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The second moment of $\mathrm{GL}(n) \times \mathrm{GL}(n)$ Rankin–Selberg *L*-functions

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Abstract

We prove an asymptotic expansion of the second moment of the central values of the $\mathrm{GL}(n) \times \mathrm{GL}(n)$ Rankin–Selberg *L*-functions $L(1/2, \pi \otimes \pi_0)$ for a fixed cuspidal automorphic representation π_0 over the family of π with analytic conductors bounded by a quantity that is tending to infinity. Our proof uses the integral representations of the *L*-functions, period with regularised Eisenstein series and the invariance properties of the analytic newvectors.

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1. Introduction

The asymptotic evaluation of higher moments of the central *L*-values carries important arithmetic information: for example, subconvex bounds or nonvanishing results for the central *L*-values. This evaluation becomes more and more difficult as the moment, the degrees of the *L*-functions or the rank of the underlying group increase.

Obtaining subconvex bounds – that is, proving bounds of the form

$$L(1/2, \pi) \ll C(\pi)^{1/4-\delta}, \quad \delta > 0,$$

[1]where $C(\pi)$ is the *analytic conductor* of an automorphic representation π, is an extremely difficult problem with respect to the current technology. A narrow, but important for applications, class of automorphic representations suffers from yet another major technical difficulty named *conductor-drop*. These representations are usually functorially lifted from smaller groups and have unusually small analytic conductors.

For example, if π varies over automorphic representations for $\mathrm{PGL}_2(\mathbb{Q})$ with $C(\pi)$ being of size T, then the size of the analytic conductor of the Rankin–Selberg convolution $\pi \otimes \tilde{\pi}$ is roughly T^2, where $\tilde{\pi}$ is the contragredient of π, whereas $C(\pi \otimes \pi')$ has size T^4 if $C(\pi')$ is of size T^2 but π' is *away* from $\tilde{\pi}$. That is, the PGL(4)-subfamily of $\pi \otimes \tilde{\pi}$ shows the conductor-drop phenomena. Another example of a family that sees conductor-dropping is the PGL(3)-family of $\mathrm{Sym}^2\pi$, where π varies over a PGL(2) family (the subconvexity problem for this family is directly related to the arithmetic quantum unique ergodicity problem for $\mathrm{SL}_2(\mathbb{R})$). This happens due to one of the Langlands parameters of $\mathrm{Sym}^2\pi$ being extremely small compared to the others. The families defined by the Plancherel balls with a large radius (for example, dilated) or high centre often exclude these narrow classes. Thus moment estimates over these families do not usually become fruitful to yield a subconvex bound of an *L*-function that has conductor drop; see, for example, [2, 31, 30].

One naturally interesting and important family of automorphic representations can be given by representations with growing conductors: for example,

$$\mathcal{F}_X := \{\pi \text{ automorphic representation for } \mathrm{PGL}_n(\mathbb{Z}) \mid C(\pi) < X\},$$

with $X \to \infty$. The family $\mathcal{F}_X$, unlike the families defined by the Plancherel balls, is indifferent to the conductor-drop issue. So a Lindelöf-consistent estimate for a high enough moment over the family $\mathcal{F}_X$ will likely produce a subconvex estimate even for the *L*-functions suffering from conductor-drop. Here, by *Lindelöf-consistent* (also called *Lindelöf on average*) estimate for the $2k$th moment, we mean the estimate

$$\mathbb{E}_{\mathcal{F}_X} |L(1/2, \pi)|^{2k} \ll_\epsilon X^\epsilon,$$

where $\mathbb{E}$ denotes the average. On the other hand, a more interesting and difficult question would be to find an asymptotic formula of (a suitably weighted and smoothened version of) the above average whose leading term is believed to be a polynomial in $\log X$.

However, the family $\mathcal{F}_X$ becomes quite large as X tends to infinity. One informally has $|\mathcal{F}_X| \asymp X^{n-1}$; see [6] for the corresponding nonarchimedean analogue. This is why, to obtain a subconvex bound of an *L*-function attached to an element in $\mathcal{F}_X$, one needs to evaluate quite a high moment asymptotically or at least estimate in the Lindelöf-consistent manner. For example, we need to estimate an amplified $4(n-1)$th moment over $\mathcal{F}_X$ even to break the convexity barrier. Unfortunately, the current technology is not advanced enough to tackle such a high moment of these *L*-functions due to the large size of the conductors. Hence a natural, informal question arises regarding the race between the sizes of the conductors of the *L*-functions and the families: *as a function of* n, *how high of a moment can be asymptotically evaluated (or estimated in a Lindelöf-consistent manner) over the family* $\mathcal{F}_X$?

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[1]The generalised Lindelöf hypothesis predicts that any $\delta < 1/4$ is achievable.

Such a question has been addressed in the literature for low-rank groups. We may try to guess an answer to our proposed informal question by looking at the small number of examples in low ranks. For $n = 2$ in [22], the authors obtained an asymptotic formula of the 4th moment over a family in the nonarchimedean conductor aspect and restricted only to the holomorphic forms. In [3], the authors proved a Lindelöf-consistent upper bound of the 6th moment in the nonarchimedean conductor aspect for $n = 3$. These are the best possible estimates so far for small n, which allows us to wonder whether the $2n$th moment can be asymptotically evaluated over the family $\mathcal{F}_X$. However, if we work on the $\mathrm{GL}(n)$ rather than on the $\mathrm{PGL}(n)$ family – that is, if we do an extra central average – we expect that an asymptotic formula of the $2n + 2$th moment is achievable.

Our primary motivation is to prove an asymptotic formula for the $2n$th moment of the central L-values for $\mathrm{PGL}(n)$ with $n \geq 3$, over the family $\mathcal{F}_X$, using the integral representations of the L-functions and spectral theory. If π is an automorphic representation for $\mathrm{PGL}(n)$, then

$$L(1/2, \pi)^n = L(1/2, \pi \otimes E_0),$$

where E_0 is the minimal Eisenstein series for $\mathrm{PGL}(n)$ with trivial Langlands parameters and $\otimes$ denotes the Rankin–Selberg convolution. Thus evaluating the $2n$th moment of $L(1/2, \pi)$ is the same as evaluating the second moment of $L(1/2, \pi \otimes E_0)$. However, the approach of the integral representations and the spectral decomposition encounters severe analytic difficulties due to the growth of E_0 near the cusp: for example, E_0 fails to be square integrable in the fundamental domain. To avoid this particular technical difficulty, we may replace E_0 with a fixed cusp form and try to evaluate their second moment asymptotically.

Let $n \geq 3$. In this article, we evaluate the second moment of the central Rankin–Selberg L-values $L(1/2, \pi \otimes \pi_0)$, where π varies over a family of automorphic representations for $\mathrm{PGL}_n(\mathbb{Q})$ that are unramified at all the finite places and the archimedean conductors are growing to infinity. Here π_0 is a fixed cuspidal representation for $\mathrm{PGL}_n(\mathbb{Q})$, which is again unramified at all the finite places. Below we informally describe our main theorem.

