1

The Radon Transform in the Plane

In this chapter we will study basic properties of the Radon transform in the plane. In this setting it is possible to give precise results on uniqueness, stability, reconstruction, and range characterization for the related inverse problem. We will also discuss the normal operator and show that it is an elliptic pseudodifferential operator. These results will act as model cases for the corresponding geodesic X-ray transform results in Chapters 4, 7, 8, and 9. The results are rather classical, and we refer to Helgason (1999) and Natterer (2001) for more detailed treatments (see also Kuchment (2014) for a more recent reference). The chapter concludes with another classical topic: the Funk transform on the 2-sphere.

1.1 Uniqueness and Stability

The *X-ray transform* If of a function f in $\mathbb{R}^n$ encodes the integrals of f over all straight lines, whereas the *Radon transform* Rf encodes the integrals of f over $(n-1)$-dimensional affine planes. We will focus on the case $n = 2$, where the two transforms coincide. There are many ways to parametrize the set of lines in $\mathbb{R}^2$. We will parametrize lines by their normal vector ω and signed distance s from the origin.

Definition 1.1.1 If $f \in C_c^\infty(\mathbb{R}^2)$, the *Radon transform* of f is the function

$$Rf(s,\omega) := \int_{-\infty}^{\infty} f\left(s\omega + t\omega^\perp\right) dt, \quad s \in \mathbb{R},\ \omega \in S^1.$$

Here S^1 is the unit circle, $\omega^\perp$ is the vector in S^1 obtained by rotating ω counterclockwise by 90°, and $C_c^\infty(\mathbb{R}^2)$ denotes the set of smooth compactly supported functions in $\mathbb{R}^2$.

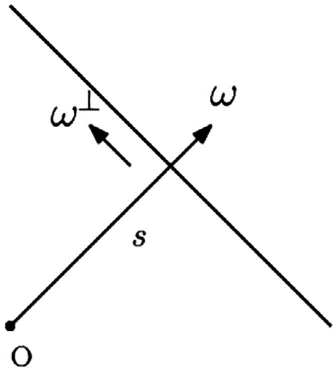

Figure 1.1 Parallel-beam geometry.

Remark 1.1.2 The parametrization of lines by (s, ω) as above is called the *parallel-beam geometry* (see Figure 1.1) and is commonly used for the Radon transform in the plane. When studying the geodesic X-ray transform in Chapter 4 we will however use a different parametrization, the *fan-beam geometry*, which is customary in that context.

The Radon transform arises in medical imaging in the context of *X-ray computed tomography*. In this imaging method, X-rays are sent through the patient from various locations and angles, and one measures how much the rays are attenuated. The measurements correspond to integrals of the unknown attenuation coefficient in the body along straight lines. Moreover, the imaging is often carried out in two-dimensional cross sections of the body, and the idealized measurements (corresponding to X-rays sent from all locations and angles) correspond exactly to the two-dimensional Radon transform. This leads to the basic inverse problem in X-ray computed tomography.

Inverse problem: Determine the attenuation function f in $\mathbb{R}^2$ from X-ray measurements encoded by the Radon transform Rf.

It is easy to see that given any $f \in C_c^\infty(\mathbb{R}^2)$, one has $Rf \in C^\infty(\mathbb{R} \times S^1)$ and for each $\omega \in S^1$ the function $Rf(\,\cdot\,, \omega)$ is compactly supported in $\mathbb{R}$. Moreover, the Radon transform enjoys the following invariance under translations:

$$R(f(\,\cdot\, - s_0\omega))(s, \omega) = Rf(s - s_0, \omega).$$

Exercise 1.1.3 Prove the properties for R stated in the previous paragraph.

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The translation invariance suggests that the Radon transform should behave well under Fourier transforms. Indeed, there is a well-known relation between Rf and the Fourier transform $\hat{f} = \mathscr{F} f$ given by the *Fourier slice theorem*. Here, for $h \in C_c^\infty(\mathbb{R}^n)$ we use the convention

$$\hat{h}(\xi) = \mathscr{F} h(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} h(x)\,dx, \qquad \xi \in \mathbb{R}^n.$$

Recall the following facts regarding the Fourier transform in $\mathbb{R}^n$ (see e.g. Hörmander (1983–1985, chapter 7) for more details):

1. The Fourier transform is bounded $L^1(\mathbb{R}^n) \to L^\infty(\mathbb{R}^n)$.
2. The Fourier transform is bijective $\mathscr{S}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n)$, where $\mathscr{S}(\mathbb{R}^n)$ is the *Schwartz space* consisting of all $f \in C^\infty(\mathbb{R}^n)$ so that $x^\alpha \partial^\beta f \in L^\infty(\mathbb{R}^n)$ for all $\alpha, \beta \in \mathbb{N}_0^n$.
3. Any $f \in \mathscr{S}(\mathbb{R}^n)$ can be recovered from its Fourier transform $\hat{f}$ by the Fourier inversion formula
$$f(x) = \mathscr{F}^{-1}\hat{f}(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \hat{f}(\xi)\,d\xi, \qquad x \in \mathbb{R}^n.$$
4. For $f, g \in \mathscr{S}(\mathbb{R}^n)$ one has the Parseval identity
$$\int_{\mathbb{R}^n} \hat{f}(\xi)\overline{\hat{g}(\xi)}\,d\xi = (2\pi)^n \int_{\mathbb{R}^n} f(x)\overline{g(x)}\,dx,$$
and the Plancherel formula
$$\|\hat{f}\|_{L^2(\mathbb{R}^n)} = (2\pi)^{n/2} \|f\|_{L^2(\mathbb{R}^n)}.$$
5. The Fourier transform converts derivatives to polynomials:
$$(D_j f)\hat{} = \xi_j \hat{f}(\xi), \tag{1.1}$$
where $D_j = \frac{1}{i}\frac{\partial}{\partial x_j}$.

Exercise 1.1.4 Show that R maps $\mathscr{S}(\mathbb{R}^2)$ to $C^\infty(\mathbb{R} \times S^1)$. A more precise result will be given in Theorem 1.2.3.

We will denote by $(Rf)\tilde{}(\,\cdot\,,\omega)$ the Fourier transform of Rf with respect to s. The following theorem states that the one-dimensional Fourier transform $(Rf)\tilde{}(\,\cdot\,,\omega)$ is equal to the slice of the two-dimensional Fourier transform $\hat{f}$ along the line $\sigma \mapsto \sigma\omega$.

Theorem 1.1.5 (Fourier slice theorem) *If $f \in C_c^\infty(\mathbb{R}^2)$, then*

$$(Rf)\tilde{}(\sigma,\omega) = \hat{f}(\sigma\omega).$$

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Proof Parametrizing $\mathbb{R}^2$ by $y = s\omega + t\omega^\perp$, we have

$$(Rf)\tilde{}(\sigma,\omega) = \int_{-\infty}^{\infty} e^{-i\sigma s}\left[\int_{-\infty}^{\infty} f\left(s\omega + t\omega^\perp\right) dt\right] ds$$
$$= \int_{\mathbb{R}^2} e^{-i\sigma y\cdot\omega} f(y)\, dy = \hat{f}(\sigma\omega). \qquad \square$$

This result gives uniqueness in the inverse problem for the Radon transform:

Theorem 1.1.6 (Uniqueness) *If $f_1, f_2 \in C_c^\infty(\mathbb{R}^2)$ are such that $Rf_1 = Rf_2$, then $f_1 = f_2$.*

Proof Since R is linear, it is enough to write $f = f_1 - f_2$ and to show that $Rf \equiv 0$ implies $f \equiv 0$. But if $Rf \equiv 0$, then $\hat{f} \equiv 0$ by Theorem 1.1.5 and consequently $f \equiv 0$ by Fourier inversion. $\square$

In fact, it is easy to prove a quantitative version of the above uniqueness result stating that if $Rf_1 \approx Rf_2$, then $f_1 \approx f_2$ (in suitable norms). Given any $s \in \mathbb{R}$, we will employ the Sobolev norms

$$\|f\|_{H^s(\mathbb{R}^2)} := \left\|\left(1+|\xi|^2\right)^{s/2}\hat{f}(\xi)\right\|_{L^2(\mathbb{R}^2)},$$
$$\|Rf\|_{H_T^s(\mathbb{R}\times S^1)} := \left\|\left(1+\sigma^2\right)^{s/2}(Rf)\tilde{}(\sigma,\omega)\right\|_{L^2(\mathbb{R}\times S^1)}.$$

Exercise 1.1.7 If $m \geq 0$ is an integer, use the Plancherel theorem for the Fourier transform to show that

$$\|f\|_{H^{2m}(\mathbb{R}^2)} \sim \sum_{|\alpha|\leq 2m} \|\partial^\alpha f\|_{L^2(\mathbb{R}^2)},$$
$$\|Rf\|_{H_T^{2m}(\mathbb{R}\times S^1)} \sim \sum_{j=0}^{2m} \left\|\partial_s^j Rf\right\|_{L^2(\mathbb{R}\times S^1)},$$

where $A \sim B$ means that $cA \leq B \leq CA$ for some constants $c, C > 0$ which are independent of f.

