simeq \bigoplus_{j=1}^{\lfloor d/2 \rfloor - 1} i_\sharp \Sigma^{-1} \oplus \mathrm{fib}(\Sigma^{(d-1)\sigma - d} \xrightarrow{g} \Sigma^{d\sigma - d}).$$

We now analyze the nontrivial map $g \colon \Sigma^{(d-1)\sigma - d} \to \Sigma^{d\sigma - d}$. On the level of commutative monoids, this corresponds to the inclusion

$$\mathbb{Z}(u_1 - u_d) \oplus \cdots \oplus \mathbb{Z}(u_{d-1} - u_d) \oplus \mathbb{N}(-e_d) \oplus \mathbb{N}(e_{d+1} - e_d) \oplus \cdots \oplus \mathbb{N}(e_n - e_d)$$
$$\to \mathbb{Z}u_1 \oplus \cdots \oplus \mathbb{Z}u_d \oplus \mathbb{N}u_{d+1} \oplus \cdots \oplus \mathbb{N}u_n,$$

where $u_i := e_i - e_{d+1}$ for $i \in \{1, \ldots, n\} - \{d+1\}$ and $u_{d+1} := e_{d+1}$. As the 0-entry for $\mathrm{B}^{\mathrm{di}}\mathbb{N}$ is just a point, we only need to study the homomorphism $\mathbb{Z}^{d-1} \to \mathbb{Z}^d$ given by

$$(a_1, \ldots, a_{d-1}) \mapsto (a_1, \ldots, a_{n-1}, -a_1 - \cdots - a_{d-1}).$$

In the degree $(0,\ldots,0)$, this is easily seen to induce via Proposition 4.2.8 the map $(S^\sigma)^{\times d-1} \to (S^\sigma)^{\times d}$ given by

$$(x_1,\ldots,x_{d-1}) \mapsto (x_1,\ldots,x_{d-1},x_1^{-1}\cdots x_{d-1}^{-1}).$$

After a further cancellation of base points, we are left with studying the map $h\colon (S^\sigma)^{\wedge d-1} \to (S^\sigma)^{\wedge d}$. This is an equivariant cofibration, as it is the $(S^\sigma)^{\wedge d-1}$-suspension of the pushout of the cofibration $G \times S^0 \to G \times I$ along the projection $G \times S^0 \to (G/G) \times S^0$. Hence, unstably the equivariant (homotopy) cofiber of f is given by $(S^\sigma)^{\wedge d}/h((S^\sigma)^{\wedge d-1}) \simeq S^d \vee S^d$, where $G = \mathbb{Z}/2$ acts on the latter by switching the spheres. Thus the stable equivariant (homotopy) fiber $\mathrm{fib}(g)$ is given by $\Sigma^{-d-1}\Sigma^\infty(S^d \vee S^d) \simeq i_\sharp \Sigma^{-1}$ using equation (A.15). This completes the induction argument for even d.

Together with equation (5.6), we obtain an equivalence

$$\mathrm{THR}(\mathbb{P}_{\mathbb{S}}^n) \simeq \begin{cases} \mathbb{S} \oplus \bigoplus_{j=1}^{\lfloor n/2 \rfloor} i_\sharp \mathbb{S} & \text{if } n \text{ is even,} \\ \mathbb{S} \oplus \bigoplus_{j=1}^{\lfloor n/2 \rfloor} i_\sharp \mathbb{S} \oplus \Sigma^{n(\sigma-1)} & \text{if } n \text{ is odd.} \end{cases}$$

Use Propositions 4.2.15 and 5.2.5 for the standard cover of $\mathbb{P}^n$ to obtain an equivalence

$$\mathrm{THR}(X \times \mathbb{P}^n) \simeq \mathrm{THR}(X) \wedge \mathrm{THR}(\mathbb{P}_{\mathbb{S}}^n)$$

whenever X is an affine scheme with involution. Use Proposition 5.2.5 again to generalize this equivalence to the case when X is a separated scheme with involution. Hence, to obtain the desired equivalence, it suffices to obtain an equivalence

$$\mathrm{THR}(X) \wedge i_\sharp \mathbb{S} \simeq i_\sharp \mathrm{THH}(X).$$

This follows from Propositions Appendix A.1.10 and 2.1.3. $\qquad\square$

Note that this result is compatible with the projective bundle theorem for the oriented theory THH, see [10], after applying i^* and using Propositions 3.4.7 and Appendix A.2.7.

For all $X \in \mathbf{Sch}_{\mathbb{Z}/2}$ and integers m, we set

$$\mathrm{THO}^{[m]}(X) := (\Sigma^{m(1-\sigma)}\mathrm{THR}(X))^{\mathbb{Z}/2}. \tag{5.8}$$

Recall that $(-)^{\mathbb{Z}/2}$ commutes with Σ^1 but not with Σ^σ.

Looking at fixed points, the result becomes

$$\mathrm{THO}^{[m]}(X \times \mathbb{P}^n) \simeq \begin{cases} \mathrm{THO}^{[m]}(X) \oplus \bigoplus_{j=1}^{\lfloor n/2 \rfloor} \mathrm{THH}(X) & \text{if } n \text{ is even,} \\ \mathrm{THO}^{[m]}(X) \oplus \bigoplus_{j=1}^{\lfloor n/2 \rfloor} \mathrm{THH}(X) \oplus \mathrm{THO}^{[m-n]}(X) & \text{if } n \text{ is odd.} \end{cases}$$

$$\tag{5.9}$$

Remark 5.2.7. As for $n = 1$ in the previous subsection, the formula for $\mathrm{THO}(X \times \mathbb{P}^n)$ corresponds to the one for $\mathbf{KO}(X \times \mathbb{P}^n)$. Indeed, for higher-dimensional projective spaces $\mathbb{P}^n$ with trivial involution, $\mathbf{KO} = \mathbf{KR}$ has been recently computed by [50] and [36]. Analyzing the arguments of [50], one sees that for even n the results for $\mathbf{KR}(\mathbb{P}^n)$ with and without involution are the same. However, when trying to compute THR of higher-dimensional $\mathbb{P}^n$ with involution, the standard cover of $\mathbb{P}^n$ will not respect the involution.

For the one-dimensional $\mathbb{P}^\sigma$ we used the square in equation (5.2) instead, and for $(\mathbb{P}^n, \tau)$ in general one would have to construct more complicated cubes that respect the involution τ.

Appendix A. Equivariant homotopy theory

The purpose of this appendix is to review equivariant homotopy theory. Throughout this section, G is a finite group.

A.1. ∞-categories of equivariant spectra

In this subsection, we review the ∞-categorical formulation of equivariant homotopy theory following Bachmann and Hoyois [4]. We restrict to finite groupoids, although Bachmann and Hoyois deal more generally with profinite groupoids. This approach will be compared to more classical references like [29] in Section Appendix A.2 below.

Definition Appendix A.1.1. Let FinGpd denote the 2-category of finite groupoids, that is those with only finitely many objects and morphisms. A morphism in FinGpd is called a *finite covering* if its fibers (which by our assumptions have only finitely many objects) automatically are sets, that is, do not have nontrivial automorphisms. Recall that the fiber of a 1-morphism $f \colon Y \to X$ over a point $*$ in X is $Y \times_X *$. For $X \in \mathrm{FinGpd}$, let Fin_X denote the category of finite coverings of X. There is an equivalence between Fin_{BG} and the category of finite G-sets by [4, Lemma 9.3].

For a morphism $f \colon Y \to X$ in FinGpd, there is an adjunction

$$f^* \colon \mathrm{Fin}_X \rightleftarrows \mathrm{Fin}_Y : f_*, \tag{A.1}$$

where f^* sends $V \in \mathrm{Fin}_X$ to $V \times_X Y$. If f is a finite covering, then there is an adjunction

$$f_\sharp \colon \mathrm{Fin}_Y \rightleftarrows \mathrm{Fin}_X : f^*, \tag{A.2}$$

where $f_\sharp$ sends $V \in \mathrm{Fin}_Y$ to V, compare the paragraph preceding [4, section 9.2].