Theorem 1.1 (Informal version). *Let $n \geq 3$ and π_0 be a cuspidal automorphic representation for $\mathrm{PGL}_n(\mathbb{Z})$ (i.e., unramified at the finite places), which is tempered at ∞. Let π vary over the generic automorphic representations in $\mathcal{F}_X$. Then we have an asymptotic formula of the following (weighted) average*

$$\mathbb{E}_{\substack{\pi \in \mathcal{F}_X \\ \text{generic}}} \left[\frac{|L(1/2, \pi \otimes \pi_0)|^2}{L(1, \pi, \mathrm{Ad})} + \text{continuous} \right] = n \frac{\zeta(n/2)^2}{\zeta(n)} L(1, \pi_0, \mathrm{Ad}) \log X + O_{\pi_0}(1),$$

as X tends to infinity.

For the actual formal statement, we refer to Theorem 2.1.

Remark 1.2. In Theorem 1.1, by 'continuous', we mean the corresponding terms from the generic noncuspidal spectrum. In the actual statement – that is, Theorem 2.1 – we do a specific weighted average over the full generic automorphic spectrum such that the weights are uniformly bounded away from zero on the cuspidal spectrum with analytic conductors bounded by X. Consequently, we also need to change the harmonic weight $L(1, \pi, \mathrm{Ad})$ by an equivalent arithmetic factor for the noncuspidal spectrum.

This is the first instance of an asymptotic evaluation of the second moment of a family of L-functions with arbitrary high degree. In general, for a pair of groups $H \leq G$ and their representations π and Π, respectively, it is an interesting question to asymptotically evaluate moments of the central L-values of the Rankin–Selberg product $\Pi \otimes \pi$ (if defined). Previously, in [31], Nelson–Venkatesh asymptotically evaluated the first moment keeping Π fixed and letting π vary over a dilated Plancherel ball when (G, H) are Gan–Gross–Prasad pairs and, more interestingly, allowing arbitrary weights in the spectral side. More recently, in [30], Nelson proved a Lindelöf-consistent upper bound of the first moment for the groups $(G, H) = (\mathrm{U}(n + 1), \mathrm{U}(n))$ in the nonsplit case, keeping π fixed and letting Π vary over a

Plancherel ball with high centre. Both [31, 30] assume that the family of $L(s, \Pi \otimes \pi)$ does not show any conductor-dropping. The method in [30] also yields an asymptotic formula with power savings of a specific weighted first moment over this family. Blomer in [1] obtained a Lindelöf-consistent upper bound of the second moment for $G = H = \mathrm{GL}(n)$, keeping Π a fixed cuspidal representation and letting π vary in a Plancherel ball. On the contrary to [31, 30], he proves a Lindelöf-consistent upper bound when the family of $L(s, \Pi \otimes \pi)$ *shows* conductor-dropping. However, his method does not yield an asymptotic formula.

There have been quite a few results for asymptotic formulas and upper bounds on rank ≤ 2 and degree ≤ 4. In particular, we refer to [4]: the authors prove an asymptotic formula for $\mathrm{GL}(2) \times \mathrm{GL}(2)$ Rankin–Selberg L-functions, fixing one of the representations but with an extra average over the centre of $\mathrm{GL}(2)$. In [10], an asymptotic formula for the sixth moment of the L-values attached to holomorphic cusp forms for $\mathrm{GL}(2)$ is achieved, but again with an extra average over the centre of $\mathrm{GL}(2)$.

1.1. Sketch of the proof

Our point of departure is similar to [1] and [32]. We use the spectral decomposition of $\mathrm{PGL}(n)$ and integral representations of the L-functions. We start by choosing $\phi_0 \in \pi_0$ such that the Whittaker function W_0 of ϕ_0 is an *analytic newvector*; see Section 2.9 for a brief description of the analytic newvectors. Such W_0 in the Kirillov model of π_0 can be described by a fixed bump function. Let $\mathrm{Eis}(f_s)$ be the maximal Eisenstein series $\mathrm{PGL}_n(\mathbb{Z})$ attached to a generalised principal series vector f_s. Also let X be a large real number and x be the diagonal element in $\mathrm{PGL}_n(\mathbb{R})$ given by $\mathrm{diag}(X, \ldots, X, 1)$. We translate the Eisenstein series by x to obtain $\mathrm{Eis}(f_s)(.x)$.

For this subsection, let $\mathbb{X} := \mathrm{PGL}_n(\mathbb{Z}) \backslash \mathrm{PGL}_n(\mathbb{R})$ and N be the maximal unipotent of the upper triangular matrices in $\mathrm{PGL}_n(\mathbb{R})$. We start by writing the inner product

$$\langle \phi_0 \mathrm{Eis}(f_{1/2})(.x), \phi_0 \mathrm{Eis}(f_{1/2})(.x) \rangle = \langle |\phi_0|^2, |\mathrm{Eis}(f_{1/2})|^2(.x) \rangle, \tag{1.1}$$

where all the inner products above are the usual L^2-inner product on the fundamental domain $\mathbb{X}$. Note that both of the sides of equation (1.1) are absolutely convergent as ϕ_0 decays rapidly at the cusps.

We use Parseval's identity on the left-hand side over $\mathrm{PGL}(n)$. A typical term corresponding to an automorphic representation π in the spectral sum would look like

$$\left| \int_{\mathbb{X}} \phi_0(g) \overline{\phi(g)} \mathrm{Eis}(f_{1/2})(gx) dg \right|^2 = \frac{|L(1/2, \tilde{\pi} \otimes \pi_0)|^2}{L(1, \pi, \mathrm{Ad})} |Z_x(f_{1/2}, W, W_0)|^2,$$

and

$$Z_x(f_s, W, W_0) = \int_{N \backslash \mathrm{PGL}_r(\mathbb{R})} W_0(g) \overline{W(g)} f_s(gx) dg$$

is the local zeta integral.

We choose f_s such that $f_{1/2}\left[\begin{pmatrix} I & \\ c & 1 \end{pmatrix} \right]$ is supported on $|c| < \tau$ for some $\tau > 0$ sufficiently small, so that $W_0 f_{1/2}(.x)$ would mimic a smoothened characteristic function of the archimedean congruence subgroup $K_0(X, \tau)$ (see equation (2.8)). If W is an *analytic newvector* (see Section 2.9), then the invariance property of W will yield that $Z_x(f_{1/2}, W, W_0) \gg 1$ if $C(\pi) < X$. We use

$$\sum_W |Z_x(f_{1/2}, W, W_0)|^2$$

as the spectral weights where in the above sum W traverses some orthonormal basis of π. We point out on the naive similarities between the spectral weight here and the one used in, for example, [18, Theorem 1]. However, the invariance property that is needed here is a bit stronger than the invariance used in

[18, Theorem 1]: we only needed invariance at points near the identity in $\mathrm{GL}_n(\mathbb{R})$ in [18], whereas here, we have to gain an invariance that is uniform for all elements in $\mathrm{GL}_{n-1}(\mathbb{R})$. The method of using the *approximate* invariance of the newvectors is similar to [32] for $\mathrm{GL}(2)$, where in the nonarchimedean aspect, the exact invariance is used. This analysis is done in Section 7.