Thus, roughly, the $H^s(\mathbb{R}^2)$ norm of f measures the size of the first s derivatives of f in L^2 (this holds by Exercise 1.1.7 when s is an even integer, and remains true for any real number $s \geq 0$ with a suitable interpretation of fractional derivatives). A similar statement holds for the H_T^t norm of Rf, with the difference that the H_T^t norm only involves derivatives in the s variable but not in ω.

Theorem 1.1.8 (Stability) *If $s \in \mathbb{R}$, then for any $f_1, f_2 \in C_c^\infty(\mathbb{R}^2)$ one has*

$$\|f_1 - f_2\|_{H^s(\mathbb{R}^2)} \leq \frac{1}{\sqrt{2}}\|Rf_1 - Rf_2\|_{H_T^{s+1/2}(\mathbb{R}\times S^1)}.$$

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Proof Let $f = f_1 - f_2$. Using polar coordinates, we obtain that

$$\begin{aligned}\|f\|^2_{H^s(\mathbb{R}^2)} &= \big\|\big(1+|\xi|^2\big)^{s/2}\hat{f}\big\|^2_{L^2(\mathbb{R}^2)} = \int_0^\infty \int_{S^1} (1+\sigma^2)^s |\hat{f}(\sigma\omega)|^2 \sigma \, d\omega \, d\sigma \\ &= \frac{1}{2}\int_{-\infty}^\infty \int_{S^1} (1+\sigma^2)^s |\hat{f}(\sigma\omega)|^2 |\sigma| \, d\omega \, d\sigma \\ &= \frac{1}{2}\int_{-\infty}^\infty \int_{S^1} (1+\sigma^2)^s |(Rf)\tilde{}(\sigma,\omega)|^2 |\sigma| \, d\omega \, d\sigma. \end{aligned} \tag{1.2}$$

In particular, since $|\sigma| \leq (1+\sigma^2)^{1/2}$, this implies the stability estimate

$$\|f\|^2_{H^s(\mathbb{R}^2)} \leq \frac{1}{2}\|Rf\|^2_{H_T^{s+1/2}(\mathbb{R}\times S^1)}. \qquad \square$$

If f is supported in a fixed compact set, the previous inequality can be reversed.

Theorem 1.1.9 (Continuity) *Let $s \in \mathbb{R}$ and let $K \subset \mathbb{R}^2$ be compact. There is a constant $C_K > 0$ so that for any $f \in C_c^\infty(\mathbb{R}^2)$ with $\operatorname{supp}(f) \subset K$ one has*

$$\|Rf\|_{H_T^{s+1/2}(\mathbb{R}\times S^1)} \leq C_K \|f\|_{H^s(\mathbb{R}^2)}.$$

Exercise 1.1.10 Prove Theorem 1.1.9 when $s \geq 0$ by splitting the last integral in (1.2) into two parts, one over $\{|\sigma| \leq 1\}$ and the other over $\{|\sigma| > 1\}$.

Exercise 1.1.11 Prove Theorem 1.1.9 for all $s \in \mathbb{R}$. This requires the Sobolev duality assertion $|\int_{\mathbb{R}^n} fh\,dx| \leq \|f\|_{H^s}\|h\|_{H^{-s}}$.

Remark 1.1.12 Theorem 1.1.9 implies that the Radon transform extends as a bounded map

$$R\colon H_K^s\big(\mathbb{R}^2\big) \to H_T^{s+1/2}\big(\mathbb{R}\times S^1\big),$$

where $H_K^s(\mathbb{R}^2) = \{f \in H^s(\mathbb{R}^2)\,;\, \operatorname{supp}(f) \subset K\}$. In fact one may replace the $H_T^{s+1/2}$ norm on the right by the $H^{s+1/2}$ norm (see for instance Natterer (2001, Theorem II.5.2)). Thus, in a sense, the Radon transform in the plane is smoothing of order $1/2$ (it adds $1/2$ derivatives). We also observe that Theorems 1.1.8 and 1.1.9 yield the two-sided inequality

$$\sqrt{2}\|f\|_{H^s} \leq \|Rf\|_{H_T^{s+1/2}(\mathbb{R}\times S^1)} \leq C_K\|f\|_{H^s}, \qquad f \in H_K^s\big(\mathbb{R}^2\big).$$

1.2 Range and Support Theorems

We will next consider the range characterization problem: which functions in $\mathbb{R}\times S^1$ are of the form Rf for some $f \in C_c^\infty(\mathbb{R}^2)$? There is an obvious restriction: one has

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$$Rf(-s,-\omega) = Rf(s,\omega), \tag{1.3}$$

i.e. Rf is always *even*. Another restriction comes from studying the *moments*

$$\mu_k(Rf)(\omega) = \int_{-\infty}^{\infty} s^k (Rf)(s,\omega)\, ds, \qquad k \geq 0,\ \omega \in S^1.$$

It is easy to see that

$$\text{for any } k \geq 0,\ \mu_k(Rf) \text{ is a homogeneous polynomial of degree } k \text{ in } \omega. \tag{1.4}$$

This means that $\mu_k(Rf)(\omega) = \sum_{j_1,\dots,j_k=1}^{2} a_{j_1\cdots j_k}\omega_{j_1}\cdots\omega_{j_k}$ for some constants $a_{j_1\cdots j_k}$.

Exercise 1.2.1 Prove that Rf always satisfies (1.3) and (1.4).

It turns out that these conditions (called *Helgason–Ludwig range conditions*) are essentially the only restrictions. We will first consider range characterization on $\mathscr{S}(\mathbb{R}^2)$. To do this, we need to define a Schwartz space on $\mathbb{R} \times S^1$.

Definition 1.2.2 The space $\mathscr{S}(\mathbb{R}\times S^1)$ is the set of all $\varphi \in C^\infty(\mathbb{R}\times S^1)$ so that $(1+s^2)^k \partial_s^l (P\varphi) \in L^\infty(\mathbb{R}\times S^1)$ for all integers $k, l \geq 0$ and for all differential operators P on S^1 with smooth coefficients. We write $\mathscr{S}_H(\mathbb{R}\times S^1)$ for the set of all functions $\varphi \in \mathscr{S}(\mathbb{R}\times S^1)$ that satisfy the Helgason–Ludwig conditions, i.e. (1.3) and (1.4).

The following result is a Radon transform analogue of the fact that the Fourier transform is bijective $\mathscr{S}(\mathbb{R}^2) \to \mathscr{S}(\mathbb{R}^2)$.

Theorem 1.2.3 (Range characterization on Schwartz space) *The Radon transform is bijective* $\mathscr{S}(\mathbb{R}^2) \to \mathscr{S}_H(\mathbb{R}\times S^1)$.

The proof of Theorem 1.2.3 is outlined in the following exercises (the proof may also be found in Helgason (1999)).

Exercise 1.2.4 Show that R maps $\mathscr{S}(\mathbb{R}^2)$ into $\mathscr{S}_H(\mathbb{R}\times S^1)$.

Exercise 1.2.5 Show that R is injective on $\mathscr{S}(\mathbb{R}^2)$. (It is enough to verify that the Fourier slice theorem holds for Schwartz functions.)

Exercise 1.2.6 Given $\varphi \in \mathscr{S}_H(\mathbb{R}\times S^1)$, show that there exists $f \in \mathscr{S}(\mathbb{R}^2)$ with $Rf = \varphi$ as follows:

(i) By the Fourier slice theorem one should have $\hat{f}(\sigma\omega) = \tilde{\varphi}(\sigma,\omega)$. Motivated by this, define the function F on $\mathbb{R}^2 \setminus \{0\}$ by

$$F(\xi) := \tilde{\varphi}(|\xi|, \xi/|\xi|), \qquad \xi \in \mathbb{R}^2 \setminus \{0\}.$$

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(One wants to eventually show that $F = \hat{f}$ for the required function f.) Show that F is C^∞ in $\mathbb{R}^2 \setminus \{0\}$.