Example Appendix A.1.2. If i is the obvious morphism of groupoids $\mathrm{pt} \to BG$, then $i_\sharp E = \coprod_G E$ and $(i_\sharp, i^*)$ is the usual free-forgetful adjunction between sets and G-sets. On the other hand, for any finite set E the G-set $i_* E$ is isomorphic to $\prod_G E$ with G acting by permuting the indices.

For $p \colon BG \to \mathrm{pt}$, we have $p_* E = E^G$ for every finite G-set E. Note that $p_\sharp E$ is not defined as p is not a finite covering, although for finite G-sets E the left adjoint to p^* exists, and is given by the orbit set E/G.

Definition Appendix A.1.3. For a category $\mathcal{C}$ with pullbacks, let $\mathrm{Span}(\mathcal{C})$ denote the category of spans, whose objects are the same as $\mathcal{C}$, whose morphisms are given by the diagrams $(X \xleftarrow{f} Y \xrightarrow{p} Z)$ in $\mathcal{C}$, and whose compositions of morphisms are given by pullbacks. A morphism $(X \xleftarrow{f} Y \xrightarrow{p} Z)$ is called a *forward morphism* (resp. *backward morphism*) if $f = \mathrm{id}$ (resp. $p = \mathrm{id}$). The notion of spans can be generalized to the case when $\mathcal{C}$ is an ∞-category; see [5, section 5] for the details.

Construction Appendix A.1.4. In [4, section 9.2], Bachmann and Hoyois construct three functors $\mathcal{H}, \mathcal{H}_\bullet$ and $\mathcal{SH}$ on FinGpd by certain presheaves on Fin_X, and then further refine these to functors

$$\mathcal{H}^\otimes, \mathcal{H}_\bullet^\otimes, \mathcal{SH}^\otimes \colon \mathrm{Span}(\mathrm{FinGpd}) \to \mathrm{CAlg}(\mathrm{Cat}_\infty), \, (X \xleftarrow{f} Y \xrightarrow{p} Z) \mapsto p_\otimes f^* \qquad \text{(A.3)}$$

together with natural transformations

$$\mathcal{H}^\otimes \xrightarrow{(-)_+} \mathcal{H}_\bullet^\otimes \xrightarrow{\Sigma^\infty} \mathcal{SH}^\otimes. \qquad \text{(A.4)}$$

Let us explain parts of their construction.

For an ∞-category $\mathcal{C}$ with finite coproducts, let $\mathcal{P}_\Sigma(\mathcal{C})$ denote the ∞-category of presheaves of spaces which transform finite coproducts into finite products. For $X \in$ FinGpd, we set $\mathcal{H}(X) := \mathcal{P}_\Sigma(\mathrm{Fin}_X)$. Then we set $\mathcal{H}_\bullet(X) := \mathcal{H}(X)_*$, which is the ∞-category of pointed objects in $\mathcal{H}(X)$. As claimed in [4, p. 81], for $X = \mathrm{B}G$ these yield the usual ∞-categories of G-spaces and pointed G-spaces.

For a morphism $f \colon Y \to X$, the functor f^* for $\mathcal{H}$ and $\mathcal{H}_\bullet$ is induced by equation (A.1), and f^* admits a right adjoint f_*. For $\mathcal{H}_\bullet$, $f_\otimes$ is a symmetric monoidal functor preserving sifted colimits such that $f_\otimes(V_+) \simeq f_*(V)_+$ for $V \in \mathrm{Fin}_X$. If f is a finite covering, f^* for $\mathcal{H}$ and $\mathcal{H}_\bullet$ admits a left adjoint $f_\sharp$.

We obtain $\mathcal{SH}(X)$ from $\mathcal{H}_\bullet(X)$ by $\otimes$-inverting $p_\otimes(S^1)$ for all finite coverings $p \colon Y \to X$. The functor f^* for $\mathcal{SH}$ is induced by that for $\mathcal{H}_\bullet$. This admits a right adjoint f_*, and this admits a left adjoint $f_\sharp$ if f is a finite covering. The functor $f_\otimes$ for $\mathcal{SH}$ is the unique symmetric monoidal functor preserving sifted colimits such that the square

$$
\begin{array}{ccc}
\mathcal{H}_\bullet(Y) & \xrightarrow{f_\otimes} & \mathcal{H}_\bullet(X) \\
\Sigma^\infty \downarrow & & \downarrow \Sigma^\infty \\
\mathcal{SH}(Y) & \xrightarrow{f_\otimes} & \mathcal{SH}(X)
\end{array}
$$

commutes. Furthermore, if f has connected fibers, then $f_\otimes$ preserves colimits.

Proposition Appendix A.1.5. *Suppose $X_1, \ldots, X_n \in$ FinGpd. Then there exists a canonical equivalence*

$$\mathcal{SH}(X_1 \amalg \cdots \amalg X_n) \simeq \mathcal{SH}(X_1) \times \cdots \times \mathcal{SH}(X_n).$$

Proof. This is a consequence of [4, Lemma 9.6]. $\qquad \square$

There is an alternative construction only inverting S^1: By [4, Proposition 9.11], we have an equivalence

$$\mathcal{SH}(X) \simeq \mathrm{Sp}(\mathcal{P}_\Sigma(\mathrm{Span}(\mathrm{Fin}_X))). \qquad \text{(A.5)}$$

Proposition Appendix A.1.6. *Suppose $X \in$ FinGpd. Then the family*

$$\{\Sigma^n \Sigma^\infty V_+ : V \in \mathrm{Fin}_X, n \in \mathbb{Z}\}$$

compactly generates $\mathcal{SH}(X)$. In other words, the functor $\mathrm{Map}_{\mathcal{SH}(X)}(\Sigma^n \Sigma^\infty V_+, -)$ preserves filtered colimits for all $V \in \mathrm{Fin}_X$ and $n \in \mathbb{Z}$, and the family of functors

$$\{\mathrm{Map}_{\mathcal{SH}(X)}(\Sigma^n \Sigma^\infty V_+, -) : V \in \mathrm{Fin}_X, n \in \mathbb{Z}\} \tag{A.6}$$

is conservative.

Proof. Using equation (A.5), we see that a map $\mathcal{G} \to \mathcal{F}$ in $\mathcal{SH}(X)$ is an equivalence if and only if the induced map $\mathcal{G}(V) \to \mathcal{F}(V)$ is an equivalence for all $V \in \mathrm{Fin}_X$. This is further equivalent to saying that the induced morphism $\Omega^\infty \Sigma^{-n} \mathcal{G}(V) \to \Omega^\infty \Sigma^{-n} \mathcal{F}(V)$ is an equivalence for all $V \in \mathrm{Fin}_X$ and $n \in \mathbb{Z}$. This proves that equation (A.6) is conservative. For the other claim, we reduce to the case when $X = \mathrm{B}G$ using Proposition Appendix A.1.5 since every finite groupoid is equivalent to a finite disjoint union of classifying spaces. Then combine [39, Lemma I.5.3] and [42, Proposition 1.4.4.1(3)] to conclude. See also [32, Theorem 9.4.3]. $\square$

Let Fold_X denote the full subcategory of Fin_X consisting of the finite fold maps

$$(\mathrm{id}, \dots, \mathrm{id}) \colon X \amalg \cdots \amalg X \to X,$$

Proposition Appendix A.1.7. *Let $f \colon X^{\amalg n} \to X$ be the n-fold map, where $X \in \mathrm{FinGpd}$ and $n \geq 1$ is an integer. Then the composite*

$$\mathcal{SH}(X)^{\times n} \xrightarrow{\simeq} \mathcal{SH}(X^{\amalg n}) \xrightarrow{f_\otimes} \mathcal{SH}(X)$$

is the n-fold smash product.