We now explain how we proceed to give an asymptotic expansion of the right-hand side of equation (1.1). The heuristic idea, at least to obtain an upper bound, is to make the change of variables in the period of the right-hand side of equation (1.1) to write it as

$$\int_{\mathbb{X}} |\phi_0(gx^{-1})|^2 |\mathrm{Eis}(f_{1/2})(g)|^2 dg$$

and then bound this period by

$$\leq \|\phi_0\|_{L^\infty(\mathbb{X})}^2 \int_{\mathbb{X}} |\mathrm{Eis}(f_{1/2})(g)|^2 dg.$$

But unfortunately, $\mathrm{Eis}(f_{1/2})$ (barely!) fails to be square integrable on $\mathbb{X}$. That is why we have to regularise the period. We adopt the regularising techniques of Zagier [34]; see also [27, 29]. First we deform $|\mathrm{Eis}(f_{1/2})|^2$ as $\mathrm{Eis}(f_{1/2+s})\overline{\mathrm{Eis}(f_{1/2})}$ for s lying in some generic position with very small $\Re(s)$. From the Fourier expansions of the Eisenstein series, we can pick off the nonintegrable terms in the product $\mathrm{Eis}(f_{1/2+s})\overline{\mathrm{Eis}(f_{1/2})}$ and call their sum F_s. Then we construct a regularised Eisenstein series by

$$\tilde{E}(s, .) := \mathrm{Eis}(f_{1/2+s})\overline{\mathrm{Eis}(f_{1/2})} - \mathrm{Eis}(F_s).$$

We will check that $\tilde{E}(s, .)$ lies in $L^2(\mathbb{X})$. Consequently, we regularise the period as

$$\langle |\phi_0(.x^{-1})|^2, |\mathrm{Eis}(f_{1/2})|^2 \rangle = \lim_{s \to 0} \langle |\phi_0(.x^{-1})|^2, \overline{\mathrm{Eis}(f_{1/2})}\mathrm{Eis}(f_{1/2+s}) \rangle$$
$$= \lim_{s \to 0} \langle |\phi_0(.x^{-1})|^2, \tilde{E}(s, .) \rangle + \lim_{s \to 0} \langle |\phi_0(.x^{-1})|^2, \mathrm{Eis}(F_s) \rangle.$$

We call the first summand the *regularised term*, which, upon rigorous application of the heuristic above, can be proved to be of bounded size. The second summand is called the *degenerate term* and yields the main term.

Up to some nonarchimedean factors involving $L(1, \pi_0, \mathrm{Ad})$, the degenerate term is of the form

$$\partial_{s=0}Z_x(f_{1/2}f_{1/2+s}, W_0, \overline{W_0}) - \partial_{s=0}Z_x(Mf_{1/2}Mf_{1/2+s}, W_0, \overline{W_0}),$$

where M is certain intertwining operator that arises in the constant term of a maximal parabolic Eisenstein series. One main difficulty of the paper is asymptotically evaluating the above two derivatives. The first one is comparatively easy to understand as one can apply the support condition of $f_{1/2}f_{1/2+s}\left[\begin{pmatrix} I & \\ c & 1 \end{pmatrix}x\right]$, which is concentrated on $c = O(1/X)$ and the approximate invariance of W_0. The second one is more technical to analyse. The intertwined vector $Mf_{1/2+s}\left[\begin{pmatrix} I & \\ c & 1 \end{pmatrix}x\right]$, which on the matrices $\begin{pmatrix} I & \\ c & 1 \end{pmatrix}$ essentially mimics a Fourier transform of $f_{1/2+s}$, has support of size $c = O(X)$. So we cannot get away with just the invariance properties of W_0. In this case, we understand a more detailed shape of the intertwined vectors via the Iwasawa decomposition on the matrices of the form $\begin{pmatrix} I & \\ cX & 1 \end{pmatrix}$. This analysis is done in Section 5.

On the other hand, to analyse the regularised term, we understand the growth of the (degenerate) Fourier terms of $\mathrm{Eis}(f_s)$ for s being close to $0, 1/2$ or 1. This analysis relies on the analytic properties

of the intertwining operators attached to various Weyl elements and functional analytic properties of the Eisenstein series. This analysis is done in Section 6.

Remark 1.3. We remark that our method of proof, which is uniform for $n \geq 3$, can also be made to work for $n = 2$ with a slight modification with a modified main term (the statement of our theorem does not in any way make sense for $n = 2$). The main terms in the asymptotic expansion are the artefacts of the nonintegrable terms among the product of the constant terms in the Fourier expansion of $\mathrm{Eis}(f_{1/2})$ and $\mathrm{Eis}(f_{1/2+s})$. The constant term of $\mathrm{Eis}(f_s)$ looks like $\sum_w M_w f_s$, where M_w are certain intertwining operators and w runs over a set of Weyl elements attached to the underlying parabolic subgroup (see Section 3). If $n \geq 3$, then the nonintegrable terms in the above-mentioned product are of the form $f_{1/2} f_{1/2+s}$ and $M f_{1/2} M f_{1/2+s}$, where M is the intertwiner attached to the relative long Weyl element. In particular, the off-diagonal terms of the from $f_{1/2} M f_{1/2+s}$ are integrable. Such a phenomenon does not happen for $n = 2$. In this case (where the maximal Eisenstein series is also a minimal Eisenstein series), the off-diagonal terms are also nonintegrable.

As described in the sketch of the proof, eventually we need to deform the principal series vector to regularise the Eisenstein series. The number of deformations needed in the Langlands parameters of the associated principal series vector depends on the number of nonintegrable terms in the product of the constant terms. For $n \geq 3$, we need to deform only one of the parameters of the principal series vector to regularise the corresponding maximal parabolic Eisenstein series. However, for $n = 2$, for the reasons stated above, to regularise the Eisenstein series, we need to deform two (i.e., both) of the parameters. This modification will produce more degenerate terms, and consequently, a different main term with a different constant will appear; see [4].

1.2. What's next?

As we have described above, the motivating question for us is to find an asymptotic expansion of the $2n$th moment of the central L-values for $\mathrm{PGL}(n)$, and to do that, we need to replace ϕ_0 with a minimal Eisenstein series E_0 with trivial Langlands parameters. As, in particular, E_0 is not in L^∞, our current proof obviously fails (see the sketch of the proof), and that is why we need to regularise E_0 as well. However, this regularisation increases the analytic difficulties manyfold. We need to employ a regularised version of the spectral decomposition (and Parseval), as in, for example, [29, 27], to follow the same strategy as in the sketch of the proof of the main theorem. On the other hand, regularising both the Eisenstein series involved in the period $\langle |E_0|^2, |\mathrm{Eis}(f_{1/2})(.x)|^2 \rangle$ will introduce many more degenerate terms, which will typically have higher-order poles at the critical point. This will likely yield a higher power of $\log X$ in the main term. It will be interesting to see if the constant appearing in the main term is the same as predicted by the random matrix models; see [9]. However, we leave this to future work.