(ii) Show that F is Schwartz near infinity, i.e. $\xi^\alpha \partial^\beta F \in L^\infty(\mathbb{R}^2 \setminus B(0,1))$ for $\alpha, \beta \in \mathbb{N}_0^n$.

(iii) Show that F can be extended continuously near 0, by using the fact that $\mu_0\varphi(\omega)$ is homogeneous of degree 0 (i.e. a constant).

(iv) Use the fact that each $\mu_k\varphi$ is homogeneous of degree k to show that F can be extended as a C^∞ function near 0.

(v) Now that F is known to be in $\mathscr{S}(\mathbb{R}^2)$, let f be the inverse Fourier transform of F and show that $Rf = \varphi$.

There is a similar range characterization for the Radon transform when rapid decay is replaced by compact support conditions.

Theorem 1.2.7 (Range characterization on $C_c^\infty(\mathbb{R}^2)$) *The map R is bijective $C_c^\infty(\mathbb{R}^2) \to \mathscr{D}_H(\mathbb{R} \times S^1)$, where*

$$\mathscr{D}_H(\mathbb{R} \times S^1) = \mathscr{S}_H(\mathbb{R} \times S^1) \cap C_c^\infty(\mathbb{R} \times S^1).$$

In fact, Theorem 1.2.7 is an immediate consequence of Theorem 1.2.3 and the following fundamental result:

Theorem 1.2.8 (Helgason support theorem) *Let f be a continuous function on $\mathbb{R}^2$ such that $|x|^k f \in L^\infty(\mathbb{R}^2)$ for any $k \geq 0$. If $A > 0$ and if $Rf(s,\omega) = 0$ whenever $|s| > A$ and $\omega \in S^1$, then $f(x) = 0$ whenever $|x| > A$.*

The above result will not be needed later, and we refer to Helgason (1999) for its proof. However, we will prove a closely related result following Strichartz (1982), Andersson and Boman (2018).

Theorem 1.2.9 (Local uniqueness) *Let B be a ball in $\mathbb{R}^2$, and let $f \in C_c(\mathbb{R}^2)$ be supported in $\overline{B}$. Let $x_0 \in \partial B$ and let L_0 be the tangent line to ∂B through x_0. If f integrates to zero along any line L in a small neighbourhood of L_0, then $f = 0$ near x_0.*

Proof We will prove the result assuming that $f \in C_c^\infty(\mathbb{R}^2)$ and that f is supported in $\overline{B}$ (the general case is given as an exercise). After a translation and rotation we may assume that $x_0 = 0$, $B \subset \{x_2 \geq 0\}$, and L_0 is the x_1-axis. It is convenient to use a slightly different parametrization of lines and to consider the operator

$$Pf(\xi,\eta) = \int_{-\infty}^{\infty} f(t, \xi t + \eta)\,dt, \qquad \xi,\eta \in \mathbb{R}.$$

The assumption implies that $Pf(\xi,\eta) = 0$ for (ξ,η) in some neighbourhood V of $(0,0)$. Since $f \in C_c^\infty(\mathbb{R}^2)$, we may take derivatives in ξ so that

$$\partial_\xi Pf(\xi,\eta) = \int_{-\infty}^{\infty} t\partial_{x_2} f(t,\xi t+\eta)\,dt = \partial_\eta P(x_1 f)(\xi,\eta).$$

Since $Pf(\xi,\eta) = 0$ for $(\xi,\eta) \in V$, we have $P(x_1 f)(\xi,\eta) = c(\xi)$ in V. But taking η negative and using the support condition for f gives $c(\xi) = 0$ for ξ close to 0, i.e. $P(x_1 f)(\xi,\eta) = 0$. Repeating this argument gives

$$P\big(x_1^k f\big)(\xi,\eta) = 0 \quad \text{near } (0,0) \text{ for any } k \geq 0.$$

In particular, choosing $\xi = 0$ gives

$$\int_{-\infty}^{\infty} t^k f(t,\eta)\,dt = 0 \quad \text{for } \eta \text{ near 0 whenever } k \geq 0.$$

This means that all moments of $f(\,\cdot\,,\eta)$ vanish, and it follows that $f(\,\cdot\,,\eta) = 0$ for η near 0 (see the following exercise). Thus f vanishes in a neighbourhood of 0. □

Exercise 1.2.10 If $f \in C_c(\mathbb{R})$ and $\int_{-\infty}^{\infty} t^k f(t)\,dt = 0$ for any $k \geq 0$, show that $f = 0$. (You may use the Weierstrass approximation theorem.)

Exercise 1.2.11 Prove Theorem 1.2.9 for functions $f \in C_c(\mathbb{R}^2)$ supported in $\overline{B}$. Hint: consider mollifications $f_\varepsilon(x) = \int_{\mathbb{R}^2} f(x-y)\varphi_\varepsilon(y)\,dy$ where $\varphi_\varepsilon(x) = \varepsilon^{-n}\varphi(x/\varepsilon)$ is a standard mollifier, and show that the Radon transform of f_ε vanishes along certain lines when ε is small.

Remark 1.2.12 Theorem 1.2.9 is valid with the same proof also when B is a strictly convex domain in $\mathbb{R}^2$. Similarly, the Helgason support theorem (Theorem 1.2.8) can be phrased so that if f satisfies the given decay properties and integrates to zero over any line disjoint from a compact convex set K, then $f = 0$ outside K. Theorem 1.2.9 follows from this version of the Helgason support theorem after redefining f suitably.

1.3 The Normal Operator and Singularities

1.3.1 Normal Operator

We will now proceed to study the *normal operator* R^*R of the Radon transform, where the formal adjoint R^* is defined with respect to the natural L^2

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inner products on $\mathbb{R}^2$ and $\mathbb{R} \times S^1$. The formula for R^* is obtained as follows: if $f \in C_c^\infty(\mathbb{R}^2)$, $h \in C^\infty(\mathbb{R} \times S^1)$, one has

$$\begin{aligned}(Rf, h)_{L^2(\mathbb{R}\times S^1)} &= \int_{-\infty}^{\infty} \int_{S^1} Rf(s,\omega)\overline{h(s,\omega)}\, d\omega\, ds \\ &= \int_{-\infty}^{\infty} \int_{S^1} \int_{-\infty}^{\infty} f\big(s\omega + t\omega^\perp\big)\overline{h(s,\omega)}\, dt\, d\omega\, ds \\ &= \int_{\mathbb{R}^2} f(y) \left(\int_{S^1} \overline{h(y\cdot\omega,\omega)}\, d\omega \right) dy.\end{aligned}$$

Thus R^* is the *backprojection operator*

$$R^*\colon C^\infty(\mathbb{R} \times S^1) \to C^\infty(\mathbb{R}^2), \quad R^*h(y) = \int_{S^1} h(y\cdot\omega,\omega)\, d\omega.$$

The following result shows that the normal operator R^*R corresponds to multiplication by $\frac{4\pi}{|\xi|}$ on the Fourier side, and gives an inversion formula for reconstructing f from Rf.

Theorem 1.3.1 (Normal operator) *One has*

$$R^*R = 4\pi|D|^{-1} = \mathscr{F}^{-1}\left\{\frac{4\pi}{|\xi|}\mathscr{F}(\,\cdot\,)\right\},$$

and f can be recovered from Rf by the formula

$$f = \frac{1}{4\pi}|D|R^*Rf.$$

Remark 1.3.2 Above we have written, for $\alpha \in \mathbb{R}$,

$$|D|^\alpha f := \mathscr{F}^{-1}\{|\xi|^\alpha \hat{f}(\xi)\}.$$

The notation $(-\Delta)^{\alpha/2} = |D|^\alpha$ is also used.