Proof. One can show an analogous claim for $\mathcal{H}_\bullet$ as in [4, Theorem 3.3(6)]. To obtain the claim for $\mathcal{SH}$, use [4, Lemma 4.1]. $\square$

Definition Appendix A.1.8 ([4, Definition 9.14]). Suppose $X \in \mathrm{FinGpd}$. A *normed X-spectrum* is a section of $\mathcal{SH}^\otimes$ over $\mathrm{Span}(\mathrm{Fin}_X)$ that is co-Cartesian over the backward morphisms. Let $\mathrm{NAlg}(\mathcal{SH}(X))$ denote the ∞-category of normed X-spectra.

Mapping X to pt yields an equivalence between Fold_X and $\mathrm{Fold}_{\mathrm{pt}}$, and forgetting the map to pt $\mathrm{Fold}_{\mathrm{pt}}$ is obviously equivalent to the category of finite sets. By [4, Corollary C.2], $\mathrm{CAlg}(\mathcal{SH}(X))$ is equivalent to the ∞-category of sections of $\mathcal{SH}^\otimes$ over $\mathrm{Span}(\mathrm{Fold}_X)$ that is co-Cartesian over the backward morphisms. Hence, there is a forgetful functor

$$\mathrm{NAlg}(\mathcal{SH}(X)) \to \mathrm{CAlg}(\mathcal{SH}(X)), \tag{A.7}$$

which is conservative and preserves colimit and limits as in [4, Proposition 7.6(3)]. This is an equivalence if $X = \mathrm{pt}$ since $\mathrm{Fold}_{\mathrm{pt}} \simeq \mathrm{Fin}_{\mathrm{pt}}$. The forgetful functor

$$\mathrm{CAlg}(\mathcal{SH}(X)) \to \mathcal{SH}(X) \tag{A.8}$$

is conservative by [42, Lemma 3.2.2.6]. It follows that the composite forgetful functor $\mathrm{NAlg}(\mathcal{SH}(X)) \to \mathcal{SH}(X)$ is conservative too.

Suppose $X \in \mathrm{FinGpd}$ and $R \in \mathrm{CAlg}(\mathcal{SH}(X))$. There is an induced commutative square

$$
\begin{array}{ccc}
\mathrm{CAlg}(\mathrm{Mod}_R) & \xrightarrow{U} & \mathrm{CAlg}(\mathcal{SH}(X)) \\
\downarrow & & \downarrow \\
\mathrm{Mod}_R & \longrightarrow & \mathcal{SH}(X).
\end{array}
$$

The vertical functors are the forgetful functors, which is symmetric monoidal according to [42, Example 3.2.4.4, Proposition 3.2.4.10]. Note that the symmetric monoidal structure on $\mathrm{CAlg}(-)$ is given by the coproduct. The monoidal product in $\mathcal{SH}(X)$ (resp. Mod_R) is denoted by $\wedge$ (resp. $\wedge_R$). Then we have the induced monoidal products on $\mathrm{CAlg}(\mathcal{SH}(X))$ and $\mathrm{CAlg}(\mathrm{Mod}_R)$. There is an equivalence between $\mathrm{CAlg}(\mathrm{Mod}_R)$ and the ∞-category of R-algebras $\mathrm{CAlg}(\mathcal{SH}(X))^{/R}$ by [42, Corollary 3.4.1.7].

The coproduct in $\mathrm{NAlg}(\mathcal{SH}(X))$ is also denoted by $\wedge$. Since the forgetful functor $\mathrm{NAlg}(\mathcal{SH}(X)) \to \mathrm{CAlg}(\mathcal{SH}(X))$ preserves colimits, the notation $\wedge$ on $\mathrm{CAlg}(\mathcal{SH}(X))$ and $\mathrm{NAlg}(\mathcal{SH}(X))$ is compatible.

Proposition Appendix A.1.9. *Let*

$$
\begin{array}{ccc}
Y' & \xrightarrow{g'} & Y \\
f' \downarrow & & \downarrow f \\
X' & \xrightarrow{g} & X
\end{array}
$$

be a Cartesian square in FinGpd *such that f is a finite covering. For $\mathcal{SH}$, the natural transformation*

$$
f'_\sharp g'^* \to g^* f_\sharp
$$

given by the composite

$$
f'_\sharp g'^* \xrightarrow{ad} f'_\sharp g'^* f^* f_\sharp \xrightarrow{\simeq} f'_\sharp f'^* g^* f_\sharp \xrightarrow{ad'} g^* f_\sharp
$$

is an equivalence.

Proof. As usual, Proposition Appendix A.1.6 allows us to reduce to showing that the induced map $f'_\sharp g'^* \Sigma^n \Sigma^\infty W_+ \to g^* f_\sharp \Sigma^n \Sigma^\infty W_+$ is an equivalence for every $W \in \mathrm{Fin}_Y$ and integer n. This follows from the fact that the composite of the induced morphisms

$$
Y' \times_Y W \to Y' \times_Y (Y \times_X W) \xrightarrow{\cong} Y' \times_{X'} (X' \times_X W) \to X' \times_X W
$$

is an isomorphism. $\qquad\square$

Proposition Appendix A.1.10. *Let $f \colon Y \to X$ be a finite covering. Then for $\mathcal{F} \in \mathcal{SH}(Y)$ and $\mathcal{G} \in \mathcal{SH}(X)$, there exists a canonical equivalence*

$$
f_\sharp(\mathcal{F} \wedge f^*\mathcal{G}) \simeq f_\sharp \mathcal{F} \wedge \mathcal{G}.
$$

Proof. As usual, Proposition Appendix A.1.6 allows us to reduce to the case when $\mathcal{F} = \Sigma^m \Sigma^\infty V_+$ and $\mathcal{G} = \Sigma^n \Sigma^\infty W_+$ for some $V \in \mathrm{Fin}_Y$, $W \in \mathrm{Fin}_X$ and $m,n \in \mathbb{Z}$. In this case, the canonical isomorphism

$$V \times_Y (W \times_X Y) \cong V \times_X W$$

gives the desired equivalence.   □

Let $f \colon Y \to X$ be a morphism in FinGpd. The formulation equation (A.3) tells that the functor f^* is symmetric monoidal. Hence, we obtain an induced adjoint pair

$$f^* : \mathrm{CAlg}(\mathcal{SH}(X)) \rightleftarrows \mathrm{CAlg}(\mathcal{SH}(Y)) : f_* \tag{A.9}$$

by [42, Remark 7.3.2.13]. The formulation of these functors provided in [42, Proposition 7.3.2.5] shows that the two squares in

$$
\begin{array}{ccc}
\mathrm{CAlg}(\mathcal{SH}(X)) & \underset{f_*}{\overset{f^*}{\rightleftarrows}} & \mathrm{CAlg}(\mathcal{SH}(Y)) \\
\downarrow & & \downarrow \\
\mathcal{SH}(X) & \underset{f_*}{\overset{f^*}{\rightleftarrows}} & \mathcal{SH}(Y)
\end{array}
\tag{A.10}
$$

commute, where the vertical functors are the forgetful functors.

If f has connected fibers, then we noted that $f_\otimes$ preserves colimits. Hence, we similarly obtain a functor

$$f_\otimes : \mathrm{CAlg}(\mathcal{SH}(Y)) \to \mathrm{CAlg}(\mathcal{SH}(X)) \tag{A.11}$$

and a commutative square

$$
\begin{array}{ccc}
\mathrm{CAlg}(\mathcal{SH}(Y)) & \overset{f_\otimes}{\longrightarrow} & \mathrm{CAlg}(\mathcal{SH}(X)) \\
\downarrow & & \downarrow \\
\mathcal{SH}(Y) & \overset{f_\otimes}{\longrightarrow} & \mathcal{SH}(X).
\end{array}
\tag{A.12}
$$

The following should be compared with equation (A.16).