It is natural to speculate what happens for the second moment of the Rankin–Selberg L-functions for other $(\mathrm{GL}(n), \mathrm{GL}(m))$ pairs with $m \neq n$ and the $\mathrm{GL}(m)$ form being fixed (cuspidal or Eisenstein). If $m < n$, we believe that the problems become simpler than the $m = n$ case as the degrees, and hence conductors, become lower. Similarly, for $m > n$, we expect the problems to be much more difficult for high degree and conductor size. In particular, it will be very interesting to see if we can push the method in this paper at least to the case $m = n + 1$ case. More interestingly, if $n = 3$ and the fixed form is a minimal Eisenstein series, then we will have a Lindelöf-consistent eighth moment (the convexity barrier) of L-functions of $\mathrm{PGL}(3)$ over the family $\mathcal{F}_X$.

Remark 1.4. We briefly remark that one may try to explicate the constant contribution of the asymptotic expansion in the main theorem and obtain a power-saving error term as in [4]. One possible way to obtain finer asymptotics in the regularised part is to spectrally expand the period $\langle |\phi_0|^2, \tilde{E}_s(.x) \rangle$ over the $\mathrm{PGL}(n)$ automorphic spectrum. Then one may use the existence of a spectral gap and explicit decay of the matrix coefficient for $n \geq 3$ to obtain that $\langle |\phi_0|^2, \phi \rangle \langle \phi, \tilde{E}_s(.x) \rangle$, at least for a tempered ϕ, will decay polynomially in X. However, it is not yet clear to us how to explicate the constant term and get an error term with polynomial saving in the degenerate part; see Remark 5.4.

2. Basic notations and preliminaries

2.1. Basic notations

We use adèlic language. Let $r \geq 3$. For any ring R by $G(R)$, we denote the set of points $\mathrm{GL}_r(R)/R^\times$. In this paper, R will denote the adèles $\mathbb{A}$ over $\mathbb{Q}$ or the local fields $\mathbb{R}$, $\mathbb{Q}_p$ or rational numbers $\mathbb{Q}$ or the local ring $\mathbb{Z}_p$. We drop the ring R from the notation $G(R)$ if the ring is clear from the context.

Let N be the maximal unipotent subgroup of G consisting of upper triangular matrices. For $q \in \mathbb{A}^{r-1}$, we define a character of $N(\mathbb{A})$ by

$$\psi_q(n(x)) = \psi_0\left(\sum_{i=1}^{r-1} q_i x_{i,i+1}\right), \quad n(x) := (x_{i,j})_{i,j},$$

where ψ_0 is an additive character of $\mathbb{Q}\backslash\mathbb{A}$. We abbreviate $\psi_{(1,\dots,1)}$ by ψ. We call ψ_q *nondegenerate* if $q_i \neq 0$ for $1 \leq i \leq r-1$; otherwise, we call ψ_q *degenerate*.

Let A be the set of diagonal matrices in G, which we identify with $\begin{pmatrix} A_{r-1} & \\ & 1 \end{pmatrix}$, where A_{r-1} is the set of diagonal matrices in $\mathrm{GL}(r-1)$. We parametrise elements of A_{r-1} as $a(y) := \mathrm{diag}(y_1 \dots y_{r-1}, \dots, y_{r-1})$ Let $K := \prod_{p \leq \infty} K_p$ be the standard maximal compact in $G(\mathbb{A})$, where $K_p := G(\mathbb{Z}_p)$ for $p < \infty$ and $K_\infty := \mathrm{PO}_r(\mathbb{R})$.

For any factorisable function f on $G(\mathbb{A})$ by f_p, we denote the pth component of f, which is a function on $G(\mathbb{Q}_p)$.

2.2. Domains and measures

We fix Haar measures on G and its subgroups, and a G-invariant measure of $N\backslash G$. If the subgroup is compact, then we normalise the Haar measure to be a probability measure. Let δ denote the modular character on A. It is defined by

$$\delta(a(y)) := \prod_{j=1}^{r-1} |y_j|^{j(r-1-j)}$$

and is trivially extended to NA.

To integrate over $N(\mathbb{R})\backslash G(\mathbb{R})$, we use two different types of coordinates according to efficiency. The first one is Bruhat (with respect to the standard maximal parabolic) coordinates. First, note that the set of elements of the form $\begin{pmatrix} h & b^t \\ c & 0 \end{pmatrix}$ with $h \in \mathrm{GL}_{r-1}(\mathbb{R})$ and row vectors $b, c \in \mathbb{R}^{r-1}$ has zero measure in $\mathrm{GL}_r(\mathbb{R})$ with respect to its Haar measure. Thus while integrating over $G(\mathbb{R})$, we integrate over the points of the form

$$\begin{pmatrix} h & b^t \\ & 1 \end{pmatrix}\begin{pmatrix} \mathrm{I}_{r-1} & \\ c & 1 \end{pmatrix}, \quad h \in \mathrm{GL}_{r-1}(\mathbb{R}) \text{ and } b, c \in \mathbb{R}^{r-1} \text{ row vectors.}$$

Similarly, the set of points of the form

$$g = \begin{pmatrix} h & \\ & 1 \end{pmatrix}\begin{pmatrix} \mathrm{I}_{r-1} & \\ c & 1 \end{pmatrix}, \quad h \in N_{r-1}(\mathbb{R})\backslash\mathrm{GL}_{r-1}(\mathbb{R}) \text{ and } c \in \mathbb{R}^{r-1} \text{ row vector}$$

has full measure in $N(\mathbb{R})\backslash G(\mathbb{R})$. We use these coordinates to integrate over $N(\mathbb{R})\backslash G(\mathbb{R})$ using the invariant measure

$$dg = \frac{dh}{|\det(h)|} dc,$$

where dc denotes the Lebesgue measure and dh is the $\mathrm{GL}_{r-1}(\mathbb{R})$-invariant Haar measure on $N_{r-1}(\mathbb{R})\backslash\mathrm{GL}_{r-1}(\mathbb{R})$. Description of the above invariant measure follows from [23, eq. (5.14)] and discussion above that. However, there is a more direct way to see this. Let $\phi \in C_c(G(\mathbb{R}))$ be measurable. Then it follows from [14, Proposition 1.4.3] that

$$\int_{G(\mathbb{R})} \phi(g)dg = \int_{\mathbb{R}^{n^2-1}} \phi\left[\begin{pmatrix} h & b^t \\ c & 1 \end{pmatrix}\right] \frac{\prod_{i,j} d_L h_{ij} \prod_i d_L b_i \prod_i d_L c_i}{\left|\det\left[\begin{pmatrix} h & b^t \\ c & 1 \end{pmatrix}\right]\right|^n},$$

where $d_L x$ denotes the Lebesgue measure on $\mathbb{R}$. Noting that

$$\begin{pmatrix} h & b^t \\ c & 1 \end{pmatrix} = \begin{pmatrix} \mathrm{I}_{r-1} & b^t \\ & 1 \end{pmatrix}\begin{pmatrix} h - b^t c & \\ & 1 \end{pmatrix}\begin{pmatrix} \mathrm{I}_{r-1} & \\ c & 1 \end{pmatrix},$$

we can write

$$\int_{G(\mathbb{R})} \phi(g)dg = \int_{\mathbb{R}^{n^2-1}} \phi\left[\begin{pmatrix} \mathrm{I}_{r-1} & b^t \\ & 1 \end{pmatrix}\begin{pmatrix} h - b^t c & \\ & 1 \end{pmatrix}\begin{pmatrix} \mathrm{I}_{r-1} & \\ c & 1 \end{pmatrix}\right] \frac{\prod_{i,j} d_L h_{ij} \prod_i d_L b_i \prod_i d_L c_i}{|\det(h - b^t c)|^n}.$$

