AN IMPROVED RESULT CONCERNING SINGULAR MANIFOLDS OF DIFFERENCE POLYNOMIALS

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1. Introduction. Let $\mathfrak{K}$ be a difference field of characteristic 0, $\mathfrak{M}$ an irreducible manifold of effective order n over $\mathfrak{K}\{y\}$, and F an algebraically irreducible difference polynomial in $\mathfrak{K}\{y\}$ of effective order $n + k$, $k > 0$, which vanishes on $\mathfrak{M}$. In an earlier paper **(2**, p. 447**)** I gave necessary conditions, restated below as (a), (b), and (c) of the main theorem, for $\mathfrak{M}$ to be an essential singular manifold of F. These conditions are analogous to the low power criterion of Ritt **(1**, p. 65**)** for the corresponding problem of differential algebra. Like that criterion they depend, in the special case that $\mathfrak{M}$ is the manifold of y, only on which power products appear effectively in F. Unlike the low power criterion, however, conditions (a), (b), and (c) are only necessary, not sufficient. I have proved the following results **(2**, p. 459; **4)** concerning sufficiency:

(1) *if* $k = 1$, *the conditions are never satisfied, so that* $\mathfrak{M}$ *is not an essential singular manifold of* F;

(2) *if* $k = 2$, $n = 0$, *the condition is both necessary and sufficient;*

(3) *if* $k > 2$, *the condition is not sufficient, even if* $n = 0$. *Moreover, no condition dependent only on which power products of* y *and its transforms appear effectively in* F *is sufficient in the special case that* $\mathfrak{M}$ *is the manifold of* y.

I shall now show that the restriction $n = 0$ may be removed from (2). Hence, there is a close analogy with the situation in differential algebra described by the low power theorem in the case that the effective order of the difference polynomial F exceeds by 2 the effective order of the manifold $\mathfrak{M}$, but only a partial analogy in all other cases.

The proof for $k = 2$ and "general" n is based on a preparation theorem suggested by the preparation theorem used by Ritt for differential polynomials. The preparation theorem of difference algebra (restricted to the case that $\mathfrak{K}$ is inversive) consists of the relations (3) and (4) of § 5 between F and the first polynomial A of the characteristic set of the reflexive prime difference ideal with manifold $\mathfrak{M}$. The conditions (a), (b), and (c) of the main theorem are equivalent to the conditions (α) and (β), stated in § 8, for (3) and (4). These conditions in turn imply that $\mathfrak{M}$ is an essential singular manifold of F in the case $k = 2$. The proof of this is accomplished by a minor modification of the power series method used in **(4)** for the special case $n = 0$ and the conditions (a), (b), and (c).

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2. The *weight function* $f(\theta)$ of a term

$$\sigma y^{a_0} y_1^{a_1} \ldots y_r^{a_r}, \qquad \sigma \neq 0, \sigma \in \mathfrak{K},$$

of a difference polynomial of $\mathfrak{K}\{y\}$ is defined to be the polynomial $a_0 + a_1\theta + \ldots + a_r\theta^r$. The indeterminate θ is called the *weight parameter*. The *weight* of this term for a value τ of the weight parameter is $f(\tau)$. If an element c whose transform is defined to be c^τ is substituted formally into a term, then the exponent of c in the result is the weight of the term for the value τ of the weight parameter.

Let F and $\mathfrak{M}$ be as in § 1. Denote by α a generic zero of $\mathfrak{M}$ and by Σ the reflexive prime difference ideal with manifold $\mathfrak{M}$. Let A be an algebraically irreducible polynomial in Σ of effective order n—if $\mathfrak{K}$ is inversive, A is the first polynomial of a characteristic set of Σ or one of its transforms. If P is any polynomial of $\mathfrak{K}\{y\}$, then $\bar{P}$ is to denote the polynomial of $\mathfrak{K} <\alpha>\{z\}$ which is obtained from P by the substitution $y = z + \alpha$, and P^* the polynomial consisting of the terms of least degree of $\bar{P}$. We can now state the main theorem.

THEOREM. *In order that $\mathfrak{M}$ be an essential singular manifold of F it is necessary that:*

(a) *there exist a term of $\bar{F}$ which is of lower weight than any other term for every positive value $\tau < 1$ of the weight parameter,*

(b) *there exist a term of $\bar{F}$ which is of lower weight than any other term for every value $\tau > 1$ of the weight parameter,*

(c) *every solution of F^* be a solution of A^*.*

These conditions are sufficient if $k = 2$.

It only remains to prove sufficiency in the case $k = 2$. The rest of this paper is devoted mainly to this proof. In the last section a method is given for testing conditions (a), (b), and (c) constructively if a beginning of a characteristic sequence of the ideal Σ is known.