Proof of Theorem 1.3.1 The proof is based on computing the inner product $(Rf, Rg)_{L^2(\mathbb{R}\times S^1)}$ using the Parseval identity, the Fourier slice theorem, symmetry, and polar coordinates:

$$\begin{aligned}(R^*Rf, g)_{L^2(\mathbb{R}^2)} &= (Rf, Rg)_{L^2(\mathbb{R}\times S^1)} \\ &= \int_{S^1} \left[\int_{-\infty}^{\infty} (Rf)(s,\omega)\overline{(Rg)(s,\omega)}\, ds \right] d\omega \\ &= \frac{1}{2\pi} \int_{S^1} \left[\int_{-\infty}^{\infty} (Rf)\tilde{\ }(\sigma,\omega)\overline{(Rg)\tilde{\ }(\sigma,\omega)} \right] d\sigma\, d\omega \\ &= \frac{1}{2\pi} \int_{S^1} \left[\int_{-\infty}^{\infty} \hat{f}(\sigma\omega)\overline{\hat{g}(\sigma\omega)} \right] d\sigma\, d\omega\end{aligned}$$

$$= \frac{2}{2\pi} \int_{S^1} \left[\int_0^\infty \hat{f}(\sigma\omega) \overline{\hat{g}(\sigma\omega)} \right] d\sigma \, d\omega$$
$$= \frac{2}{2\pi} \int_{\mathbb{R}^2} \frac{1}{|\xi|} \hat{f}(\xi) \overline{\hat{g}(\xi)} \, d\xi$$
$$= \left(4\pi \mathscr{F}^{-1} \left\{ \frac{1}{|\xi|} \hat{f}(\xi) \right\}, g \right)_{L^2(\mathbb{R}^2)} . \qquad \square$$

The same argument, based on computing $(|D_s|^{1/2} Rf, |D_s|^{1/2} Rg)_{L^2(\mathbb{R}\times S^1)}$ instead of $(Rf, Rg)_{L^2(\mathbb{R}\times S^1)}$, leads to the famous *filtered backprojection* (FBP) inversion formula:

Theorem 1.3.3 (Filtered backprojection) *If $f \in C_c^\infty(\mathbb{R}^2)$, then*

$$f = \frac{1}{4\pi} R^* |D_s| Rf,$$

where $|D_s| Rf$ is the inverse Fourier transform of $|\sigma| (Rf)\tilde{}$ with respect to σ.

The FBP formula is efficient to implement and gives accurate reconstructions when one has complete X-ray data and relatively small noise, and hence FBP (together with its variants) has been commonly used in X-ray CT scanners.

1.3.2 Recovery of Singularities

We will later study X-ray transforms in more general geometries. In such cases, exact reconstruction formulas such as FBP are often not available. However, it will be important that some structural properties of the normal operator may still be valid. In particular, Theorem 1.3.1 implies that the normal operator is an *elliptic pseudodifferential operator* of order -1 in $\mathbb{R}^2$. The theory of pseudodifferential operators (i.e. *microlocal analysis*) then immediately yields that the *singularities* of f are uniquely determined from the knowledge of Rf. For the benefit of those readers who are not familiar with these notions, we will give a short presentation partly without proofs.

For a reference to distribution theory, see Hörmander (1983–1985, vol. I), and for wave front sets, see Hörmander (1983–1985, chapter 8). Sobolev wave front sets are considered in Hörmander (1983–1985, section 18.1).

We first define compactly supported distributions.

Definition 1.3.4 Define the set of *compactly supported distributions* in $\mathbb{R}^n$ as

$$\mathscr{E}'(\mathbb{R}^n) = \bigcup_{s\in\mathbb{R}} H_c^s(\mathbb{R}^n),$$

where $H_c^s(\mathbb{R}^n)$ is the set of compactly supported elements in $H^s(\mathbb{R}^n)$.

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This definition coincides with the more standard ones defining $\mathscr{E}'(\mathbb{R}^n)$ as the dual of $C^\infty(\mathbb{R}^n)$ with a suitable topology, or as the compactly supported distributions in $\mathscr{D}'(\mathbb{R}^n)$. By Remark 1.1.12, the Radon transform R is well defined in $\mathscr{E}'(\mathbb{R}^2)$. We also recall that the Fourier transform maps $\mathscr{E}'(\mathbb{R}^n)$ to $C^\infty(\mathbb{R}^n)$.

We next discuss the singular support of u, which consists of those points x_0 such that u is not a smooth function in any neighbourhood of x_0. We also consider the Sobolev singular support, which also measures the 'strength' of the singularity (in the L^2 Sobolev scale).

Definition 1.3.5 (Singular support) We say that a function or distribution u in $\mathbb{R}^n$ is C^∞ *(respectively* H^α*) near* x_0 if there is $\varphi \in C_c^\infty(\mathbb{R}^n)$ with $\varphi = 1$ near x_0 such that φu is in $C^\infty(\mathbb{R}^n)$ (respectively in $H^\alpha(\mathbb{R}^n)$). We define

$$\operatorname{sing\,supp}(u) = \mathbb{R}^n \setminus \{x_0 \in \mathbb{R}^n;\ u \text{ is } C^\infty \text{ near } x_0\},$$
$$\operatorname{sing\,supp}^\alpha(u) = \mathbb{R}^n \setminus \{x_0 \in \mathbb{R}^n;\ u \text{ is } H^\alpha \text{ near } x_0\}.$$

Example 1.3.6 Let $D_1, \ldots, D_N$ be bounded domains with C^∞ boundary in $\mathbb{R}^n$ so that $\overline{D}_j \cap \overline{D}_k = \emptyset$ for $j \neq k$, and define

$$u = \sum_{j=1}^N c_j \chi_{D_j},$$

where $c_j \neq 0$ are constants, and χ_{D_j} is the characteristic function of D_j. Then

$$\operatorname{sing\,supp}^\alpha(u) = \emptyset \quad \text{for } \alpha < 1/2,$$

since $u \in H^\alpha$ for $\alpha < 1/2$, but

$$\operatorname{sing\,supp}^\alpha(u) = \bigcup_{j=1}^N \partial D_j \text{ for } \alpha \geq 1/2,$$

since u is not $H^{1/2}$ near any boundary point. Thus in this case the singularities of u are exactly at the points where u has a jump discontinuity, and their strength is precisely $H^{1/2}$. Knowing the singularities of u can already be useful in applications. For instance, if u represents some internal medium properties in medical imaging, the singularities of u could determine the location of interfaces between different tissues. On the other hand, if u represents an image, then the singularities in some sense determine the 'sharp features' of the image.

Next we discuss the *wave front set*, which is a more refined notion of a singularity. For example, if $f = \chi_D$ is the characteristic function of a bounded strictly convex C^∞ domain D and if $x_0 \in \partial D$, one could think that f is in some sense smooth in tangential directions at x_0 (since f restricted to a tangent hyperplane is identically zero, except possibly at x_0), but that f is not smooth in normal directions at x_0 since in these directions there is a jump. The wave front set is a subset of $T^*\mathbb{R}^n \setminus 0$, the cotangent space with the zero section removed:

$$T^*\mathbb{R}^n \setminus 0 := \{(x,\xi)\,;\, x,\xi \in \mathbb{R}^n, \xi \neq 0\}.$$

Definition 1.3.7 (Wave front set) Let u be a distribution in $\mathbb{R}^n$. We say that u is (microlocally) C^∞ *(respectively* H^α*) near* (x_0,ξ_0) if there exist $\varphi \in C_c^\infty(\mathbb{R}^n)$ with $\varphi = 1$ near x_0 and $\psi \in C^\infty(\mathbb{R}^n \setminus \{0\})$ so that $\psi = 1$ near ξ_0 and ψ is homogeneous of degree 0, such that

$$\text{for any } N \text{ there is } C_N > 0 \text{ so that } |\psi(\xi)(\varphi u)\hat{\ }(\xi)| \leq C_N(1+|\xi|)^{-N}$$

(respectively $\mathscr{F}^{-1}\{\psi(\xi)(\varphi u)\hat{\ }(\xi)\} \in H^\alpha(\mathbb{R}^n)$). The *wave front set* $WF(u)$ (respectively H^α *wave front set* $WF^\alpha(u)$) consists of those points (x_0,ξ_0) where u is not microlocally C^∞ (respectively H^α).

Example 1.3.8 The wave front set of the function u in Example 1.3.6 is

$$WF(u) = \bigcup_{j=1}^{N} N^*(D_j),$$

where $N^*(D_j)$ is the conormal bundle of D_j,

$$N^*(D_j) := \{(x,\xi)\,;\, x \in \partial D_j \text{ and } \xi \text{ is normal to } \partial D_j \text{ at } x\}.$$

The wave front set describes singularities more precisely than the singular support, since one always has

$$\pi(WF(u)) = \operatorname{sing\,supp}(u), \tag{1.5}$$

where $\pi : (x,\xi) \mapsto x$ is the projection to x-space.