Fixing b, c and changing variables $h_{ij} \mapsto h_{ij} + (b^t c)_{ij}$, we can write the above as

$$\int_{G(\mathbb{R})} \phi(g)dg = \int_{\mathbb{R}^{n^2-1}} \phi\left[\begin{pmatrix} \mathrm{I}_{r-1} & b^t \\ & 1 \end{pmatrix}\begin{pmatrix} h & \\ & 1 \end{pmatrix}\begin{pmatrix} \mathrm{I}_{r-1} & \\ c & 1 \end{pmatrix}\right] \frac{\prod_{i,j} d_L h_{ij} \prod_i d_L b_i \prod_i d_L c_i}{|\det(h)|^n}.$$

Noting that $\frac{\prod_{i,j} d_L h_{ij}}{|\det(h)|^{n-1}} = dh$ and taking the $N(\mathbb{R})$-quotient on the left, we deduce the invariant measure on $N(\mathbb{R})\backslash G(\mathbb{R})$.

On the other hand, when we integrate on $N_{r-1}(\mathbb{R})\backslash\mathrm{GL}_{r-1}(\mathbb{R})$, we use Iwasawa coordinates. We write

$$N_{r-1}(\mathbb{R})\backslash\mathrm{GL}_{r-1}(\mathbb{R}) \ni h = a(y)k, \quad a(y) \in A_{r-1}, k \in K_{r-1},$$

where K_{r-1} is the standard maximal compact in GL_{r-1}, with the measure

$$dh = \frac{\prod_i d^\times y_i}{\delta(a(y))} dk,$$

where dk is the probability Haar measure on K_{r-1}.

Let $\mathbb{X} := G(\mathbb{Q})\backslash G(\mathbb{A})$. We fix a fundamental domain $\mathbb{X}$ in $G(\mathbb{A})$ of the form

$$D \times K_f, \quad D \subseteq G(\mathbb{R}), K_f := \prod_{p<\infty} K_p$$

that is contained in a Siegel domain of the form $\mathbb{S} \times K_f$, where

$$\mathbb{S} := \left\{ G(\mathbb{R}) \ni g = n(x)\begin{pmatrix} a(y) & \\ & 1 \end{pmatrix}k \mid |x_{i,j}| < 1, |y_i| > y_0, k \in K_\infty \right\}, \tag{2.1}$$

where $y_0 > 0$ is an explicit constant dependent only on the group. The above follows from strong approximation for $\mathrm{GL}(n)$ and [14, §1.3].

We equip $\mathbb{X}$ with the $G(\mathbb{A})$-invariant probability measure that in Iwasawa coordinates is given by

$$\mathbb{X} \ni g = n(x)\begin{pmatrix} a(y) & \\ & 1 \end{pmatrix}k, \quad dg = \prod_{j,k} dx_{j,k} \frac{\prod_i d^\times y_i}{\delta(a(y))|\det(a(y))|} dk,$$

where $n(x) \in N(\mathbb{R})$ and $dx_{i,j}$ is the usual Lebesgue measure. Note that $\delta\begin{pmatrix} a(y) & \\ & 1 \end{pmatrix} = \delta(a(y))|\det(a(y))|$.

2.3. *Automorphic representations*

We briefly describe the classes of local and global representations that are relevant in this paper. We refer to [28], [12, §5] for details.

Let $\mathring{\mathbb{X}}$ be the isomorphism class of irreducible unitary automorphic representations that are unramified at all finite places and appear in the spectral decomposition of $L^2(\mathbb{X})$. Similarly, by $\mathring{\mathbb{X}}_{gen}$, we denote the subclass of *generic* representations in $\mathring{\mathbb{X}}$: that is, the class of representations that have (unique) Whittaker models.

We first mention the Langlands description for $\mathring{\mathbb{X}}_{gen}$. We take a partition $r = r_1 + \cdots + r_k$. Let π_j be a unitary cuspidal automorphic representation for $GL_{r_j}(\mathbb{Q})$ (if $r_j = 1$, we take π_j to be a unitary Hecke character). Consider the normalised parabolic induction Π from the Levi $GL(r_1) \times \cdots \times GL(r_k)$ to G of the tensor product $\pi_1 \otimes \cdots \otimes \pi_k$. There exists a unique irreducible constituent of Π, which we denote by the isobaric sum $\pi_1 \boxplus \cdots \boxplus \pi_k$. Then the Langlands classification says that every element in $\mathring{\mathbb{X}}_{gen}$ is isomorphic to an isobaric sum $\boxplus_{j=1}^{k'} \pi'_j$ for some partition $r = \sum_{j=1}^{k'} r'_j$ and some cuspidal representation π'_j of $GL(r'_j)$.

We recall from [28] that we call $\pi' \in \mathring{\mathbb{X}}$ a *discrete series* if π' appears discretely in the spectral decomposition of $L^2(\mathbb{X})$. The elements of π' are square-integrable automorphic forms for $G(\mathbb{Q})$. Mœglin–Waldspurger classified the discrete series for $G(\mathbb{Q})$ via the iterated residues of generic automorphic forms; see [28]. The Langlands description of $\mathring{\mathbb{X}}$ says that every element in $\mathring{\mathbb{X}}$ is isomorphic to an isobaric sum $\boxplus_{j=1}^{k'} \pi'_j$ for some partition $r = \sum_{j=1}^{k'} r'_j$ and some discrete series representation π'_j of $GL(r'_j)$.

We fix an automorphic Plancherel measure $d\mu_{aut}$ on $\mathring{\mathbb{X}}$ compatible with the invariant probability measure on $\mathbb{X}$. If π is a discrete series, then $d\mu_{aut}(\pi)$ is absolutely continuous to the counting measure at π. On the other hand, if π is an Eisenstein series induced from a twisted discrete series $\pi'|.|^\lambda$, where π' is a discrete series on a Levi subgroup M and λ lies in the purely imaginary dual of the Cartan subalgebra of M, then $d\mu_{aut}(\pi)$ is absolute continuous to the product of the counting measure at π' and the Lebesgue measure at λ.