3. Proof of a lemma. The first lemma to be proved concerns polynomial rings, the second, difference rings.

LEMMA 1. *Let $\mathfrak{K}$ be a field of characteristic* 0, Π *a prime ideal in the polynomial ring $\mathfrak{R} = \mathfrak{K}[u_1, \ldots, u_q; x_1, \ldots, x_r]$, the u_i forming a parametric set for* Π. *Let $A_1, \ldots, A_r$ be a characteristic set for Π with A_i introducing x_i. Let F be a polynomial of $\mathfrak{R}$. Then there exists a polynomial S of $\mathfrak{R}$ which is not in* Π *and an integer t such that*

$$SF = \sum_{i=1}^{t} L_i A_1^{p_{i,1}} A_2^{p_{i,2}} \ldots A_r^{p_{i,r}}, \tag{1}$$

the L_i being polynomials of R which are not in Π, *and the $p_{i,j}$, $i = 1, \ldots, t$; $j = 1, \ldots, r$, constituting t distinct sets of non-negative integers.*

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Proof. A polynomial of $\mathfrak{R}$ is said to be of *class* k if it effectively involves x_k but no x_i, $i > k$. The conclusion of the lemma follows trivially if F is free of all the x_i. We shall prove by induction on the class that it is valid for other polynomials in the following strengthened form: Let a_i denote the degree of A_i in x_i, $i = 1, \ldots, r$. Then if F is of class k and degree f in x_k it is possible to find a relation (1) in which S and the L_i are free of the x_i, $i > k$, and the power products in the A_i which occur on the right-hand side of (1) involve no A_i, $i > k$, and are of degree in A_k less than or equal to the greatest integer h_k not exceeding f/a_k.

The strengthened result holds if F is of class 1. For, if $h \geqslant 0$ is the greatest power of A_1 which divides F, we may write $F = LA_1{}^h$. This expression is of the form (1) and meets the added conditions.

Let F be of class $k > 1$, and assume the strengthened conclusion to have been proved for all polynomials of lower class. Let f be as before. If $f < a_k$, we use the expression

$$F = F_0 + F_1 x_k + \ldots + F_f x_k^f,$$

each F_i being free of x_i, $i \geqslant k$. For each F_i, $0 \leqslant i \leqslant f$, we find an expression of the form of (1):

$$S_i F_i = \sum_{j=1}^{t_i} L_{ij} A_1^{p_{i,j,1}} \ldots A_{k-1}^{p_{i,j,k-1}}, \tag{2}$$

where S_i and the L_{ij} are not in Π and are of class less than k.

Let S be the product of the S_i. Substituting from the expressions (2) into

$$SF = (S/S_0)S_0F_0 + (S/S_1)S_1F_1x_k + \ldots + (S/S_f)S_fx_k^f$$

and combining terms involving equal power products of the A_i we obtain an expression of the form (1) for SF. The coefficients L_i of this expression are polynomials in x_k of degree less than a_k, with coefficients free of x_i, $i \geqslant k$, and not in Π. Hence, the L_i themselves are not in Π. Clearly, S is not in Π, and A_k does not appear in the power products; so that the strengthened conclusion is valid for F.

We now suppose that $f \geqslant a_k$, and that the strengthened conclusion has been demonstrated for all polynomials of class k and degree less than f in x_k. Applying the division algorithm to F and $A_k^{h_k}$ we find a relation

$$JF - MA_k^{h_k} = R,$$

where J, M, and R are polynomials free of x_i, $i > k$, J is not in Π, M is of degree less than f in x_k, and R is of degree less than $h_k a_k$ in x_k. By the assumption made at the beginning of this paragraph there exists a polynomial S_1 not in Π and free of x_i, $i > k$, such that S_1R is a linear combination of power products of $A_1, \ldots, A_k$ of degree less than h_k in A_k, the coefficients of these power products being polynomials not in Π and free of x_i, $i > k$. By the case $f < a_k$ previously disposed of there exists a polynomial S_2 not in Π and free of x_i, $i > k$, such that S_2M is a linear combination of power products of

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$A_1, \ldots, A_{k-1}$, the coefficients of these power products being polynomials not in Π and free of x_i, $i > k$.

Let $S = S_1 S_2 J$. In

$$SF = S_1 S_2 M A_k^{h_k} + S_2 S_1 R$$

we substitute the expressions for $S_2 M$ and $S_1 R$ just described. There results an expression for SF as a linear combination of power products of $A_1, \ldots, A_k$. Those power products obtained from $S_2 S_1 R$ are of degree less than h_k in A_k and therefore distinct from the power products obtained from

$$S_1 S_2 M A_k^{h_k}.$$

One can verify immediately that the expression for SF has the properties prescribed in the strengthened form of the conclusion to Lemma I.