Proposition Appendix A.1.11. *Let* $f \colon Y \to X$ *be a finite covering in* FinGpd. *Then there is an induced adjunction*

$$f_\otimes : \mathrm{NAlg}(\mathcal{SH}(Y)) \rightleftarrows \mathrm{NAlg}(\mathcal{SH}(X)) : f^*.$$

Furthermore, the two squares in

$$
\begin{array}{ccc}
\mathrm{NAlg}(\mathcal{SH}(Y)) & \underset{f^*}{\overset{f_\otimes}{\rightleftarrows}} & \mathrm{NAlg}(\mathcal{SH}(X)) \\
\downarrow & & \downarrow \\
\mathcal{SH}(Y) & \underset{f^*}{\overset{f_\otimes}{\rightleftarrows}} & \mathcal{SH}(X)
\end{array}
\tag{A.13}
$$

commute, where the vertical functors are the forgetful functors.

Proof. Apply [4, Theorem 8.5] to the case when $\mathcal{C} := \mathrm{FinGpd}$, $\mathcal{A} := \mathcal{SH}^{\otimes}$ and right (resp. left) is the class of all morphisms (finite coverings) in FinGpd. Then $\mathrm{Sect}(\mathcal{A}_X)$ in the reference is precisely $\mathrm{NAlg}(\mathcal{SH}(X))$ for $X \in \mathrm{FinGpd}$. Hence, as observed in [4, Remark 8.6], we have the desired adjunction such that the two squares in equation (A.13) commutes. $\square$

Proposition Appendix A.1.12. *Let* $f\colon Y \to X$ *be a finite covering in* FinGpd. *Then there is an induced adjunction*

$$f^* \colon \mathrm{NAlg}(\mathcal{SH}(X)) \rightleftarrows \mathrm{NAlg}(\mathcal{SH}(Y)) \colon f_*.$$

Furthermore, the two squares in

$$
\begin{array}{ccc}
\mathrm{NAlg}(\mathcal{SH}(X)) & \underset{f_*}{\overset{f^*}{\rightleftarrows}} & \mathrm{NAlg}(\mathcal{SH}(Y)) \\
\downarrow & & \downarrow \\
\mathcal{SH}(X) & \underset{f_*}{\overset{f^*}{\rightleftarrows}} & \mathcal{SH}(Y)
\end{array}
\tag{A.14}
$$

commute, where the vertical functors are the forgetful functors.

Proof. Let us imitate the proof of [4, Theorem 8.2]. By [4, Corollary C.21(2)], there is an induced adjunction

$$f^* \colon \mathrm{Span}(\mathrm{Fin}_X) \rightleftarrows \mathrm{Span}(\mathrm{Fin}_Y) \colon f_\sharp.$$

Let $\mathcal{SH}^{\otimes}|\mathrm{Span}(\mathrm{Fin}_X)$ be the restriction of $\mathcal{SH}^{\otimes}$ to $\mathrm{Span}(\mathrm{Fin}_X)$. For every $V \in \mathrm{Span}(\mathrm{Fin}_X)$, let $f_V \colon V \times_X Y \to V$ be the projection. The functor $f_V^* \colon \mathcal{SH}(V) \to \mathcal{SH}(V \times_X Y)$ admits the right adjoint f_{V*}. Apply [4, Proposition 8.16] to the co-Cartesian fibration $\mathcal{SH}^{\otimes}|\mathrm{Span}(\mathrm{Fin}_X)$ to obtain the desired adjunction.

Let us review the descriptions of f^* and f_* for NAlg in this reference. The functor f^* for NAlg used here is the same as the functor f^* for NAlg in Proposition Appendix A.1.11. Suppose $B \in \mathrm{NAlg}(\mathcal{SH}(Y))$. For $V \in \mathrm{Span}(\mathrm{Fin}_X)$, the section $(f_*B)(V)$ is given by $f_{V*}(B(V \times_X Y))$. In particular, the section $(f_*B)(X)$ is given by $f_*(B(X))$. Hence, the two squares in equation (A.14) commute. $\square$

For abbreviation, we set

$$\mathrm{Sp}_G := \mathcal{SH}(\mathrm{B}G), \; \mathrm{CAlg}_G := \mathrm{CAlg}(\mathcal{SH}(\mathrm{B}G)), \text{ and } \mathrm{NAlg}_G := \mathrm{NAlg}(\mathcal{SH}(\mathrm{B}G)).$$

This notation is further justified by Remark Appendix A.2.3.

A.2. Equivariant orthogonal spectra

The purpose of this section is to review equivariant homotopy theory using model categories. Our references for that are [23], [44], [29] and [53]. We will also review the comparison between ∞-categorical and model categorical constructions of equivariant spectra. Consequently, we may apply certain known constructions and results for equivariant orthogonal spectra to ∞-categories as discussed in the previous subsection.

Let BG denote the associated finite groupoid. In this subsection, we are interested in the obvious morphisms

$$\mathrm{B}H \xrightarrow{i} \mathrm{B}G \xrightarrow{p} *,$$

where $H \to G$ is an inclusion.

Definition Appendix A.2.1. Let Sp^O denote the category of orthogonal spectra. For a finite group G, let Sp^O_G denote the category of orthogonal G-spectra. Recall that an *orthogonal G-spectrum* is an orthogonal spectrum with a G-action. A *morphism of orthogonal G-spectra* is a morphism of underlying orthogonal spectra that is compatible with the G-actions.

The definition of orthogonal G-spectra in [29] is *different* from the above one, but the two categories are equivalent. See [53, Remark 2.7] for the details.

Definition Appendix A.2.2. The category Sp^O_G admits a symmetric monoidal model structure; see [29, Propositions B.63, B.76]. We denote the (model) category of commutative monoids in Sp^O by CAlg^O. According to [29], we sometimes denote it also by **Comm**, and the (model) category of commutative monoids in Sp^O_G by **Comm$_\mathbf{G}$**. We refer to [29, section A.1.2] for the details. According to [29, Proposition B.129], **Comm**$_G$ has a nice model structure. A morphism in **Comm**$_G$ is a weak equivalence (resp. fibration) precisely when its underlying morphism in Sp^O_G is a weak equivalence (resp. fibration). The (underived) coproduct in **Comm**$_G$ is denoted by $\wedge$.

Remark Appendix A.2.3. As observed in the preceding paragraphs of [4, Lemma 9.6], Sp_G is equivalent to the underlying ∞-category of the model category of symmetric G-spectra. This is equivalent to the underlying ∞-category of Sp^O_G by [43]. See also [4, Remark 9.12] for another ∞-description. Furthermore, as observed in [4, after Definition 9.14], NAlg_G is equivalent to the underlying ∞-category of the model category of G-$\mathbb{E}_\infty$-rings, which is equivalent to the underlying ∞-category of **Comm**$_G$. We refer to [24] for a comparison of different models, rectification results and further references.

Construction Appendix A.2.4. Let H be a subgroup of G, with the inclusion map $H \to G$. (For $H = \mathrm{pt}$, compare Example Appendix A.1.2.) Let us review several functors from [29, sections 2.2.3, 2.5.1]. The *norm functor*

$$N_H^G \colon \mathrm{Sp}^O_H \to \mathrm{Sp}^O_G$$

sends $Y \in \mathrm{Sp}^O_H$ to $\bigwedge_{i \in G/H} Y$ with a suitable G-action. If $H = \mathrm{pt}$, we often simply write N^G.