For any $\pi \in \mathring{\mathbb{X}}$, we denote the pth component of π by π_p for $p \le \infty$. The generalised Ramanujan conjecture predicts that if π is cuspidal, then π_p is tempered for all $p \le \infty$. In this paper, we assume that certain cuspidal representations are ϑ-*tempered* at the archimedean place, whose definition we recall below.

First we describe the Langlands description of the generic representations of $G(\mathbb{R})$. Let $r' \in \{1, 2\}$ and σ be an *essentially* square integrable (square integrable mod centre) representation of $GL_{r'}(\mathbb{R})$. That is, if $r' = 2$, then σ is a discrete series of $GL_2(\mathbb{R})$; and if $r' = 1$, then σ is a unitary character of $GL_1(\mathbb{R})$. By the Langlands classification, we know that any *generic* unitary irreducible representation ξ of $G(\mathbb{R})$ is isomorphic to a normalised parabolic induction of

$$\sigma_1 |\det|^{s_1} \otimes \cdots \otimes \sigma_k |\det|^{s_k},$$

from a Levi subgroup attached to a partition of $r = \sum_{j=1}^{k} r_j$ with $r_j \in \{1, 2\}$, where σ_j is an essentially square integrable representation of $GL(r_j)$ and $s_j \in \mathbb{C}$ with $\sum_{j=1}^{k} r_j s_j = 0$ and $\Re(s_1) \ge \cdots \ge \Re(s_k)$.

Let $\vartheta \ge 0$. We say that ξ is ϑ-*tempered* if all such s_i have real parts in $[-\vartheta, \vartheta]$. By [26], if π is cuspidal, then π_∞ is ϑ-tempered with $\vartheta = 1/2 - 1/(1 + r^2)$.

We denote the analytic conductor of π by $C(\pi)$. Note that as $\pi \in \mathring{\mathbb{X}}$ is unramified at all the finite places, we have $C(\pi) = C(\pi_\infty)$. If $\{\mu_i\} \in \mathbb{C}^r$ are the Langlands parameters of π_∞, then we define (see [15]) $C(\pi_\infty) := \prod_{i=1}^{r} (1 + |\mu_i|)$.

2.4. Maximal Eisenstein series

Let P be the standard parabolic subgroup in G attached to the $r = (r - 1) + 1$ partition. We choose a generalised principal series vector

$$f_s \in \mathcal{I}_{r-1,1}(s) := \mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})} |\det|^s \boxplus |.|^{-(r-1)s}, \quad s \in \mathbb{C},$$

by

$$f_s(g) = f_{s,\Phi}(g) := \int_{\mathbb{A}^\times} \Phi(t e_r g)|\det(tg)|^s d^\times t, \tag{2.2}$$

where $\Phi \in \mathcal{S}(\mathbb{A}^r)$ is a Schwartz–Bruhat, factorisable function and $e_r = (0, \ldots, 0, 1) \in \mathbb{A}^r$. The integral in equation (2.2) converges for $\Re(s) > 1/r$ and then can be extended meromorphically to the whole complex plane. By $\hat{\Phi}$, we denote Fourier transform of Φ that is defined by

$$\hat{\Phi}(x) := \int_{\mathbb{A}^r} \Phi(u)\psi_0(x_1 u_1 + \cdots + +x_r u_r)du.$$

We abbreviate $f_{s,\hat{\Phi}}$ as $\hat{f}_s$. We record the following transformation property of f_s, which can be seen from equation (2.2):

$$f_s\left[\begin{pmatrix} h \\ & 1 \end{pmatrix} g\right] = |\det(h)|^s f_s(g), \quad h \in \mathrm{GL}_{r-1}(\mathbb{A}), g \in G(\mathbb{A}). \tag{2.3}$$

Finally, we define the maximal Eisenstein series associated to f_s by

$$\mathrm{Eis}(f_s)(g) := \sum_{\gamma \in P(\mathbb{Q}) \backslash G(\mathbb{Q})} f_s(\gamma g).$$

The above definition is valid for s in a right half plane and then can be extended to all of $\mathbb{C}$ by meromorphic continuation. From [21, §4], [11, §2.3.1], we know that for $f_{s,\Phi} \in \mathcal{I}_{r-1,1}(s)$ with some $\Phi \in \mathcal{S}(\mathbb{A}^r)$, the maximal parabolic Eisenstein series $\mathrm{Eis}(f_{s,\Phi})$ has at most simple poles at $s = 0$ and $s = 1$. The residues at these poles are independent of g.

Let $\tilde{P}$ be the maximal parabolic subgroup in G attached to the partition $r = 1 + (r - 1)$ (the *associate parabolic* to P). We can similarly construct an associated Eisenstein series from a vector $\tilde{f}_s \in \mathcal{I}_{1,r-1}(s)$ defined analogously. All of the properties of an Eisenstein series associated to P hold analogously for the same associated to $\tilde{P}$.

2.5. Genericity and Kirillov model

We briefly review the Whittaker and Kirillov models of a generic representation of G over a local field; see [16] for details. In this subsection, we only work locally, without mentioning the underlying local field. Fix a nondegenerate additive character ψ of $N < G$. Consider the space of Whittaker functions on G by

$$\mathcal{W}(G) := \left\{ W \in C^\infty(G) \,\middle|\, \begin{matrix} W(ng) = \psi(n)W(g), n \in N, g \in G; \\ W \text{ grows at most polynomially in } g \end{matrix} \right\},$$

on which G acts by right translation.

We call an irreducible representation π of G *generic* if there exists a G-equivariant embedding $\pi \hookrightarrow \mathcal{W}(G)$. For generic π, we identify π with its image in $\mathcal{W}(G)$, which we call the *Whittaker model* of π under this embedding.

It is known (for example, see [16] for the case of an archimedean local field) from the theory of the Kirillov model that if π is an irreducible generic representation of PGL(r), then

$$\pi \ni W \mapsto \left\{ \mathrm{GL}(r-1) \ni g \mapsto W\left[\begin{pmatrix} g & \\ & 1 \end{pmatrix} \right] \right\}$$

is injective and the space of the restricted Whittaker functions in the right-hand side, which is called the *Kirillov model*, is isomorphic to π as well. It is also known that the space $C_c^\infty(N_{r-1}(\mathbb{R})\backslash\mathrm{GL}_{r-1}(\mathbb{R}), \psi)$ is contained in π under this realisation; see [16, Proposition 5].

If π is also unitary, then we can give a unitary structure on its Whittaker model by the inner product

$$\langle W_1, W_2 \rangle := \int_{N_{r-1}(\mathbb{R})\backslash\mathrm{GL}_{r-1}(\mathbb{R})} W_1\left[\begin{pmatrix} h & \\ & 1 \end{pmatrix} \right] \overline{W_2\left[\begin{pmatrix} h & \\ & 1 \end{pmatrix} \right]} dh;$$

that is, we have $\langle W_1(.g), W_2(.g) \rangle = \langle W_1, W_2 \rangle$ for $g \in G$.