4. A second lemma.

LEMMA II. *Let $\mathfrak{K}$ be a difference field of characteristic* 0, *A an algebraically irreducible difference polynomial of order and effective order n in the difference ring $\mathfrak{K}\{y\}$, and $C^{(0)}$ $(= A)$, $C^{(1)}$, $C^{(2)}, \ldots$, a characteristic sequence of a non-singular component Σ of $\{A\}$. There exist difference polynomials $M^{(0)}$, $M^{(1)}$, $M^{(2)}, \ldots$, of orders not exceeding $n, n+1, n+2, \ldots$, respectively, which are not in Σ, and for which each product $C^{(k)}$, $M^{(k)}$ $k = 0, 1, \ldots$, is a linear combination of $A, A_1, \ldots, A_k$ with coefficients of order not exceeding $n+k$, while the coefficient of A_k is not in Σ.*

Proof. We choose $M^{(0)} = 1$. We suppose $M^{(1)}, \ldots, M^{(k-1)}$ to have been found and demonstrate the existence of $M^{(k)}$.

Since A_k has remainder 0 with respect to the chain $C^{(0)}, \ldots, C^{(k)}$ there is a relation

$$TC^{(k)} = JA_k + L^{(0)}C^{(0)} + \ldots + L^{(k-1)}C^{(k-1)}, \qquad J \notin \Sigma.$$

Multiplying both sides of this equation by $M^{(0)} \ldots M^{(k-1)}$, replacing each $M^{(i)}C^{(i)}$, $i = 0, \ldots, k-1$, by the appropriate linear combination of $A, \ldots, A_i$, and putting $TM^{(0)} \ldots M^{(k-1)} = M^{(k)}$ there results

$$M^{(k)}C^{(k)} = N^{(0)}A + \ldots + N^{(k)}A_k,$$

with $N^{(k)} = JM^{(0)} \ldots M^{(k-1)} \notin \Sigma$. Since the formal partial derivative $\partial A_k / \partial y_{n+k}$ is not in Σ, it is immediately seen by differentiation of $M^{(k)}C^{(k)}$ that $M^{(k)}$, too, is not in Σ. This proves Lemma II.

5. The preparation process. Let Σ be a reflexive prime difference ideal of order n in the ring $\mathfrak{K}\{y\}$, where the difference field $\mathfrak{K}$ is inversive and of characteristic 0, and let $F \in \mathfrak{K}\{y\}$ be of order $n+k$, $k \geqslant 0$. The following theorem provides two expressions for F in terms of the first polynomial A of the characteristic set of Σ.

PREPARATION THEOREM. *There exist difference polynomials S, T, of order at most $n+k$, which are not in Σ, and positive integers s, t such that*

$$SF = \sum_{i=1}^{s} L^{(i)} A^{p_{i,0}} A_1^{p_{i,1}} \ldots A_k^{p_{i,k}}, \tag{3}$$

$$TF = \sum_{l=1}^{t} N^{(i)} A^{q_{i,0}} A_1^{q_{i,1}} \ldots A_k^{q_{i,k}}. \tag{4}$$

Here the $p_{i,j}$ are non-negative integers, no two sets $p_{a,j}$, $p_{b,j}$ $(a \neq b, j = 0, \ldots, k)$ are identical, the $q_{i,j}$ have a similar description, and the $L^{(i)}$ are difference polynomials of order not exceeding $n+k$. Those $L^{(i)}$ which are coefficients of terms whose power products in $A, A_1, \ldots, A_k$ are of least weight for any positive value of the weight parameter not exceeding 1 are not in Σ. Also the $N^{(i)}$ are difference polynomials of order not exceeding $n+k$, while those $N^{(i)}$ which are coefficients of terms whose power products in A, $A_1, \ldots, A_k$ are of least weight for any value of the weight parameter not less than 1 are not in Σ.

6. Proof of (3). Let $C^{(0)}$ $(= A)$, $C^{(1)}, \ldots, C^{(k)}$ be the first $k+1$ polynomials of a characteristic sequence of Σ. They are the characteristic set of the prime ideal

$$\Sigma' = \Sigma \cap \mathfrak{R}\,[y, y_1, \ldots, y_{n+k}].$$

According to Lemma I there exists a relation

$$RF = \sum_{i=1}^{r} P^{(i)} (C^{(0)})^{a_{i,0}} \ldots (C^{(k)})^{a_{i,k}}, \tag{5}$$

where the $a_{i,j}$ have a description similar to that of the $p_{i,j}$ of (3), and R and the $P^{(i)}$ are polynomials of $\mathfrak{R}[y, y_1, \ldots, y_{n+k}]$ which are not in Σ', hence not in Σ.

Let polynomials $M^{(i)}$ be chosen in accordance with Lemma II. Putting $a = \max(a_{i,j})$, let $Q = R(M^{(0)} \ldots M^{(k)})^a$. Then, using (5) and substituting for the $C^{(i)}M^{(i)}$ the linear combinations described in Lemma II, one finds a relation

$$QF = \sum_{i=1}^{q} J^{(i)} A^{r_{i,0}} A_1^{r_{i,1}} \ldots A_k^{r_{i,k}}, \qquad Q \notin \Sigma, \tag{3*}$$

where the $r_{i,j}$ have a description similar to that of the $p_{i,j}$ of (3). We shall show that those $J^{(i)}$ which are coefficients of terms whose power products in the A_i are of least weight for any positive value of the weight parameter less than 1 are not in Σ.