The *restriction functor*

$$i^* \colon \mathrm{Sp}^O_G \to \mathrm{Sp}^O_H$$

sends $X \in \mathrm{Sp}_G^O$ to X, and the action is the restriction of the G-action to H. Its left adjoint $i_\sharp$ and right adjoint i_* send $X \in \mathrm{Sp}_H^O$ to

$$\bigvee_{i \in G/H} X_i \text{ and } \prod_{i \in G/H} X_i \tag{A.15}$$

respectively, where $X_i := H_i \wedge_H X$ and $H_i \subset G$ is the coset indexed by G. According to [29, Proposition B.72], i^* is a left and right Quillen functor. Hence, we have Quillen adjunctions

$$i_\sharp : \mathrm{Sp}_H^O \rightleftarrows \mathrm{Sp}_G^O : i^* \text{ and } i^* : \mathrm{Sp}_G^O \rightleftarrows \mathrm{Sp}_H^O : i_*.$$

We also have the functor

$$\iota : \mathrm{Sp}^O \to \mathrm{Sp}_G^O$$

imposing the trivial G-action. The *fixed point functor*

$$(-)^G : \mathrm{Sp}_G^O \to \mathrm{Sp}^O$$

sends an orthogonal G-spectrum $(X_0, X_1, \dots)$ to $(X_0^G, X_1^G, \dots)$. There is a Quillen adjunction

$$\iota : \mathrm{Sp}^O \rightleftarrows \mathrm{Sp}_G^O : (-)^G.$$

Construction Appendix A.2.5. For the definition of the *geometric fixed point functor*

$$\Phi^G : \mathrm{Sp}_G^O \to \mathrm{Sp}^O,$$

we refer to [29, section B.10.1]. There is yet another functor

$$\Phi_M^G : \mathrm{Sp}_G^O \to \mathrm{Sp}^O,$$

in [29, Definition B.190], which is called the *monoidal geometric fixed point functor*. This is lax monoidal and preserves cofibrations and acyclic cofibrations; see [29, sections B.10.3, B.10.4].

According to [29, Proposition B.201], there is a zig-zag of weak equivalences between $\Phi^G(X)$ and $\Phi_M^G(X)$ whenever $X \in \mathrm{Sp}_G^O$ is cofibrant.

Remark Appendix A.2.6. The functor $i^* : \mathrm{Sp}_H^O \to \mathrm{Sp}_G^O$ is a model for the functor of ∞-categories $i^* : \mathrm{Sp}_H \to \mathrm{Sp}_G$ since their values on $\Sigma^n \Sigma^\infty X_+$ are equivalent for all $X \in \mathrm{Fin}_H$ and integers n. Likewise, the functor $\iota : \mathrm{Sp}^O \to \mathrm{Sp}_G^O$ is a model for the functor $p^* : \mathrm{Sp} \to \mathrm{Sp}_G$, where $p : BG \to \mathrm{pt}$. It follows by the uniqueness of ∞-adjoints that the fixed point functor $(-)^G : \mathrm{Sp}_G^O \to \mathrm{Sp}^O$ is a model for the functor $p_* : \mathrm{Sp}_G^O \to \mathrm{Sp}^O$. We have similar comparison results for $i_\sharp$ and i_*.

According to [4, Remark 9.10], the norm functor $N_H^G : \mathrm{Sp}_H^O \to \mathrm{Sp}_G^O$ is a model for the functor $i_\otimes : \mathrm{Sp}_H \to \mathrm{Sp}_G$, and the geometric fixed point functor $\Phi^G : \mathrm{Sp}_G^O \to \mathrm{Sp}^O$ is a model for the functor $p_\otimes : \mathrm{Sp}_G \to \mathrm{Sp}$. It follows that Φ_M^G is a model for $p_\otimes$ too.

The functors $N_H^G \colon \mathrm{Sp}_H^O \to \mathrm{Sp}_G^O$ and $i^* \colon \mathrm{Sp}_H^O \to \mathrm{Sp}_G^O$ are symmetric monoidal. By [29, Proposition 2.27], they induce a Quillen adjunction

$$N_H^G \colon \mathbf{Comm}_H \rightleftarrows \mathbf{Comm}_G \colon i^*. \tag{A.16}$$

In short, comparing the adjunctions between $\mathbf{Comm}_G$ with those for the underlying spectra, N_H^G already exists for spectra, but only becomes a left adjoint to i^* in $\mathbf{Comm}_G$, replacing $i_\sharp$.

Compare the diagram in [29, Proposition A.56] with equation (A.12) and use the conservativity of the forgetful functor $\mathrm{NAlg}_G \to \mathrm{Sp}_G$ to show that $N_H^G \colon \mathbf{Comm}_H \to \mathbf{Comm}_G$ is a model for the functor $i_\otimes \colon \mathrm{NAlg}_H \to \mathrm{NAlg}_G$. Then $i^* \colon \mathbf{Comm}_G \to \mathbf{Comm}_H$ is a model for the functor $i^* \colon \mathrm{NAlg}_G \to \mathrm{NAlg}_H$ by adjunction.

Proposition Appendix A.2.7. *We have the following equivalences of functors between ∞-categories* Sp_G *for appropriate* G:

(1) $p_\otimes i_\otimes \simeq \mathrm{id}$ *if* $H = e$,

(2) $i^* p^* \simeq \mathrm{id}$ *if* $H = e$,

(3) $i_\sharp \xrightarrow{\simeq} i_*$,

(4) $\mathrm{id} \xrightarrow{\simeq} p_\otimes p^*$,

(5) $p_\otimes i_\sharp \simeq 0$ *if* $G \neq e$,

(6) $i^* i_\otimes \simeq (-)^{\wedge [G:H]}$,

(7) $i^* i_\sharp \simeq (-)^{\oplus [G:H]}$.

Proof. The first two follow from $pi = \mathrm{id}$. The next three follow from [29, Propositions B.56, B.182, B.192]. For (6), consider the Cartesian square

$$
\begin{array}{ccc}
G/H \times \mathrm{B}H & \xrightarrow{\ q\ } & \mathrm{B}H \\
{\scriptstyle q}\downarrow & & \downarrow{\scriptstyle i} \\
\mathrm{B}H & \xrightarrow{\ i\ } & \mathrm{B}G,
\end{array}
$$

where q is the $|G/H|$-fold map. If $\bar{i}$ and $\bar{q}$ denote the forward morphisms in $\mathrm{Span}(\mathrm{FinGpd})$ associated with i and q, then $\bar{i}i \simeq q\bar{q}$. Hence, we have an equivalence $i^* i_\otimes \simeq q_\otimes q^*$. Together with Proposition Appendix A.1.7, we obtain the desired equivalence. For (7), Proposition Appendix A.1.9 gives an equivalence $i^* i_\sharp \simeq q_\sharp q^*$. The functor q^* can be identified with the diagonal functor $\mathrm{Sp}_H \to (\mathrm{Sp}_H)^{\times G/H}$, and the functor $q_\sharp$ can be identified with the $[G:H]$-fold direct sum $(\mathrm{Sp}_H)^{\times G/H} \to \mathrm{Sp}_H$. Use these facts to conclude. $\qquad\square$

Remark Appendix A.2.8. Recall that the forgetful functors $\mathrm{CAlg}_G \to \mathrm{Sp}_G$ and $\mathrm{NAlg}_G \to \mathrm{Sp}_G$ are conservative. Together with equations (A.10) and (A.12), we see that Proposition Appendix A.2.7(2),(4) holds for CAlg_G for appropriate G. Similarly, together with equation (A.13), we see that Proposition Appendix A.2.7(2),(6) holds for NAlg_G for appropriate G.

Lemma Appendix A.2.9. *Suppose $R \in \mathrm{CAlg}$, $M \in \mathrm{Mod}_R$ and $L \in \mathrm{Mod}_{p^*R}$. If M and R are connective, then there exists a canonical equivalence of R-modules*

$$M \wedge_R p_* L \simeq p_*(p^* M \wedge_{p^* R} L).$$

Proof. We have maps

$$p^*(M \wedge_R p_* L) \xrightarrow{\simeq} p^* M \wedge_{p^* R} p^* p_* L \to p^* M \wedge_{p^* R} L,$$

where the second map is induced by the counit map $p^* p_* L \to L$. By adjunction, we obtain $M \wedge_R p_* L \to p_*(p^* M \wedge_{p^* R} L)$. We only need to show that this is an equivalence in Sp after forgetting the module structures.