2.6. Zeta integrals

We review the theory of the GL$(r) \times$ GL(r) Rankin–Selberg integral. We refer to [11] for details. We choose $\phi_0 \in \pi_0$ with a factorisable Whittaker function W_0 and a maximal Eisenstein series Eis(f_s) attached to some vector f_s in the generalised Principal series $\mathcal{I}_{r-1,1}(s)$, as defined in Section 2.4. Let $\pi \in \hat{\mathbb{X}}$ and $\phi \in \pi$ be any automorphic form. One defines the global Rankin–Selberg integral [11, §2.3.2] of ϕ_0 and ϕ by

$$\Psi(f_s, \phi_0, \bar{\phi}) := \int_{G(\mathbb{Q})\backslash G(\mathbb{A})} \phi_0(g)\overline{\phi(g)}\mathrm{Eis}(f_s)(g)dg,$$

where $s \in \mathbb{C}$ is such that Eis(f_s) is regular. As ϕ_0 is cuspidal, the above integral converges absolutely. For s in a right half plane performing a standard unfolding-folding, one gets

$$\Psi(f_s, \phi_0, \bar{\phi}) = \int_{P(\mathbb{Q})\backslash G(\mathbb{A})} \phi_0(g)\overline{\phi(g)}f_s(g)dg.$$

The above integral representation of $\Psi(f_s, \phi_0, \bar{\phi})$ has a meromorphic continuation to all $s \in \mathbb{C}$. It is known that if π and π_0 are cuspidal, then the only possible poles of Ψ are simple and can occur at $\Re(s) = 0, 1$.

We may choose f_s to be factorisable, which can be done by choosing $\Phi \in \mathcal{S}(\mathbb{A}^r)$, as in Section 2.4, to be factorisable. Furthermore, if we assume that π is generic and $\phi \in \pi$ has a factorisable Whittaker function W_ϕ, then for all $s \in \mathbb{C}$, the global zeta integral is *Eulerian* – that is, factors in local zeta integrals

$$\Psi(f_s, \phi, \bar{\phi}) = \Psi_\infty(f_{s,\infty}, W_{0,\infty}, \overline{W_{\phi,\infty}}) \prod_{p<\infty} \Psi_p(f_{s,p}, W_{0,p}, \overline{W_{\phi,p}}),$$

where the local zeta integral Ψ_∞ is defined by

$$\begin{aligned}
\Psi_\infty(f_{s,\infty}, W_{0,\infty}, \overline{W_{\phi,\infty}}) &:= \int_{N(\mathbb{R})\backslash G(\mathbb{R})} W_{0,\infty}(g)\overline{W_{\phi,\infty}(g)}f_{s,\infty}(g)dg \\
&= \int_{N(\mathbb{R})\backslash\mathrm{GL}_r(\mathbb{R})} W_{0,\infty}(g)\overline{W_{\phi,\infty}(g)}\Phi_\infty(e_r g)|\det(g)|^s dg,
\end{aligned}$$

for s being in some right half plane and then can be meromorphically continued to the whole complex plane. Similarly, the nonarchimedean zeta integral Ψ_p is defined by replacing ∞ with p and $\mathbb{R}$ with $\mathbb{Q}_p$.

It is known that if $\pi_{0,p}$ and π_p are unitary and ϑ_0 and ϑ tempered, respectively, then the above integral representation of Ψ_p is valid for $\Re(s) \geq 1/2$ if $\vartheta + \vartheta_0 < 1/2$ and $p \leq \infty$ (this can be seen in the archimedean case from the bounds of the Whittaker functions in Lemma 7.2).

We record the local functional equation satisfied by Ψ_∞. From [11, Theorem 3.2], we have

$$\int_{N(\mathbb{R})\backslash\mathrm{GL}_r(\mathbb{R})} \tilde{W}_{0,\infty}(g)\tilde{W}_{\phi,\infty}(g)\hat{\Phi}_\infty(e_r g)|\det(g)|^{1-s}dg$$

$$= \gamma_\infty(s, \pi_{0,\infty} \otimes \pi_\infty, \psi) \int_{N(\mathbb{R})\backslash\mathrm{GL}_r(\mathbb{R})} W_{0,\infty}(g)W_{\phi,\infty}(g)\Phi_\infty(e_r g)|\det(g)|^s dg.$$

Here $\tilde{W}$ denotes the contragredient Whittaker function of W defined by $\tilde{W}(g) := W(wg^{-t})$, where w is the long Weyl element in $G(\mathbb{R})$ and $\gamma_\infty(.,.,\psi)$ denotes the local archimedean γ-factor. As the additive character ψ is fixed throughout the paper, we drop ψ from the notation of γ_∞. Folding the above integrals over $\mathbb{R}^\times$, we can also rewrite the local functional equation as

$$\Psi_\infty(\hat{f}_{1-s,\infty}, \tilde{W}_{0,\infty}, \overline{\tilde{W}_\infty}) = \gamma_\infty(s, \pi_{0,\infty} \otimes \bar{\pi}_\infty)\Psi_\infty(f_{s,\infty}, W_{0,\infty}, \overline{W_\infty}), \tag{2.4}$$

for any $W_\infty \in \pi_\infty$, and f is related to Φ according to equation (2.2). From the definition of the γ-factors (see [11, p. 120]), one can check that $|\gamma_\infty(1/2, \Pi)| = 1$ if Π is unitary.

2.7. Plancherel formula

We refer to [27, §2.2] for a more detailed discussion of the Plancherel formula.

Recall the automorphic Plancherel measure $d\mu_{\mathrm{aut}}$ on $\hat{\mathbb{X}}$ from Section 2.3. Let $\phi_1, \phi_2 \in C^\infty(\mathbb{X})$ with rapid decay at all cusps. We record a Plancherel formula (i.e., a spectral decomposition) of the inner product between ϕ_1 and ϕ_2,

$$\langle\phi_1, \phi_2\rangle = \int_{\hat{\mathbb{X}}} \sum_{\phi \in \mathcal{B}(\pi)} \langle\phi_1, \phi\rangle\langle\phi, \phi_2\rangle d\mu_{\mathrm{aut}}(\pi), \tag{2.5}$$

where $\mathcal{B}(\pi)$ is an orthonormal basis of π and

$$\langle f_1, f_2\rangle := \int_{\mathbb{X}} f_1(g)\overline{f_2(g)}dg.$$

The identity equation (2.5) is independent of choice of $\mathcal{B}(\pi)$.

Rapid decay properties of ϕ_i imply that all the inner products on the right-hand side of equation (2.5) converge. One can show by the trace class property in $L^2(\mathbb{X})$ of some inverse Laplacian that the right-hand side of equation (2.5) converges absolutely.

2.8. Spectral weights

Let $\pi, \pi_0 \in \hat{\mathbb{X}}_{\mathrm{gen}}$ with $W_{0,\infty} \in \pi_{0,\infty}$ and $f_s \in \mathcal{I}_{r-1,1}(s)$. We define the spectral weight

$$J(f_{s,\infty}W_{0,\infty}, \pi_\infty) := \sum_{W_\infty \in \mathcal{B}(\pi_\infty)} |\Psi_\infty(f_{s,\infty}, W_{0,\infty}, \overline{W_\infty})|^2. \tag{2.6}$$

Here $\mathcal{B}(\pi_\infty)$ is an orthonormal basis of π_∞. The sum in the right-hand side of equation (2.6) is absolutely convergent and is independent of choice of $\mathcal{B}(\pi_\infty)$; see [5, Appendix 4].