To the ith term of (5) we assign a weight function $w_i(\theta) = a_{i,0} + a_{i,1}\theta + \ldots + a_{i,k}\theta^k$. We consider a positive value $\tau < 1$ of the weight parameter. Then, upon the substitutions prescribed above to obtain (3*), the ith term of (5) gives rise to a term $T^{(i)}$ of the form

$$K^{(i)} A_0{}^{a_{i,0}} \ldots A_k{}^{a_{i,k}}.$$

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$K^{(i)} \notin \Sigma$, and other terms which, because their power products are formed from that of $T^{(i)}$ by replacing one or more of the A_j by lower transforms of A, are of greater weight than $T^{(i)}$. The coefficients of these terms may be in Σ.

If, in particular, the ith term of (5) is one of the terms whose power products are of least weight for the value τ of the weight parameter, then no term of (5) will yield a term of lower weight than $T^{(i)}$. Those terms of (5) of the same weight as $T^{(i)}$ will yield terms of this weight but with different power products. Hence, $T^{(i)}$ will actually be a term of (3*), and one of least weight for the value τ of the weight parameter. Clearly, all terms of (3*) of least weight for the value τ of the weight parameter arise in the same way as $T^{(i)}$ and, hence, have coefficients which are not in Σ. Hence, (3*) has the property claimed for (3) in the preparation theorem, except possibly for terms of least weight for the value 1 of the weight parameter, that is, for terms of least degree in the A_i.

Because the weight function is continuous, at least one of the terms of least degree of (3*) is a term of least weight for a value of the weight parameter less than 1, and hence, has a coefficient which is not in Σ.

Suppose the term

$$J^{(i)} A^{r_{i,0}} A_1^{r_{i,1}} A_k^{r_{i,k}}$$

is a term of (3*) which is of least degree, but that $J^{(i)} \in \Sigma$. Following the procedure used to obtain (3*) we find an equation

$$PJ^{(i)} = \sum_{j=1}^{p} H^{(j)} A^{s_{j,0}} A_1^{s_{j,1}} \dots A_k^{s_{j,k}}, \qquad P \notin \Sigma. \tag{6}$$

No term of the right-hand side of (6) is free of the A_i. For such a term would be of least weight for values of the weight parameter less than 1; hence, its coefficient would not be in Σ. This would yield a contradiction to the fact that $J^{(i)} \in \Sigma$.

Let $Q' = QP$. Multiplying both sides of (3*) by P, and substituting for $PJ^{(i)}$ from (6) we obtain an expression for $Q'F$ whose terms are those of the right-hand side of (3*) multiplied by P, except that the ith term of (3*) has been replaced by terms whose power products are multiples of its power product by power products of positive degree. Consequently, the terms of this expression which are of least weight for values of the weight parameter less than 1 have coefficients which are not in Σ, and the number of terms of least degree with coefficients in Σ is less than the number of such terms in (3*). Continuing the procedure just described, we obtain the equation (3).

7. Proof of (4). We define the difference field $\mathfrak{K}'$ to be the difference field whose elements are those of $\mathfrak{K}$ with the same addition and multiplication operations, but with transforming defined to be the inverse of the transforming operation of $\mathfrak{K}$. Let $\mathfrak{L}$ denote the inversive extension of $\mathfrak{K}\langle\alpha\rangle$, where α denotes a generic zero of Σ. We define $\mathfrak{L}'$ to have the same relation to $\mathfrak{L}$ as $\mathfrak{K}'$ to $\mathfrak{K}$. Then $\mathfrak{L}'$ is an extension of $\mathfrak{K}'$.

Let $P \in \mathfrak{K}\{y\}$ be of order at most $n + k$. We define P' to be the polynomial of $\mathfrak{K}'\{z\}$ obtained by replacing each y_i in P by z_{n+k-i}. The operation $'$ produces a one-one correspondence between difference polynomials of order at most $n + k$ in $\mathfrak{K}\{y\}$ and in $\mathfrak{K}'\{z\}$. In particular, $B = (A_k)'$ is of order n, and B_h, $0 \leqslant h \leqslant k$, is $(A_{k-h})'$. α_{n+k} (where the subscript refers to transforming in $\mathfrak{L}$) is a generic zero of a reflexive prime difference ideal Σ' of $\mathfrak{K}'\{z\}$ whose characteristic set begins with B. The correspondence produced by $'$ maps the polynomials of Σ of order not exceeding $n + k$ onto the polynomials of Σ' of order not exceeding $n + k$.

Since (3) has been established, we know that there exists a relation

$$T'F' = \sum_{i=1}^{t} N^{(i)\prime} B^{q_{i,k}} B_1^{q_{i,k-1}} \ldots B_k^{q_{i,0}}, \tag{4'}$$

meeting requirements corresponding to those imposed on (3). Now (4′) yields (4) on application of the inverse of the correspondence produced by $'$. Clearly, $T \notin \Sigma$. It remains only to show that the $N^{(i)}$ have the stated property.