We now go back to the Radon transform. If one is mainly interested in the singularities of the image function f, then instead of using FBP to reconstruct the whole function f from Rf it is possible to use the even simpler *backprojection method*: just apply the backprojection operator R^* to the data Rf. Since R^*R is an elliptic pseudodifferential operator, the singularities are completely recovered:

Theorem 1.3.9 *If* $f \in \mathscr{E}'(\mathbb{R}^2)$*, then*

$$\operatorname{sing\,supp}(R^*Rf) = \operatorname{sing\,supp}(f),$$
$$\mathrm{WF}(R^*Rf) = \mathrm{WF}(f).$$

Moreover, for any $\alpha \in \mathbb{R}$ one has

$$\operatorname{sing\,supp}^{\alpha+1}(R^*Rf) = \operatorname{sing\,supp}^{\alpha}(f),$$
$$\mathrm{WF}^{\alpha+1}(R^*Rf) = \mathrm{WF}^{\alpha}(f).$$

Remark 1.3.10 Since R^*R is a pseudodifferential operator of order -1, hence smoothing of order 1, one can roughly expect that R^*Rf is a kind of blurred version of f where the main singularities are still visible. The previous theorem makes this precise and shows that the singularities in R^*Rf are one Sobolev degree smoother than those in f.

1.3.3 Pseudodifferential Operators

For the proof of Theorem 1.3.9 we recall quickly some relevant definitions from microlocal analysis, based on the following example. We refer to Hörmander (1983–1985, chapter 18) and Folland (1995, chapter 8) for a detailed account on pseudodifferential operators.

Example 1.3.11 (Differential operators) Let $A = a(x, D)$ be a differential operator of order m, acting on functions $f \in \mathscr{S}(\mathbb{R}^n)$ by

$$Af(x) = a(x, D)f(x) = \sum_{|\alpha|\le m} a_\alpha(x) D^\alpha f(x),$$

where $a_\alpha \in C^\infty(\mathbb{R}^n)$. Here $D = \frac{1}{i}\nabla$, so that $D^\alpha = (\frac{1}{i}\partial_{x_1})^{\alpha_1} \cdots (\frac{1}{i}\partial_{x_n})^{\alpha_n}$.

If each a_α is a constant, i.e. $a_\alpha(x) = a_\alpha$ and $A = a(D) = \sum_{|\alpha|\le m} a_\alpha D^\alpha$, we may use (1.1) to compute the Fourier transform of Af:

$$(Af)\hat{}\,(\xi) = \sum_{|\alpha|\le m} a_\alpha \xi^\alpha \hat{f}(\xi).$$

The Fourier inversion formula gives that

$$Af(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} a(\xi) \hat{f}(\xi)\, d\xi, \tag{1.6}$$

where $a(\xi) = \sum_{|\alpha|\le m} a_\alpha \xi^\alpha$ is the *symbol* of $A(D)$.

More generally, if each a_α is a C^∞ function with $\partial^\beta a_\alpha \in L^\infty(\mathbb{R}^n)$ for all $\beta \in \mathbb{N}_0^n$, we may use the Fourier inversion formula to compute

$$\begin{aligned} Af(x) &= A\left[\mathscr{F}^{-1}\{\hat{f}(\xi)\}\right] \\ &= \sum_{|\alpha|\le m} a_\alpha(x) D^\alpha \left[(2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \hat{f}(\xi)\, d\xi\right] \end{aligned}$$

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$$= (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \left[\sum_{|\alpha|\le m} a_\alpha(x)\xi^\alpha \right] \hat{f}(\xi)\, d\xi$$

$$= (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} a(x,\xi) \hat{f}(\xi)\, d\xi, \tag{1.7}$$

where

$$a(x,\xi) := \sum_{|\alpha|\le m} a_\alpha(x)\xi^\alpha \tag{1.8}$$

is the (full) *symbol* of $A = a(x, D)$.

The above example shows that any differential operator of order m has the Fourier representation (1.7), where the symbol $a(x,\xi)$ in (1.8) is a polynomial of degree m in ξ. The following definition generalizes this setup.

Definition 1.3.12 (Pseudodifferential operators) For any $m \in \mathbb{R}$, denote by S^m (the set of *symbols* of order m) the set of all $a \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ so that for any multi-indices $\alpha, \beta \in \mathbb{N}_0^n$ there is $C_{\alpha\beta} > 0$ such that

$$\left|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)\right| \le C_{\alpha\beta}(1+|\xi|)^{m-|\beta|}, \qquad x,\xi \in \mathbb{R}^n.$$

For any $a \in S^m$, define an operator $A = \mathrm{Op}(a)$ acting on functions $f \in \mathscr{S}(\mathbb{R}^n)$ by

$$Af(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} a(x,\xi) \hat{f}(\xi)\, d\xi, \qquad x \in \mathbb{R}^n.$$

Let $\Psi^m = \{\mathrm{Op}(a)\,;\, a \in S^m\}$ be the set of *pseudodifferential operators* of order m. We say that an operator $\mathrm{Op}(a)$ with $a \in S^m$ is *elliptic* if there are $c, R > 0$ such that

$$a(x,\xi) \ge c(1+|\xi|)^m, \qquad x \in \mathbb{R}^n, |\xi| \ge R.$$

We also give the definition of *classical* pseudodifferential operators (the normal operator of the Radon transform will belong to this class):

Definition 1.3.13 We say that $a \in S^m$ is a *classical* symbol, written $a \in S^m_{\mathrm{cl}}$, if one has

$$a(x,\xi) \sim \sum_{j=0}^\infty a_{m-j}(x,\xi), \tag{1.9}$$

where $a_{m-j} \in S^{m-j}$ and a_{m-j} is homogeneous of degree $m-j$ for $|\xi|$ large, i.e.

$$a_{m-j}(x,\lambda\xi) = \lambda^{m-j} a_{m-j}(x,\xi), \qquad \lambda \ge 1, \ |\xi| \text{ large}.$$

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The asymptotic sym (1.9) means that for any $N \geq 0$ one has

$$a - \sum_{j=0}^{N} a_{m-j} \in S^{m-N-1}.$$

We write $\Psi_{\mathrm{cl}}^m = \{\mathrm{Op}(a)\,;\, a \in S_{\mathrm{cl}}^m\}$.

It is a basic fact that any $A \in \Psi^m$ is a continuous map $\mathscr{S}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n)$, when $\mathscr{S}(\mathbb{R}^n)$ is given the natural topology induced by the seminorms $f \mapsto \|x^\alpha \partial^\beta f\|_{L^\infty}$ where $\alpha, \beta \in \mathbb{N}_0^n$. By duality, any $A \in \Psi^m$ gives a continuous map $\mathscr{S}'(\mathbb{R}^n) \to \mathscr{S}'(\mathbb{R}^n)$, where $\mathscr{S}'(\mathbb{R}^n)$ is the weak* dual space of $\mathscr{S}(\mathbb{R}^n)$ (the space of *tempered distributions*). In particular, any $A \in \Psi^m$ is well defined on $\mathscr{E}'(\mathbb{R}^n)$.

It is an important fact that applying a pseudodifferential operator to a function or distribution never creates new singularities:

Theorem 1.3.14 (Pseudolocal/microlocal property) *Any $A \in \Psi^m$ has the pseudolocal property*

$$\mathrm{sing\,supp}(Au) \subset \mathrm{sing\,supp}(u),$$
$$\mathrm{sing\,supp}^{\alpha-m}(Au) \subset \mathrm{sing\,supp}^{\alpha}(u),$$

and the microlocal property

$$WF(Au) \subset WF(u),$$
$$WF^{\alpha-m}(Au) \subset WF^{\alpha}(u).$$

Proof We sketch a proof for the inclusion $\mathrm{sing\,supp}(Au) \subset \mathrm{sing\,supp}(u)$. For more details see Hörmander (1983–1985, chapter 18). Suppose that $x_0 \notin \mathrm{sing\,supp}(u)$, so we need to show that $x_0 \notin \mathrm{sing\,supp}(Au)$. By definition, there is $\psi \in C_c^\infty(\mathbb{R}^n)$ with $\psi = 1$ near x_0 so that $\psi u \in C_c^\infty(\mathbb{R}^n)$. We write

$$Au = A(\psi u) + A((1-\psi)u).$$

Since A maps the Schwartz space to itself, one always has $A(\psi u) \in C^\infty$. Thus it is enough to show that $A((1-\psi)u)$ is C^∞ near x_0. To do this, choose $\varphi \in C_c^\infty(\mathbb{R}^n)$ so that $\varphi = 1$ near x_0 and some neighbourhood of $\mathrm{supp}(\varphi)$ is contained in the set where $\psi = 1$. Define

$$Bu = \varphi A((1-\psi)u).$$

It is enough to show that B is a smoothing operator, i.e. maps $\mathscr{E}'(\mathbb{R}^n)$ to $C^\infty(\mathbb{R}^n)$.