Let $\mathcal{F}$ be the class of R-modules M such that this map is an equivalence. The functors p^*, $\wedge_R p_* L$ and $\wedge_{p^* R} L$ preserve colimits. As explained after [45, Remark 6.8], the functor p_* preserves colimits too. It follows that $\mathcal{F}$ is closed under colimits. Furthermore, $\mathcal{F}$ is closed under shifts. Since $\mathcal{F}$ contains R, $\mathcal{F}$ contains all connective R-modules by [42, Proposition 7.1.1.13]. $\qquad\square$

A.3. Mackey functors

Definition Appendix A.3.1. Recall from [30, Definition 8.2.5] that a *Mackey functor for G* (or simply *Mackey functor*) is a presheaf M on $\mathrm{Span}(\mathrm{Fin}_{BG})$ of abelian groups that transforms finite coproducts into finite products. (This is easily seen to be equivalent to more classical definitions as, e.g., recalled in [29, Definition 3.1].) For a forward (resp. backward) morphism f in $\mathrm{Span}(\mathrm{Fin}_{BG})$, $M(f)$ is called a *restriction map* (resp. *transfer map*). Let Mack_G denote the category of Mackey functors for G. We include the explicit description of Mackey functors for $\mathbb{Z}/2$ in Example 2.2.1.

For all $X \in \mathrm{Sp}_G$, $M \in \mathrm{Fin}_{BG}$ and integers n, we set

$$\underline{\pi}_n(X)(M) := \mathrm{Hom}_{\mathrm{Ho}(\mathrm{Sp}_G)}(\Sigma^n \Sigma^\infty M_+, X) \cong \mathrm{Hom}_{\mathrm{Ho}(\mathrm{Sp}_G)}(\Sigma^n, \Sigma^\infty M_+ \wedge X), \qquad \text{(A.17)}$$

where the isomorphism comes from [29, Example 2.6]. The first (resp. second) formulation is contravariant (resp. covariant) in M, and these two can be combined to produce the *equivariant homotopy group functor*

$$\underline{\pi}_n \colon \mathrm{Sp}_G \to \mathrm{Mack}_G. \qquad \text{(A.18)}$$

We refer to [29, section 3.1] for the details. For $X \in \mathrm{Sp}_G$ and an integer n, we say that X is *n-connected* if $\underline{\pi}_k(X) = 0$ for all integers $k \le n$.

Proposition Appendix A.3.2. *For every integer n, $\underline{\pi}_n$ preserves products and filtered colimits.*

Proof. The claim for products follows from the first formulation in equation (A.17). By Proposition Appendix A.1.6, $\Sigma^n \Sigma^\infty M_+$ is compact, that is, $\mathrm{Map}_{\mathrm{Sp}_G}(\Sigma^n \Sigma^\infty M_+, -)$ preserves filtered colimits. This immediately implies the claim. $\qquad\square$

Now, suppose $M \in \mathrm{Mack}_G$. According to [23, Theorem 5.3], one can associate an *equivariant Eilenberg–MacLane spectrum* $\mathrm{H}M \in \mathrm{Sp}_G^O$, which satisfies

$$\underline{\pi}_n(\mathrm{H}M) \cong \begin{cases} M & \text{if } n = 0, \\ 0 & \text{if } n \neq 0. \end{cases}$$

Furthermore, $\mathrm{H}M$ is unique up to an isomorphism in the homotopy category $\mathrm{Ho}(\mathrm{Sp}_G^O) \simeq \mathrm{Ho}(\mathrm{Sp}_G)$, and there is a canonical isomorphism

$$\mathrm{Hom}_{\mathrm{Ho}(\mathrm{Sp}_G^O)}(\mathrm{H}M, \mathrm{H}L) \cong \mathrm{Hom}_{\mathrm{Mack}_G}(M, L) \tag{A.19}$$

for all $M, L \in \mathrm{Mack}_G$. It follows that we have a functor

$$\mathrm{H} \colon \mathrm{Mack}_G \to \mathrm{Ho}(\mathrm{Sp}_G). \tag{A.20}$$

Definition Appendix A.3.3. Let n be an integer. Let $(\mathrm{Sp}_G)_{\geq n}$ (resp. $(\mathrm{Sp}_G)_{\leq n}$) denote the full subcategory of Sp_G spanned by $X \in \mathrm{Sp}_G$ such that $\underline{\pi}_k(X) = 0$ for all integers $k < n$ (resp. $k > n$). Observe that there are equivalences

$$(\mathrm{Sp}_G)_{\geq n} \simeq \Sigma^n (\mathrm{Sp}_G)_{\geq 0} \text{ and } (\mathrm{Sp}_G)_{\leq n} \simeq \Sigma^n (\mathrm{Sp}_G)_{\leq 0}. \tag{A.21}$$

Suppose $X, Y \in \mathrm{Sp}_G$. By [29, Proposition 4.11], $X \in (\mathrm{Sp}_G)_{\geq 0}$ (resp. $Y \in (\mathrm{Sp}_G)_{\leq -1}$) if and only if X is slice (-1)-positive (resp. Y is slice 0-null) in the sense of [29, Definition 4.8]. This immediately implies the vanishing

$$\mathrm{Map}_{\mathrm{Sp}_G}(X, Y) \simeq 0 \tag{A.22}$$

for $X \in (\mathrm{Sp}_G)_{\geq 0}$ and $Y \in (\mathrm{Sp}_G)_{\leq -1}$. According to [29, Remark 4.12], there is an example of $X \in \mathrm{Sp}_G$ such that X is slice 0-positive but $X \notin (\mathrm{Sp}_G)_{\geq 1}$.

Suppose $X \in \mathrm{Sp}_G$. As explained in [29, section 4.2], there exists a cofiber sequence in Sp_G

$$X' \to X \to X''$$

such that X' is slice (-1)-positive and X'' is slice 0-null, that is, $X' \in (\mathrm{Sp}_G)_{\geq 0}$ and $X'' \in (\mathrm{Sp}_G)_{\leq -1}$.

We combine what we have discussed above and recall the notion of t-structures in ∞-categories from [42, Definitions 1.2.1.1, 1.2.1.4] to deduce the following result, which is probably known to the experts.

Proposition Appendix A.3.4. *The pair of $(\mathrm{Sp}_G)_{\geq 0}$ and $(\mathrm{Sp}_G)_{\leq 0}$ forms a t-structure on Sp_G.*

This t-structure is called the *equivariant homotopy t-structure on Sp_G.* By [42, Remark 1.2.1.12], the heart $\mathrm{Sp}_G^\heartsuit := (\mathrm{Sp}_G)_{\geq 0} \cap (\mathrm{Sp}_G)_{\leq 0}$ is the nerve of an abelian category, and there is an equivalence

$$\mathrm{Sp}_G^\heartsuit \simeq \mathrm{N}(\mathrm{Ho}(\mathrm{Sp}_G^\heartsuit)). \tag{A.23}$$

For every integer n, let $\tau_{\geq n}$, $\tau_{\leq n}$ and h_n denote the truncation and homology functors.

Proposition Appendix A.3.5. *The functor of ∞-categories*

$$\mathrm{Sp}_G^{\heartsuit} \to \mathrm{N}(\mathrm{Mack}_G) \tag{A.24}$$

sending $X \in \mathrm{Sp}_G^{\heartsuit}$ to $\underline{\pi}_0(X)$ is an equivalence.