Let $T^{(i)}$ denote the ith term of (4) and

$$w_i(\theta) = q_{i,0} + q_{i,1}\theta + \ldots + q_{i,k}\theta^k$$

its weight function. The ith term $T^{(i)\prime}$ of (4′) has the weight function

$$v_i(\theta) = q_{i,0}\theta^k + \ldots + q_{i,k} = \theta^k w_i(1/\theta).$$

Hence, if $T^{(i)}$ is a term of (4) of least weight for the value $\tau \geqslant 1$ of the weight parameter, then $T^{(i)\prime}$ is a term of least weight of (4′) for the value $1/\tau \leqslant 1$ of the weight parameter. Then $N^{(i)\prime} \notin \Sigma'$, so $N^{(i)} \notin \Sigma$. This completes the proof of the preparation theorem.

8. Proof of Equivalence. We now assume that F and A as described in § 5 also satisfy the conditions (a), (b), and (c) stated in the main theorem. As in that theorem, F is to be irreducible and vanish on the irreducible manifold $\mathfrak{M}$ of order n. As in the discussion of the preparation theorem, $\mathfrak{K}$ is assumed to be inversive, and A is chosen to be the first polynomial of the characteristic set of the reflexive prime difference ideal Σ with manifold $\mathfrak{M}$. In addition, we assume that F is of order and effective order $n + k$ with $k > 0$. It will be shown that there exists a power product U of the A_i which is of positive degree and is free of A and A_k such that

(α) *the right-hand side of* (3) *contains a term* YU, $Y \notin \Sigma$, *which is the term of least weight in the* A_i *for each positive value of the weight parameter not greater than* 1,

(β) *the right-hand side of* (4) *contains a term* ZU, $Z \notin \Sigma$, *which is the term of least weight in the* A_i *for each value of the weight parameter not less than* 1.

We follow the notation used in the statement of the main theorem. Because $\partial A/\partial y$ and $\partial A/\partial y_n$ are not in Σ, A^* is a polynomial of first degree which effectively contains z and z_n. S^* and T^* are in $\mathfrak{K} <\alpha>$. We find

$$F^* = \sum{}' \ (L^{i*}/S^*)(A^*)^{p_{i,0}} \dots (A_k{}^*)^{p_{i,k}}, \tag{7}$$

$$F^* = \sum{}'' \ (N^{i*}/T^*)(A^*)^{q_{i,0}} \dots (A_k{}^*)^{q_{i,k}}, \tag{8}$$

where Σ' and Σ'' are taken over those terms of (3) and (4) respectively which are of least degree in the A_i. It follows from the preparation theorem that the L^{*i} and N^{*i} appearing in these sums are elements of $\mathfrak{K} <\alpha>$.

Suppose that Σ', say, consists of more than one term. Then the homogeneous polynomial

$$\sum{}' \ (L^{*i}/S^*)u^{p_{i,0}} \dots u_k^{p_{i,k}}$$

of the difference ring $\mathfrak{K} <\alpha>\{u\}$ is not a product of irreducible factors of effective order 0, and hence has a solution $u = \beta$, $\beta \neq 0$. The polynomial $A^* - \beta$ has a solution $z = \gamma$. Then γ is a solution of F^* but not of A^*, contrary to condition (c) of the main theorem. Hence, Σ' consists of just one term. Similarly, Σ'' consists of just one term. Because the left-hand sides of (7) and (8) are identical, and the coefficients on the right-hand sides are in $\mathfrak{K} <\alpha>$, it is clear that the same power product of the $A^*{}_i$ occurs in each of these terms. We denote by U the corresponding power product of the A_i.

Consider a value $\tau < 1$ of the weight parameter. Let w_i be the weight of the term $T^{(i)} = L^{(i)}A^{p_{i,0}} \dots A_k{}^{p_{i,k}}$ of (3). Upon substituting $y = z + \alpha$ the power product

$$A^{p_{i,0}} \dots A_k^{p_{i,k}}$$

yields a term with power product

$$z_n^{p_{i,0}} \dots z_{n+k}^{p_{i,k}}$$

of weight $\tau^n w_i$ and other terms whose weights are greater, since their power products are formed by replacing transforms of z in the indicated term by lower transforms.

If, in particular, $T^{(i)}$ is a term of (3) of least weight, $L^{(i)}$ is not in Σ, so that $T^{(i)}$ itself will yield a term with the above power product. Terms of greater weight than $T^{(i)}$ must yield only power products of z and its transforms of weight greater than $\tau^n w_i$. If

$$T^{(j)} = L^{(j)}A^{p_{j,0}} \dots A_k^{p_{j,k}}, \qquad j \neq i,$$

is also of weight w_i it yields a distinct power product of weight $\tau^n w_i$ and other power products of greater weight. Hence, one power product of weight $\tau^n w_i$ appears in the polynomial $\bar{F}$ of the main theorem for each term of (3) of least weight, and these power products are of least weight in $\bar{F}$. Since, by hypothesis, there is a unique term of $\bar{F}$ of least weight, (3) contains a unique term of least weight for the value τ of the weight parameter. By continuity of the weight function this term is the same for all values of the weight parameter less than 1. It follows at once that it is the term of (3) of least degree.