We compute the integral kernel of B:

$$\begin{aligned} Bu(x) &= (2\pi)^{-n}\varphi(x)\int_{\mathbb{R}^n} e^{ix\cdot\xi}a(x,\xi)((1-\psi)u)\hat{\ }(\xi)\,d\xi \\ &= \int_{\mathbb{R}^n} K(x,y)u(y)\,dy, \end{aligned}$$

where

$$K(x,y) = (2\pi)^{-n}\int_{\mathbb{R}^n}\varphi(x)e^{i(x-y)\cdot\xi}a(x,\xi)(1-\psi(y))\,d\xi.$$

Recall that a satisfies $|a(x,\xi)| \leq C(1+|\xi|)^m$. Thus if $m < -n$, the integral is absolutely convergent and one gets that $K \in L^\infty(\mathbb{R}^n \times \mathbb{R}^n)$. In the general case the integral may not be absolutely convergent, but it can be interpreted as an oscillatory integral or as the Fourier transform of a tempered distribution. The main point is that $|x-y| \geq c > 0$ on the support of $K(x,y)$, due to the support conditions on φ and ψ. It follows that we may write, for any $N \geq 0$,

$$e^{i(x-y)\cdot\xi} = |x-y|^{-2N}(-\Delta_\xi)^N\left(e^{i(x-y)\cdot\xi}\right),$$

and integrate by parts in ξ to obtain that

$$\begin{aligned} K(x,y) &= (2\pi)^{-n}|x-y|^{-2N} \\ &\quad\times\int_{\mathbb{R}^n}\varphi(x)e^{i(x-y)\cdot\xi}((-\Delta_\xi)^N a(x,\xi))(1-\psi(y))\,d\xi. \end{aligned} \tag{1.10}$$

If N is chosen large enough (it is enough that $m-2N < -n-1$), one has $|(-\Delta_\xi)^N a(x,\xi)| \leq C(1+|\xi|)^{-n-1}$. Thus the integral in (1.10) is absolutely convergent, and in particular $K \in L^\infty(\mathbb{R}^n \times \mathbb{R}^n)$. Taking derivatives gives that $\partial_x^\alpha\partial_y^\beta K$ is also bounded for any α and β, showing that $K \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$. It follows from the next exercise that the operator B maps into $C^\infty(\mathbb{R}^n)$. □

Exercise 1.3.15 Show that an operator $Bu(x) = \int_{\mathbb{R}^n} K(x,y)u(y)\,dy$, where $K \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$, induces a well-defined map from $\mathscr{E}'(\mathbb{R}^n)$ to $C^\infty(\mathbb{R}^n)$.

We now go back to the normal operator R^*R and the proof of Theorem 1.3.9. Theorem 1.3.1 states that R^*R has symbol $\frac{4\pi}{|\xi|}$, which would be in the symbol class S^{-1} except that the symbol is not smooth when $\xi = 0$. This can be dealt with in the following standard way.

Theorem 1.3.16 *The normal operator satisfies*

$$R^*R = Q + S,$$

where $Q \in \Psi_{\mathrm{cl}}^{-1}$ is elliptic, and S is a smoothing operator *that maps $\mathscr{E}'(\mathbb{R}^2)$ to $C^\infty(\mathbb{R}^2)$.*

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Proof Let $\psi \in C_c^\infty(\mathbb{R}^2)$ satisfy $\psi(\xi) = 1$ for $|\xi| \leq 1/2$ and $\psi(\xi) = 0$ for $|\xi| \geq 1$. Write

$$Qf = 4\pi \mathscr{F}^{-1}\left\{\frac{1-\psi(\xi)}{|\xi|}\hat{f}\right\}, \qquad Sf = 4\pi \mathscr{F}^{-1}\left\{\frac{\psi(\xi)}{|\xi|}\hat{f}\right\}.$$

Then Q is a pseudodifferential operator in Ψ_{cl}^{-1} with symbol $q(x,\xi) = \frac{1-\psi(\xi)}{|\xi|}$, hence Q is elliptic. The operator S has the required property by Lemma 1.3.17 since $\frac{\psi(\xi)}{|\xi|}$ is in $L^1(\mathbb{R}^2)$ and has compact support (the function $\xi \mapsto \frac{1}{|\xi|}$ is locally integrable in $\mathbb{R}^2$). □

Lemma 1.3.17 *If $m \in L^1(\mathbb{R}^n)$ is compactly supported, then the operator*

$$S\colon f \mapsto \mathscr{F}^{-1}\{m(\xi)\hat{f}\}$$

is smoothing in the sense that it maps $\mathscr{E}'(\mathbb{R}^n)$ to $C^\infty(\mathbb{R}^n)$.

Proof If $f \in \mathscr{E}'(\mathbb{R}^n)$ then $\hat{f} \in C^\infty(\mathbb{R}^n)$. Consequently $F(\xi) := m(\xi)\hat{f}(\xi)$ is in $L^1(\mathbb{R}^n)$ and compactly supported by the assumption on m. This implies that $Sf = \mathscr{F}^{-1}F$ is C^∞. □

We can finally prove the recovery of singularities result.

Proof of Theorem 1.3.9 We prove the claim for the singular support (the other parts are analogous). By Theorem 1.3.16, one has

$$R^*Rf = Qf + C^\infty.$$

Hence it is enough to show that $\operatorname{sing\,supp}(Qf) = \operatorname{sing\,supp}(f)$. It follows from Theorem 1.3.14 that $\operatorname{sing\,supp}(Qf) \subset \operatorname{sing\,supp}(f)$. The converse inclusion is a standard argument, which follows from the construction of an approximate inverse, or *parametrix*, for the elliptic pseudodifferential operator Q. Define

$$Ef = \mathscr{F}^{-1}\left\{(1-\chi(\xi))|\xi|\hat{f}\right\},$$

where $\chi \in C_c^\infty(\mathbb{R}^2)$ satisfies $\chi(\xi) = 1$ for $|\xi| \leq 2$. Note that $E \in \Psi^1$. Since $\psi(\xi) = 0$ for $|\xi| \geq 1$, it follows that

$$EQf = \mathscr{F}^{-1}\left\{(1-\chi(\xi))|\xi|\frac{1-\psi(\xi)}{|\xi|}\hat{f}\right\} = f - \mathscr{F}^{-1}\left\{\chi(\xi)\hat{f}\right\}.$$

Thus $EQf = f + S_1 f$, where S_1 is smoothing and maps $\mathscr{E}'(\mathbb{R}^2)$ to $C^\infty(\mathbb{R}^2)$ by Lemma 1.3.17. Hence Theorem 1.3.14 applied to E gives that

$$\operatorname{sing\,supp}(f) = \operatorname{sing\,supp}(EQf) \subset \operatorname{sing\,supp}(Qf). \qquad \square$$

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1.3.4 Visible Singularities

We conclude this section with a short discussion on more precise recovery of singularities results from limited X-ray data. This follows the microlocal approach to Radon transforms introduced in Guillemin (1975). For more detailed treatments we refer to the survey articles Quinto (2006), Krishnan and Quinto (2015).

There are various imaging situations where complete X-ray data (i.e. the function $Rf(s,\omega)$ for all s and ω) are not available. This is the case for limited angle tomography (e.g. in luggage scanners at airports, or dental applications), region of interest tomography, or exterior data tomography. In such cases explicit inversion formulas such as FBP are usually not available, but the analysis of singularities still provides a powerful paradigm for predicting which sharp features can be recovered stably from the measurements.