Proof. The functor (A.20) gives an equivalence $\mathrm{Mack}_G \simeq \mathrm{Ho}(\mathrm{Sp}_G^{\heartsuit})$. Combine with equation (A.23) to obtain a quasi-inverse of equation (A.24). □

Compose a quasi-inverse of equation (A.24) with the inclusion $\mathrm{Sp}_G^{\heartsuit} \to \mathrm{Sp}_G$ to obtain the functor of ∞-categories

$$\mathrm{H}\colon \mathrm{N}(\mathrm{Mack}_G) \to \mathrm{Sp}_G, \tag{A.25}$$

which is an upgrade of equation (A.20).

Proposition Appendix A.3.6. *The functor of ∞-categories $\mathrm{H}\colon \mathrm{N}(\mathrm{Mack}_G) \to \mathrm{Sp}_G$ preserves products and filtered colimits.*

Proof. Owing to Proposition Appendix A.3.5, it remains to check that the inclusion functor $\mathrm{Sp}_G^{\heartsuit} \to \mathrm{Sp}_G$ preserves products and filtered colimits. This follows from Proposition Appendix A.3.2. □

Proposition Appendix A.3.7. *Let H be a subgroup of G. Then the norm functor $i_{H\otimes}\colon \mathrm{Sp}_H \to \mathrm{Sp}_G$ sends $(\mathrm{Sp}_H)_{\geq 0}$ into $(\mathrm{Sp}_G)_{\geq 0}$.*

Proof. We refer to [29, Proposition 4.33]. □

A.4. Green functors

Our references for Green functors are [37] and [55].

Definition Appendix A.4.1. For Mackey functors M and L, the *box product of M and L* is defined to be

$$M \square L := \underline{\pi}_0(\mathrm{H}M \wedge \mathrm{H}L).$$

There is a purely algebraic definition of the box product of Mackey functors, which is rather explicit for $G = \mathbb{Z}/p$ and some prime p; see [37, p. 61] and [55, sections 2.2 and 2.4.3]. This is expected to coincide with the above Definition, but we won't need this.

Definition Appendix A.4.2. A *Green functor A* is a commutative monoid in the category Mack_G, that is, A is equipped with morphisms $A \square A \to A$ and $\underline{\pi}_0(\mathbb{S}) \to A$ satisfying the unital, associative and commutative axioms, where $\mathbb{S}$ denotes equivariant sphere spectrum. Let Green_G denote the category of Green functors.

An *A-module M* is an object of Mack_G equipped with an action morphism $A \square M \to M$ satisfying the module axioms.

For A-modules M and L, $M \square_A L$ is defined to be the coequalizer of the two action morphisms

$$M \square A \square L \rightrightarrows M \square L.$$

Let $\underline{\mathrm{Tor}}_*^A(M, L)$ be the derived functor of $M \square_A L$.

Proposition Appendix A.4.3. *Suppose $A \in \mathrm{CAlg}_G$, and let M and L be A-modules. Then there exists a convergent spectral sequence*

$$E^2_{p,q} := \underline{\mathrm{Tor}}^{\underline{\pi}_*(A)}_p(\underline{\pi}_*(M), \underline{\pi}_*(L))_q \Rightarrow \underline{\pi}_{p+q}(M \wedge_A L). \tag{A.26}$$

Proof. We refer to [6, section 6]. With the stronger assumption $A \in \mathrm{NAlg}_G$, this result is due to Lewis and Mandell [38, Theorem 6.6]. □

Proposition Appendix A.4.4. *Suppose $A \in \mathrm{CAlg}_G$, and let M and L be A-modules. If A, M and L are (-1)-connected, then $M \wedge_A L$ is (-1)-connected too, and there is an isomorphism*

$$\underline{\pi}_0(M) \square_{\underline{\pi}_0(A)} \underline{\pi}_0(L) \simeq \underline{\pi}_0(M \wedge_A L).$$

Proof. We refer to [6, Corollary 6.8.1]. □

Apply Proposition Appendix A.4.4 to the case when A is the equivariant sphere spectrum to obtain the symmetric monoidal structure on $(\mathrm{Sp}_G)_{\geq 0}$ that is the restriction of the symmetric monoidal structure on Sp_G. Furthermore, the functor

$$\underline{\pi}_0 \colon (\mathrm{Sp}_G)_{\geq 0} \to \mathrm{Mack}_G$$

is symmetric monoidal. Its right adjoint is the functor $\mathrm{H} \colon \mathrm{Mack}_G \to (\mathrm{Sp}_G)_{\geq 0}$ by Proposition Appendix A.3.5. Together with [42, Remark 7.3.2.13], the induced functors

$$\underline{\pi}_0 : (\mathrm{CAlg}_G)_{\geq 0} \rightleftarrows \mathrm{Green}_G : \mathrm{H} \tag{A.27}$$

form an adjoint pair, where $(\mathrm{CAlg}_G)_{\geq 0} := \mathrm{CAlg}((\mathrm{Sp}_G)_{\geq 0})$. The formulation of these functors provided in [42, Proposition 7.3.2.5] shows that the two squares in the diagram

$$
\begin{array}{ccc}
(\mathrm{CAlg}_G)_{\geq 0} & \underset{H}{\overset{\pi_0}{\rightleftarrows}} & \mathrm{Green}_G \\
{\scriptstyle U}\downarrow & & \downarrow{\scriptstyle U} \\
(\mathrm{Sp}_G)_{\geq 0} & \underset{H}{\overset{\pi_0}{\rightleftarrows}} & \mathrm{Mack}_G
\end{array}
\tag{A.28}
$$

commute, where the vertical functors are the forgetful functors.

Suppose $A \in (\mathrm{CAlg}_G)_{\geq 0}$. Let $(\mathrm{Mod}_A)_{\geq 0}$ denote the ∞-category of A-modules in $(\mathrm{Sp}_G)_{\geq 0}$. By [47, Remark 3.8], we also have adjoint functors

$$\underline{\pi}_0 : (\mathrm{Mod}_A)_{\geq 0} \rightleftarrows \mathrm{Mod}_{\underline{\pi}_0(A)} : \mathrm{H} \tag{A.29}$$

such that the two squares in the diagram

$$
\begin{array}{ccc}
(\mathrm{Mod}_A)_{\geq 0} & \underset{H}{\overset{\pi_0}{\rightleftarrows}} & \mathrm{Mod}_{\underline{\pi}_0(A)} \\
{\scriptstyle U}\downarrow & & \downarrow{\scriptstyle U} \\
(\mathrm{Sp}_G)_{\geq 0} & \underset{H}{\overset{\pi_0}{\rightleftarrows}} & \mathrm{Mack}_G
\end{array}
\tag{A.30}
$$

commute, where the vertical functors are the forgetful functors.

Remark Appendix A.4.5. The equivariant Eilenberg–MacLane spectrum of a Green functor does *not* produce an object of NAlg_G in general; see [57, Theorem 5.3, Proposition 6.1]. We need the stronger notion of Tambara functors to construct an object of NAlg_G as the equivariant Eilenberg–MacLane spectrum. We refer to [57] for the details.

A.5. Flat modules

Definition Appendix A.5.1. Let A be a Green functor. An A-module M is called *flat* if the functor $M\square_A(-)$ from the category of A-modules to the category of Mackey functors is exact. Equivalently, $\underline{\mathrm{Tor}}_s^A(-,M) = 0$ for every integer $s \geq 1$. This definition was considered in [38, section 4].

If $A \to B$ is a morphism of Green functors and M is a flat A-module, then $M\square_A B$ is a flat B-module.

Definition Appendix A.5.2. Recall from [38, section 2] that the *Burnside category* $\mathfrak{B}_G$ is defined to be the additive category whose objects are the finite G-sets and whose hom groups are given by

$$\mathrm{Hom}_{\mathfrak{B}_G}(X,Y) := \mathrm{Hom}_{\mathrm{Ho}(\mathrm{Sp}_G)}(\Sigma^\infty X_+, \Sigma^\infty Y_+)$$

for all finite G-sets X and Y.