Let the weight parameter be $\tau > 1$. Let the term

$$N^{(i)}A^{q_{i,0}}\dots A_k^{q_{i,k}}$$

of (4) have weight w_i. Upon substituting $y = z + \alpha$ into the power product

$$A^{q_{i,0}}\dots A_k^{q_{i,k}}$$

there results a term with power product

$$z^{q_{i,0}}\dots z_k^{q_{i,k}}$$

of weight w_i and other terms whose weights are greater since their power products are formed by replacing transforms of z in the indicated term by higher transforms. It follows, as above, that the term of (4) of least degree is the unique term of least weight for all values of the weight parameter exceeding 1.

Not every term on the right-hand side of (3) actually contains A, since F has no factors of order n, and A does not divide S. For sufficiently small values of the weight parameter a term free of A is certainly of lower weight than a term which contains A. Hence, U is free of A. Using (4) and considering large values of the weight parameter, we find that U is free of A_k. This completes the proof of the statements made at the beginning of this section.

It is easy, but unnecessary for the proof of the main theorem, to show that (α) and (β) are equivalent to (a), (b), and (c). Let (α) and (β) hold. For positive values less than 1 of the weight parameter, $\bar{U}$ (for explanation of the notation see the paragraph preceding the statement of the main theorem) contains a unique term of least weight. It follows from (3) and (α) that this term furnishes the unique term of least weight in $\bar{F}$. In a similar way it follows from (4) and (β) that $\bar{F}$ contains a unique term of least weight for values of the weight parameter exceeding 1. Hence, (a) and (b) hold. From either (3) or (4) there results $F^* = \gamma U^*$, $\gamma \in \mathfrak{K}\langle\alpha\rangle$. Since, clearly, U^* is a product of powers of transforms of A^*, this implies (c).

9. Completion of the proof. If $\mathfrak{M}$ is not an essential singular manifold of F there exist, as we shall see, certain formal power series solutions of F and its transforms. But we shall also see that such solutions cannot exist when $k = 2$ and the conditions (α) and (β) hold. These facts establish the main theorem. Throughout this work we maintain the restrictions of § 8. These restrictions are removed in § 14.

10. Existence of series solutions. We suppose that $\mathfrak{M}$ is not an essential singular manifold of F. Then there exists a reflexive prime difference ideal Λ containing F and properly contained in Σ. According to Lemma IV of **(3)**, Λ is of effective order greater than n, so that $A \notin \Lambda$. For any integer $r \geqslant k$

we define Σ_r and Λ_r to be the intersections of Σ and Λ respectively with the ring $\mathfrak{R}_r = \mathfrak{R}[y, \ldots, y_{n+r}]$. Let $\mathfrak{G} = \mathfrak{R}\langle\alpha\rangle$. Λ_r generates an ideal in $\mathfrak{G}[y, \ldots, y_{n+r}]$ whose radical is the intersection of prime components at least one of which admits the solution $y_i = \alpha_i$ $(i = 0, 1, \ldots, n + r)$. Let Λ_r' be such a component. Since Λ_r' and Λ_r have the same dimension **(5**, vol. 2, p. 69**)**, $\Lambda_r' \cap \mathfrak{R}_r = \Lambda_r$. Hence, $A \ldots A_r \notin \Lambda_r'$. Then **(4**, p. 526**)** Λ_r' admits a solution not annulling $A \ldots A_r$

$$y_i = \alpha_i + g_i(h), \qquad i = 0, 1, \ldots, n + r, \tag{9}$$

where h is transcendental over $\mathfrak{G}$ and the $g_i(h)$ are formal series in positive integral powers of h with coefficients algebraic over $\mathfrak{G}$.

11. Non-existence of series solutions. We now suppose that $k = 2$, and that the conditions (α) and (β) hold. We assume the existence of the solutions (9) and obtain a contradiction if r is sufficiently large.

If the series (9) is substituted into a polynomial P of $\mathfrak{R}_r$ there results a series in non-negative integral powers of h. The term of zero degree in this series results from the substitution of the α_i into P and, hence, is 0 if and only if $P \in \Sigma$. In particular, the series obtained from $A, A_1, \ldots, A_r$ are not 0 but begin with terms of positive degree. We denote the series obtained from A_i by

$$k_i(h) = a_i h^{s_i} + \ldots, \qquad i = 0, \ldots, r, \tag{10}$$

where the a_i are algebraic over $\mathfrak{G}$ and not 0, and the s_i are positive. Substitution of (9) into $F, \ldots, F_{r-k}$ $(= F_{\gamma-2})$ gives 0, while substitution into S, T, or the coefficient of the term of least degree on the right-hand side of (3) or of (4) results in a series whose term of zero degree is not 0.