We will try to explain this paradigm a little bit more, starting with an example:

Example 1.3.18 Let f be the characteristic function of the unit disk $\mathbb{D}$, i.e. $f(x) = 1$ if $|x| \leq 1$ and $f(x) = 0$ for $|x| > 1$. Then f is singular precisely on the unit circle (in normal directions). We have

$$Rf(s,\omega) = \begin{cases} 2\sqrt{1-s^2}, & |s| \leq 1, \\ 0, & |s| > 1. \end{cases}$$

Thus Rf is singular precisely at those points (s,ω) with $|s| = 1$, which correspond to those lines that are tangent to the unit circle.

There is a similar relation between the singularities of f and Rf in general, and this is explained by microlocal analysis and the interpretation of R as a Fourier integral operator (see Hörmander (1983–1985, chapter 25) for the definition and facts on Fourier integral operators):

Theorem 1.3.19 *The operator R is an elliptic Fourier integral operator of order $-1/2$. There is a precise relationship between the singularities of f and singularities of Rf.*

We will not spell out the precise relationship here, but only give some consequences. It will be useful to think of the Radon transform as defined on the set of (non-oriented) lines in $\mathbb{R}^2$. If $\mathcal{A}$ is an open subset of lines in $\mathbb{R}^2$, we consider the Radon transform $Rf|_{\mathcal{A}}$ restricted to lines in $\mathcal{A}$. Recovering f

(or some properties of f) from $Rf|_{\mathcal{A}}$ is a *limited data* tomography problem. Examples:

- If $\mathcal{A} = \{\text{lines not meeting } \overline{\mathbb{D}}\}$, then $Rf|_{\mathcal{A}}$ is called *exterior data*.
- If $0 < a < \pi/2$ and $\mathcal{A} = \{\text{lines whose angle with } x\text{-axis is} < a\}$ then $Rf|_{\mathcal{A}}$ is called *limited angle data*.

It is known that any $f \in C_c^\infty(\mathbb{R}^2 \setminus \overline{\mathbb{D}})$ is uniquely determined by exterior data (Helgason support theorem), and any $f \in C_c^\infty(\mathbb{R}^2)$ is uniquely determined by limited angle data (Fourier slice and Paley–Wiener theorems). However, both inverse problems are very unstable: inversion is not Lipschitz continuous in any Sobolev norms, but one has conditional logarithmic stability. See Koch et al. (2021) for a detailed treatment of instability issues.

The precise relationship between the singularities of f and Rf mentioned in Theorem 1.3.19 gives rise to the following notion.

Definition 1.3.20 A singularity at (x_0, ξ_0) is called *visible from* $\mathcal{A}$ if the line through x_0 in direction $\xi_0^\perp$ is in $\mathcal{A}$.

One has the following dichotomy:

- If (x_0, ξ_0) is visible from $\mathcal{A}$, then from the singularities of $Rf|_{\mathcal{A}}$ one can determine for any α whether or not $(x_0, \xi_0) \in WF^\alpha(f)$. In general, one expects the reconstruction of visible singularities to be stable.
- If (x_0, ξ_0) is not visible from $\mathcal{A}$, then this singularity is smoothed out in the measurement $Rf|_{\mathcal{A}}$. Even if $Rf|_{\mathcal{A}}$ would determine f uniquely, the inversion is not Lipschitz stable in any Sobolev norms.

1.4 The Funk Transform

In this final section we consider the X-ray transform along closed geodesics of the 2-sphere S^2 equipped with the usual metric of constant curvature 1. This is also known as the *Funk transform* (Funk, 1913). Here geodesics are great circles and they are all closed with period 2π. Manifolds all of whose geodesics are closed are called *Zoll manifolds* and the original motivation for studying the Funk transform was to describe Zoll metrics on the sphere. Our presentation follows (Guillemin, 1976, Appendix A) and it will use some basic representation theory and Fourier analysis. This is the only instance in this book in which we will consider the X-ray transform on a closed manifold.

A great circle on S^2 can be identified with a point on $S^2 \subset \mathbb{R}^3$: the correspondence associates the geodesic traveling counterclockwise through the equator with the north pole $N = (0,0,1)$. Thus we may identify the set of (oriented) closed geodesics with S^2 and consider the X-ray transform I as a map $C^\infty(S^2) \to C^\infty(S^2)$, defined by

$$I(h)(x) = \int_0^{2\pi} h(\gamma(t))\, dt,$$

where $x \in S^2$ is identified with the oriented great circle γ.

Exercise 1.4.1 Show that if h is an odd function then $I(h) = 0$.

We have a decomposition

$$C^\infty(S^2) = C^\infty_{\text{odd}}(S^2) \oplus C^\infty_{\text{even}}(S^2),$$

and the exercise asserts that $C^\infty_{\text{odd}}(S^2) \subset \ker I$. Our objective is to show the following theorem:

Theorem 1.4.2 *The kernel of the X-ray transform I on S^2 with its standard metric of constant curvature* 1 *is precisely the odd functions on S^2:*

$$\ker I = C^\infty_{\text{odd}}(S^2).$$

Moreover, $I\colon C^\infty_{\text{even}}(S^2) \to C^\infty_{\text{even}}(S^2)$ is bijective.

To prove the theorem we require some preparations. Given $f \in C^\infty(\mathbb{R}^n)$, let $\overline{f}$ denote $f|_{S^{n-1}}$. We first need a standard relationship between the Laplacian $\Delta_{\mathbb{R}^n}$ in $\mathbb{R}^n$ and the Laplacian $\Delta_{S^{n-1}}$ on the sphere S^{n-1}; its proof can be found in Gallot et al. (2004, Proposition 4.48):

$$\overline{\Delta_{\mathbb{R}^n}(f)} = \Delta_{S^{n-1}}(\overline{f}) + \overline{\frac{\partial^2 f}{\partial r^2}} + (n-1)\overline{\frac{\partial f}{\partial r}}, \tag{1.11}$$

where r is the radial coordinate.

Let

$$\mathbf{P}^n_k := \{\text{homogeneous polynomials of degree } k \text{ on } \mathbb{R}^n\},$$

and

$$\mathbf{H}^n_k := \{P \in \mathbf{P}^n_k : \Delta_{\mathbb{R}^n}(P) = 0\}$$

denote the *harmonic* homogeneous polynomials of degree k on $\mathbb{R}^n$.

We write $P \in \mathbf{P}^n_k$ as

$$P = r^k \overline{P},$$

and hence for $P \in \mathbf{P}_k^n$, (1.11) reduces to

$$\overline{\Delta_{\mathbb{R}^n}(P)} = \Delta_{S^{n-1}}(\overline{P}) + k(k+n-2)\overline{P}.$$

If $P \in \mathbf{H}_k^n$ then

$$\Delta_{S^{n-1}}(\overline{P}) = -k(k+n-2)\overline{P},$$

so that $\overline{P}$ is an eigenfunction of $\Delta_{S^{n-1}}$ with eigenvalue $-k(k+n-2)$. Write $\overline{\mathbf{P}}_k^n := \{\overline{P} \colon P \in \mathbf{P}_k^n\}$ and similarly define $\overline{\mathbf{H}}_k^n := \{\overline{P} \colon P \in \mathbf{H}_k^n\}$.

We briefly describe the representation theory we need for the orthogonal group. We define an action of $O(n)$ on $\overline{\mathbf{P}}_k^n$ by setting

$$(g \cdot \overline{P})(x) := \overline{P}(g^{-1}x)$$

for $\overline{P} \in \overline{\mathbf{P}}_k^n$ and $g \in O(n)$.

Exercise 1.4.3 Show that

$$\Delta_{S^{n-1}}(g \cdot \overline{P}) = g \cdot \Delta_{S^{n-1}}(\overline{P}),$$

and hence this action descends to give an action on $\overline{\mathbf{H}}_k^n$.

The following theorem is standard (see for instance, Sepanski (2007, Theorem 2.33)).

Theorem 1.4.4 *The set $\overline{\mathbf{H}}_k^n$ is an irreducible $O(n)$-module and for $n \geq 3$ is also an irreducible $SO(n)$-module. Moreover $L^2(S^{n-1})$ decomposes as a Hilbert space direct sum*

$$L^2(S^{n-1}) = \bigoplus_{k=0}^{\infty} \overline{\mathbf{H}}_k^n.$$

We now restrict to the case $n = 3$ and we drop the superscript n from the notation. The key observation we need is that the X-ray transform I *commutes* with the action of $SO(3)$ on S^2:

Exercise 1.4.5 Show that $I(g \cdot h) = g \cdot Ih$ for any $g \in SO(3)$ and $h \in C^\infty(S^2)$, where $(g \cdot h)(x) = h(g^{-1}x)$.