For a finite G-set X, let B^X denote the Mackey functor $\mathrm{Hom}_{\mathfrak{B}_G}(-,X)$. As explained in [38, p. 519], there is an isomorphism

$$M\square B^X(Y) \cong M(Y \times X) \tag{A.31}$$

for all finite G-sets X and Y.

Proposition Appendix A.5.3. *Let A be a Green functor. Then an A-module M is projective if and only if M is a direct summand of a direct sum of A-modules of the form $A\square B^{G/H}$, where H is a subgroup of G. Furthermore, every projective A-module is flat.*

Proof. These are nongraded versions of [38, Proposition 4.4, Theorem 4.5(c)]. See also [22, Corollary 1.5] for the first claim. $\square$

Definition Appendix A.5.4. Suppose $A \in \mathrm{CAlg}_G$. An A-module M is called *flat* if the following two conditions are satisfied:

(i) $\underline{\pi}_0(M)$ is a flat $\underline{\pi}_0(A)$-module,

(ii) the induced map

$$\underline{\pi}_n(A)\square_{\underline{\pi}_0(A)}\underline{\pi}_0(M) \to \underline{\pi}_n(M)$$

is an isomorphism for every integer n.

Proposition Appendix A.5.5. *Let $A \to B$ be a map in CAlg_G, and let M be a flat A-module. Then $B \wedge_A M$ is a flat B-module. Consequently, if we have maps $A \to B$, $A \to C$ and $B \to L$ in CAlg_G such that L is a flat B-module, then the induced map $B \wedge_A C \to L \wedge_A C$ is flat.*

Proof. The conditions (i) and (ii) in Definition Appendix A.5.4 for M imply that $\underline{\pi}_*(M)$ is a flat $\underline{\pi}_*(A)$-module. This means $\underline{\mathrm{Tor}}_p^{\underline{\pi}_*(A)}(-, \underline{\pi}_*(M))_q = 0$ for all integers $p \geq 1$ and q. Together with the convergent spectral sequence

$$E_{p,q}^2 := \underline{\mathrm{Tor}}_p^{\underline{\pi}_*(A)}(\underline{\pi}_*(B), \underline{\pi}_*(M))_q \Rightarrow \underline{\pi}_{p+q}(B \wedge_A M)$$

obtained from Proposition Appendix A.4.3, we obtain isomorphisms of Mackey functors

$$\underline{\pi}_n(B) \square_{\underline{\pi}_0(A)} \underline{\pi}_0(M) \cong (\underline{\pi}_*(B) \square_{\underline{\pi}_0(A)} \underline{\pi}_0(M))_n$$
$$\cong (\underline{\pi}_*(B) \square_{\underline{\pi}_*(A)} \underline{\pi}_*(M))_n \cong \underline{\pi}_n(B \wedge_A M)$$

for all integers n. This implies the conditions (i) and (ii) in Definition Appendix A.5.4 for $B \wedge_A M$. The second statement follows from the first applied to $B \to B \wedge_A C$. $\qquad\square$

Proposition Appendix A.5.6. *Let $f \colon A \to B$ be a flat map in CAlg_G. If the induced morphism $\underline{\pi}_0(A) \to \underline{\pi}_0(B)$ is an isomorphism, then f is an equivalence.*

Proof. Immediate from the condition (ii) in Definition Appendix A.5.4. $\qquad\square$

Proposition Appendix A.5.7. *Let A be a Green functor, and let M and L be A-modules. If M is flat, then there is an equivalence*

$$\mathrm{H}(M \square_A L) \simeq \mathrm{H}M \wedge_{\mathrm{H}A} \mathrm{H}L$$

Proof. From the convergent spectral sequence

$$E_{p,q}^2 := \underline{\mathrm{Tor}}_p^{\underline{\pi}_*(\mathrm{H}A)}(\underline{\pi}_*(\mathrm{H}M), \underline{\pi}_*(\mathrm{H}L))_q \Rightarrow \underline{\pi}_{p+q}(\mathrm{H}M \wedge_{\mathrm{H}A} \mathrm{H}L)$$

obtained from Proposition Appendix A.4.3, we have

$$M \square_A L \xrightarrow{\simeq} \underline{\pi}_0(\mathrm{H}M \wedge_{\mathrm{H}A} \mathrm{H}L) \text{ and } \underline{\pi}_k(\mathrm{H}M \wedge_{\mathrm{H}A} \mathrm{H}L) \cong 0$$

for every nonzero integer k. $\qquad\square$

Proposition Appendix A.5.8. *Let A be a Green functor. If $\mathrm{colim}_{i \in I} M_i$ is a filtered colimit of A-modules and L is an A-module, then there is a canonical equivalence*

$$\mathrm{colim}_{i \in I}(M_i \square_A L) \cong (\mathrm{colim}_{i \in I} M_i) \square_A L.$$

Proof. Since $\wedge$ commutes with colimits in each variable, we have a canonical equivalence

$$\mathrm{colim}_{i \in I}(\mathrm{H}M_i \wedge_{\mathrm{H}A} \mathrm{H}L) \simeq (\mathrm{colim}_{i \in I} \mathrm{H}M_i) \wedge_{\mathrm{H}A} \mathrm{H}L.$$

Apply $\underline{\pi}_0$ to this, and use Propositions Appendix A.3.2, Appendix A.3.6 and Appendix A.4.4 to obtain the desired equivalence. $\qquad\square$

Proposition Appendix A.5.9. *Let A be a Green functor, and let*

$$\{0 \to M_i' \to M_i \to M_i'' \to 0\}_{i \in I}$$

be a system of exact sequence of A-modules over a filtered category I. Then the induced sequence

$$0 \to \mathrm{colim}_{i \in I} M_i' \to \mathrm{colim}_{i \in I} M_i \to \mathrm{colim}_{i \in I} M_i'' \to 0$$

is exact.

Proof. We have a system of cofiber sequences in Sp_G

$$\{\mathrm{H}M_i' \to \mathrm{H}M_i \to \mathrm{H}M_i''\}_{i \in I}.$$

Take colimits and use Proposition Appendix A.3.6 to obtain a cofiber sequence

$$\mathrm{H}\operatorname*{colim}_{i \in I} M_i' \to \mathrm{H}\operatorname*{colim}_{i \in I} M_i \to \mathrm{H}\operatorname*{colim}_{i \in I} M_i''.$$

Together with the fact that cofiber sequences and exact sequences coincide in the heart of a t-structure, we deduce the claim. $\qquad\square$

Proposition Appendix A.5.10. *Let A be a Green functor. Then every filtered colimit* $\operatorname{colim}_{i \in I} M_i$ *of flat A-modules is flat.*

Proof. Let $0 \to L' \to L \to L'' \to 0$ be an exact sequence of A-modules. Then the induced sequence $0 \to M_i \square_A L' \to M_i \square_A L \to M_i \square_A L'' \to 0$ is exact, so the induced sequence

$$0 \to \operatorname*{colim}_{i \in I}(M_i \square_A L') \to \operatorname*{colim}_{i \in I}(M_i \square_A L) \to \operatorname*{colim}_{i \in I}(M_i \square_A L'') \to 0$$

is exact too by Proposition Appendix A.5.9. Combine with Proposition Appendix A.5.8 to conclude. $\qquad\square$

Proposition Appendix A.5.11. *Suppose $A \in \mathrm{CAlg}_G$. Then every filtered colimit of flat A-modules is flat. In particular, every filtered colimit of free A-modules is flat.*

Proof. Combine Propositions Appendix A.3.2 and Appendix A.5.10 to show the first claim. For the second claim, use Proposition Appendix A.3.2 to show that every free A-module is flat. $\qquad\square$

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