12. We consider the power product U of the A_i described in § 8. Since $k = 2$, $U = A_1^d$, $d > 0$. The numbering of the terms of the right-hand sides of (3) and (4) is to be chosen so that those terms whose power products are of degree less than d in A_1 precede the remaining terms, and the term with power product U is last. Let s' and t' denote the number of terms on the right-hand sides of (3) and (4) respectively whose power products are of degree less than d in A_1. Since not every term on the right-hand side of (3) or of (4) has the factor A_1, $1 \leqslant s' < s$, $1 \leqslant t' < t$.

Let

$$\begin{aligned} p_i(\theta) &= p_{i,0} + (p_{i,1} - d)\theta + p_{i,2}\theta^2, \qquad 1 \leqslant i \leqslant s', \\ q_i(\theta) &= q_{i,0} + (q_{i,1} - d)\theta + q_{i,2}\theta^2, \qquad 1 \leqslant i \leqslant t'. \end{aligned} \tag{11}$$

Since the power product A_1^d is of lower weight than the other power products on the right-hand side of (3) for positive values of the weight parameter not exceeding 1, $p_i(\theta) > 0$, $0 < \theta \leqslant 1$. Since $p_{i,1} - d < 0$, $1 \leqslant i \leqslant s'$, it follows that $p_{i,0} > 0$ for these i. Then each $p_i(\theta)$ is bounded away from 0 on the interval $0 \leqslant \theta \leqslant 1$. Similarly, $q_i(\theta) > 0$, $\theta \geqslant 1$, whence it follows that the

$q_i(\theta)$ are bounded away from 0 on this interval, and that the $q_{i,2}$, $1 \leqslant i \leqslant t'$, are all positive. Let $m > 0$ be such that

$$(12) \qquad \begin{aligned} &p_i(\theta) > m, \quad && 0 \leqslant \theta \leqslant 1, \quad && 1 \leqslant i \leqslant s', \\ &q_i(\theta) > m, \quad && \theta \geqslant 1, \quad && 1 \leqslant i \leqslant t'. \end{aligned}$$

We define a to be the maximum of the quotients $(d - q_{i,1})/q_{i,2}$, $1 \leqslant i \leqslant t'$, and b to be the maximum of the $p_{i,2}$, $1 \leqslant i \leqslant s'$, $q_{i,2}$, $1 \leqslant i \leqslant t'$. Then $a, b > 0$. Let $c = m/ab$, $d = a/c$. We shall obtain a contradiction from (9) with $r \geqslant d + 2$.

13. Upon substituting the series (9) into the right-hand side of (3) the result must be zero. This can only be so if the power product of some term of (3) other than the last term produces an expansion beginning with a power of h not higher than that with which the expansion of the last term begins. Since the last term is YA_1^d, $Y \notin \Sigma$, this means that there is an integer j_0 such that

$$p_{j_0,0}s_0 + (p_{j_0,1} - d)s_1 + p_{j_0,2}s_2 \leqslant 0.$$

From the definition of s' it is clear that $1 \leqslant j_0 \leqslant s'$. By applying similar reasoning to the right-hand side of (4) and to the transforms of orders not exceeding $r - 2$ of the right-hand sides of (3) and of (4) it follows that there exist integers j_i, k_i, $0 \leqslant i \leqslant r - 2$, such that

$$(13) \qquad \begin{aligned} &1 \leqslant j_i \leqslant s', \qquad 1 \leqslant k_i \leqslant t'; \\ &p_{j_i,0}s_i + (p_{j_i,1} - d)s_{i+1} + p_{j_i,2}s_{i+2} \leqslant 0; \\ &q_{k_i,0}s_i + (q_{k_i,1} - d)s_{i+1} + q_{k_i,2}s_{i+2} \leqslant 0. \end{aligned}$$

Let $t_i = s_{i+1}/s_i > 0$, $(i = 0, \ldots, r - 1)$. Then (13) yields, for $0 \leqslant i \leqslant r - 2$,

$$(14) \qquad \begin{aligned} &p_{j_i,0} + (p_{j_i,1} - d)t_i + p_{j_i,2}t_it_{i+1} \leqslant 0; \\ &q_{k_i,0} + (q_{k_i,1} - d)t_i + q_{k_i,2}t_it_{i+1} \leqslant 0. \end{aligned}$$