We claim that I maps $\overline{\mathbf{H}}_k$ into itself and there exist constants $c_k \in \mathbb{R}$ such that

$$I|_{\overline{\mathbf{H}}_k} = c_k \,\mathrm{Id}. \tag{1.12}$$

This is essentially a consequence of Schur's lemma (see Sepanski (2007, Theorem 2.12)) as we now explain. By Exercise 1.4.5, $I(\overline{\mathbf{H}}_k)$ is a $SO(3)$-invariant subspace. If $I(\overline{\mathbf{H}}_k)$ intersects two or more of the spaces $\overline{\mathbf{H}}_l$ nontrivially, one obtains a splitting of $\overline{\mathbf{H}}_k$ into proper $SO(3)$-invariant subspaces that

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is impossible by irreducibility. Thus $I(\overline{\mathbf{H}}_k) \subset \overline{\mathbf{H}}_l$ for some l. Since both $\overline{\mathbf{H}}_k$ and $\overline{\mathbf{H}}_l$ are irreducible, Schur's lemma yields that $I|_{\overline{\mathbf{H}}_k} : \overline{\mathbf{H}}_k \to \overline{\mathbf{H}}_l$ is either an isomorphism or $\equiv 0$. If $k \neq l$ it cannot be an isomorphism since the spaces have different dimension (Sepanski, 2007, Exercise 2.30). Thus I must map $\overline{\mathbf{H}}_k$ into itself, and Schur's lemma implies (1.12).

As we observed earlier, clearly $c_{2k+1} = 0$ for all non-negative integers k, since $\overline{\mathbf{H}}_{2k+1} \subset C^\infty_{\text{odd}}(S^2)$.

Proposition 1.4.6 *For all non-negative integers k,*

$$\begin{aligned} c_{2k} &= (-1)^k \int_0^{2\pi} (\cos\theta)^{2k}\, d\theta \\ &= 2\pi(-1)^k \frac{1\cdot 3\cdot 5\cdots(2k-1)}{2\cdot 4\cdot 6\cdots 2k}. \end{aligned}$$

Proof We take advantage of the fact that we only need to check the result on a fixed $P \in \mathbf{H}_{2k}$ of our choice and a fixed point in S^2. Consider

$$P(x,y,z) := \sum_{i=0}^{2k} a_i x^{2k-i} z^i$$

for some constants $a_i \in \mathbb{R}$. There are constraints on the coefficients a_i arising from P being harmonic:

$$\begin{aligned} 0 &= \Delta_{\mathbb{R}^n}(P) \\ &= \sum_{i=0}^{2k-2} a_i(2k-i)(2k-i-1)x^{2k-i-2}z^i + \sum_{i=2}^{2k} a_i i(i-1)x^{2k-i}z^{i-2} \\ &= \sum_{i=2}^{2k-2} [a_{i-2}(2k-i+2)(2k-i+1) + a_i i(i-1)]x^{2k-i}z^{i-2}, \end{aligned}$$

and hence

$$\frac{a_i}{a_{i-2}} = -\frac{(2k-i+2)(2k-i+1)}{i(i-1)},$$

and so,

$$\frac{a_{2k}}{a_0} = (-1)^k \frac{2k(2k-1)\cdots 2\cdot 1}{1\cdot 2\cdot 3\cdots(2k-1)2k} = (-1)^k. \tag{1.13}$$

Let $\gamma : [0, 2\pi] \to S^2$ be the great circle going around the equator, so it corresponds to the north pole N of S^2. We have

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$$\begin{aligned} I(\overline{P})(N) &= \int_0^{2\pi} P(\gamma(t))\,dt \\ &= \int_0^{2\pi} P(\cos t,\, \sin t, 0)\,dt \\ &= a_0 \int_0^{2\pi} (\cos t)^{2k}\,dt. \end{aligned}$$

But we also know that $I(\overline{P})(N) = c_{2k}P(N) = c_{2k}a_{2k}$. Thus we conclude using (1.13):

$$c_{2k} = (-1)^k \int_0^{2\pi} (\cos t)^{2k}\,dt. \tag{1.14}$$

This proves the first identity in the proposition; the second one is left as an exercise (it is a Wallis integral). □

Exercise 1.4.7 Compute the integral in (1.14).

Exercise 1.4.8 Give a shorter proof of the proposition considering the sectoral harmonic $P(x, y, z) = (x + iy)^{2k}$.

The proposition immediately proves that the kernel of I consists precisely of the odd functions; namely if $I(f) = 0$ then expanding f into harmonic polynomials and using the fact that $c_{2k} \neq 0$ for all k shows that $f \in C^\infty_{\text{odd}}(S^2)$, that is, $\ker I \subset C^\infty_{\text{odd}}(S^2)$, and we have observed that the reverse inclusion easily holds.

It will take a bit more work to prove the second assertion of Theorem 1.4.2. In the same way as we saw that the Radon transform in the plane is smoothing of order $1/2$, we shall see that the X-ray transform I on S^2 is smoothing of order $1/2$. To make this statement precise we need to define Sobolev spaces and norms. There are several ways to do this and intuitively, we think of a function f lying in $H^s(S^2)$ if it has s derivatives in L^2. For us the most convenient way to do it is to define for $f = \sum_{k=0}^\infty f_k \in L^2(S^2)$ and $s \geq 0$ that

$$\|f\|_s^2 := \sum_{k=0}^{\infty} (1 + k(k+1))^s \|f_k\|_{L^2}^2, \tag{1.15}$$

and declare that $H^s(S^2)$ is the set of $f \in L^2(S^2)$ such that $\|f\|_s < \infty$. When $s = 2m$ is an even integer this is equivalent to considering the norm $\|(-\Delta_{S^2} + 1)^m f\|_{L^2}$ and hence it captures the idea that if the norm is finite f has $2m$ derivatives in L^2. But the definition also gives meaning

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to smoothness of fractional order and it suggests that one could define the operator $(-\Delta_{S^2}+1)^{s/2}$ as

$$f \mapsto \sum_{k=0}^{\infty}(1+k(k+1))^{s/2} f_k.$$

Denote by $H^s_{\text{even}}(S^2)$ the set of even functions in $H^s(S^2)$. Now we show that with this choice of norm we have:

Theorem 1.4.9 *There is a constant $C > 1$ independent of s such that*

$$C^{-1}\|f\|_s \leq \|I(f)\|_{s+1/2} \leq C\|f\|_s$$

for all $s \geq 0$ and $f \in H^s_{\text{even}}(S^2)$.

Proof The proof is quite simple and it basically reduces to understanding the asymptotic behaviour of c_{2k} as $k \to \infty$. Using Proposition 1.4.6 together with Wallis's formula

$$\sqrt{\pi} = \lim_{k\to\infty} \frac{1}{\sqrt{k}} \frac{2\cdot 4\cdot 6\cdots 2k}{1\cdot 3\cdot 5\cdots(2k-1)},$$

we deduce that

$$c_{2k} \sim (-1)^k\sqrt{\frac{4\pi}{k}}.$$

This together with the definition of the norms in (1.15) gives the theorem right away. □

Proof of Theorem 1.4.2 Theorem 1.4.9 tells us that for $s \geq 0$ the map $I\colon H^s_{\text{even}}(S^2) \to H^{s+1/2}_{\text{even}}(S^2)$ is injective. In order to check that I is surjective, take $h \in H^{s+1/2}_{\text{even}}(S^2)$ and write $h = \sum_{k\geq 0} h_{2k}$. If we let $f := \sum_{k\geq 0} h_{2k}/c_{2k}$, then $f \in H^s_{\text{even}}(S^2)$ and $If = h$. Finally to check that I is a bijection between smooth even functions it suffices to note that $C^\infty(S^2) = \cap_{s\geq 0} H^s(S^2)$. □

Exercise 1.4.10 Consider the X-ray transform $I\colon \Omega^1(S^2) \to \Omega^1(S^2)$ acting on 1-forms on S^2 and let $\sigma\colon S^2 \to S^2$ be the antipodal map. A 1-form θ is said to be *odd* if $\sigma^*\theta = -\theta$ and *even* if $\sigma^*\theta = \theta$. Show that any odd form is in the kernel of I. Moreover, show that an even form is in the kernel of I if and only if it is exact (see Michel (1978, section 8)).

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