It follows from (12) that for each i, $0 \leqslant i \leqslant r - 2$, either $0 < t_i \leqslant 1$, and

$$p_{j_i,0} + (p_{j_i,1} - d)t_i + p_{j_i,2}t_i^2 > m;$$

or $t_i > 1$, and

$$q_{k_i,0} + (q_{k_i,1} - d)t_i + q_{k_i,2}t_i^2 > m.$$

From whichever of these inequalities is applicable it follows by subtraction of the corresponding inequality of (14) that either

$$p_{j_i,2}t_i(t_i - t_{i+1}) > m,$$

or

$$q_{k_i,2}t_i(t_i - t_{1+1}) > m.$$

In either case

$$(15) \qquad t_i(t_i - t_{i+1}) > m/b, \qquad 0 \leqslant i \leqslant r - 2.$$

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Now (14) yields

$$(q_{k_i,1} - d) + q_{k_i,2}t_{i+1} \leqslant 0, \qquad 0 \leqslant i \leqslant r-2$$

since, for every i,

$$q_{k_i,0} \geqslant 0.$$

Thus,

$$t_{i+1} \leqslant (d - q_{k_i,1})/q_{k_i,2} \leqslant a, \qquad 0 \leqslant i \leqslant r-2. \tag{16}$$

From (15) and (16) there results $t_1 \leqslant a$, $t_i - t_{i+1} > m/ba = c$, $1 \leqslant i \leqslant r-2$. Hence, $t_{r-1} < a - (r-2)c \leqslant a - dc = 0$. This is the desired contradiction.

14. Removal of the restrictions. It remains only to prove the main theorem without the restrictions of § 9. Let $\bar{\mathfrak{K}}$ be the inversive extension of $\mathfrak{K}$, C the polynomial of order n in $\bar{\mathfrak{K}}\{y\}$ and G the polynomial of order $n+k$ in $\bar{\mathfrak{K}}\{y\}$ whose transforms of the appropriate orders are A and F respectively. Let $\mathfrak{M}'$ be the irreducible manifold over $\bar{\mathfrak{K}}\{y\}$ with generic zero α, and Σ' the reflexive prime difference ideal of $\bar{\mathfrak{K}}\{y\}$ with manifold $\mathfrak{M}'$. Then $\mathfrak{M}'$ is of order n, each irreducible factor of G is of order and effective order $n+k$, and $G \in \Sigma'$.

Let H be an irreducible factor of G which is in Σ'. Then, in the notation of the main theorem, the polynomial consisting of the terms of $\bar{H}$ of least weight for some value of the weight parameter is a factor of the polynomial consisting of the terms of $\bar{G}$ of least weight for this value of the weight parameter. The latter polynomial is an inverse transform of the polynomial consisting of the terms of $\bar{F}$ of least weight.

The first polynomial D of a characteristic set of Σ' divides C. Let $C = PD$. Since α is not a solution of $\partial C/\partial y_n$, $P \notin \Sigma'$. Then $D^* = \gamma C^*$, $\gamma \in \bar{\mathfrak{K}}\langle\alpha\rangle$, $\gamma \neq 0$; and C^* is an inverse transform of A^*.

The preceding statements show that $\mathfrak{M}'$ and H satisfy the conditions (a), (b), and (c), so that $\mathfrak{M}'$ is an essential singular manifold of H according to the restricted case of the main theorem. Hence, there is a polynomial $Q \in \bar{\mathfrak{K}}\{y\}$ such that $Q \notin \Sigma'$, and, if $E \in \Sigma'$, QE vanishes on the manifold of H.

To each irreducible factor of G there corresponds a polynomial with the properties of Q. For this has just been proved for factors which vanish on $\mathfrak{M}'$, and it is evident for other factors. Let R be the product of these polynomials. Then $R \notin \Sigma'$, and, if $E \in \Sigma'$, RE vanishes on the manifold of G. Some transform S of R is in $\mathfrak{K}\{y\}$. Since $\Sigma \leqslant \Sigma'$, $S \notin \Sigma$, and, if $E \in \Sigma$, SE vanishes on the manifold of F. This proves that $\mathfrak{M}$ is a component of $\{F\}$, and, indeed, since its effective order is less than that of F, an essential singular manifold of F. The proof of the main theorem is complete.

15. Constructive methods. It is possible to determine by actual construction whether or not conditions (a), (b), and (c) are satisfied, provided one knows the first k polynomials of a characteristic sequence of Σ. (For the

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meaning to be given to "characteristic sequence" if Σ is not of equal order and effective order, see (**2**, footnote 7).) In fact, it was shown in (**2**, p. 447) that one can determine constructively whether or not (a) and (b) hold. But if (a) and (b) hold, (c) is true if and only if F^* is a product of powers of transforms (including, possibly, inverse transforms) of A^* and a factor in $\mathfrak{K}<\alpha>$. For, on the one hand, this condition clearly implies (c). On the other hand, if (a), (b), and (c) hold it follows from (α) and (β) under the conditions of the restricted form of the main theorem, and from this special case and the reasoning of § 14 in the general case, that F^* is such a product. There is no difficulty in determining constructively whether or not F^* is a product of this type. It follows, in particular, that, if $k = 2$, one can determine constructively, under the stated limitation, whether or not $\mathfrak{M}$ is an essential singular manifold of F.

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