The definition of J involves only the archimedean components of the representations and functions. In fact, one can define J for any irreducible generic unitary representation σ of $G(\mathbb{R})$ and $\beta \in C^{\infty}(N(\mathbb{R})\backslash G(\mathbb{R}), \psi_{\infty})$ with sufficient decay at infinity, by

$$J(\beta, \sigma) := \sum_{W \in \mathcal{B}(\sigma)} \left| \int_{N(\mathbb{R})\backslash G(\mathbb{R})} \beta(g)\overline{W(g)}dg \right|^2.$$

Then, using the Whittaker–Plancherel formula (see [33, Chapter 15]), one can obtain that

$$\int_{\widehat{G(\mathbb{R})}} J(\beta, \sigma)d\mu_{\mathrm{loc}}(\sigma) = \int_{N(\mathbb{R})\backslash G(\mathbb{R})} |\beta(g)|^2 dg =: \|\beta\|^2_{L^2(N(\mathbb{R})\backslash G(\mathbb{R}))}. \qquad (2.7)$$

Here $\widehat{G(\mathbb{R})}$ is the tempered unitary dual of $G(\mathbb{R})$ equipped with the local Plancherel measure $d\mu_{\mathrm{loc}}$ compatible with the chosen Haar measure on $G(\mathbb{R})$ (see [6, §4.13.2]).

2.9. Analytic newvectors

Analytic newvectors are certain approximate archimedean analogues of the classical nonarchimedean newvectors pioneered by Casselman [8] and Jacquet–Piatetski-Shapiro–Shalika [20]. Let $K_0(p^N) \subset \mathrm{PGL}_r(\mathbb{Z}_p)$ be the subgroup of matrices whose last rows are congruent to $(0, \ldots, 0, *) \bmod p^N$. Let σ be a generic irreducible representation of $\mathrm{PGL}_r(\mathbb{Q}_p)$, and let N_0 be the minimal nonnegative integer such that σ contains a nonzero vector v that is invariant by $K_0(p^{N_0})$. Let $C(\sigma)$ be the conductor of σ, which can be defined in terms of the local gamma factor attached to σ. Then the main theorem of [8, 20] states that the real number p^{N_0} is equal to $C(\sigma)$. One calls such a v a *newvector* of σ. In [20], the authors call newvectors *essential vectors*; and in some literature, the authors call them *newforms*.

In [19] the authors produce an approximate analogue of this theorem at the archimedean place. Let $X > 1$ be tending to infinity and $\tau > 0$ be sufficiently small but fixed. We define an *approximate congruence subgroup* $K_0(X, \tau) \subseteq \mathrm{PGL}_r(\mathbb{R})$, which is an archimedean analogue of the subgroup $K_0(p^N)$, in the following way: it is the image in $\mathrm{PGL}_r(\mathbb{R})$ of

$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_r(\mathbb{R}) \middle| \begin{array}{l} a \in \mathrm{GL}_{r-1}(\mathbb{R}), \quad |a - 1_{r-1}| < \tau, \quad |b| < \tau, \\ d \in \mathrm{GL}_1(\mathbb{R}), \quad |c| < \dfrac{\tau}{X}, \quad |d - 1| < \tau \end{array} \right\}. \qquad (2.8)$$

Here, $|.|$ denotes an arbitrary fixed norm on the corresponding spaces of matrices. Fix $0 \le \vartheta < 1/2$. Then in [19, Theorem 1], the authors show that for all $\epsilon > 0$, there is a $\tau > 0$ such that for all generic irreducible unitary ϑ-tempered representation π of $\mathrm{PGL}_r(\mathbb{R})$, there is a unit vector $v \in \pi$ such that

$$\|\pi(g)v - v\|_{\pi} < \epsilon \quad \text{for all } g \in K_0(C(\pi), \tau),$$

where $C(\pi)$ is the analytic conductor of π. We call such a vector v an *analytic newvector* of π.

The authors also prove that [19, Theorem 7] any unit vector v in the Kirillov model of π that can be given by a function in $C_c^{\infty}(N_{r-1}(\mathbb{R})\backslash \mathrm{GL}_{r-1}(\mathbb{R}), \psi_{\infty})^{O_{r-1}(\mathbb{R})}$ is a newvector. Moreover, v can be chosen in a way such that if W is the image of v in the corresponding Whittaker model, then also

$$|W(g) - W(1)| < \epsilon$$

for all $g \in K_0(C(\pi), \tau)$ and $W(1) \asymp 1$.

2.10. Main theorem

Theorem 2.1. *Let $r \geq 3$ and X be tending to infinity. Let π_0 be a fixed cuspidal representation in $\hat{\mathbb{X}}$ such that $\pi_{0,\infty}$ is ϑ_0-tempered for some $0 \leq \vartheta_0 < 1/(r^2 + 1)$. We define a weight function*

$$J_X : \hat{\mathbb{X}}_{\text{gen}} \to \mathbb{R}_{\geq 0},$$

as in equation (4.5), which satisfies the following properties:

○ *$J_X(\pi)$ only depends on the archimedean component of π (with an abuse of notation, we write $J_X(\pi) = J_X(\pi_\infty)$).*
○ *If π_∞ is ϑ-tempered such that $\vartheta + \vartheta_0 < 1/2$ and $C(\pi_\infty) < X$, then $J_X(\pi_\infty) \gg_{\pi_0} 1$.*
○ *$\int_{\widehat{G(\mathbb{R})}} J_X(\pi_\infty) d\mu_{\text{loc}}(\pi_\infty) = X^{r-1}$.*

And finally, we have

$$\int_{\hat{\mathbb{X}}_{\text{gen}}} \frac{|L(1/2, \tilde{\pi} \otimes \pi_0)|^2}{\ell(\pi)} J_X(\pi) d\mu_{\text{aut}}(\pi) = X^{r-1}\left(r \frac{\zeta(r/2)^2}{\zeta(r)} L(1, \pi_0, \text{Ad}) \log X + O_{\pi_0}(1)\right),$$

where $\tilde{\pi}$ is the contragredient of π. Here $\ell(\pi)$ is defined as in equation (4.3) and only depends on the nonarchimedean data of π.

If π is cuspidal, then $\ell(\pi) \asymp L(1, \pi, \text{Ad})$ with an absolute implied constant and thus $\ell(\pi) \ll_\epsilon C(\pi)^\epsilon$, which follows from [25].

We note that if $\pi \in \hat{\mathbb{X}}_{\text{gen}}$, then π_∞ is ϑ-tempered for $\vartheta < 1/2 - 1/(r^2 + 1)$, which is a result in [26]. Thus the ϑ_0-temperedness assumption of $\pi_{0,\infty}$ in Theorem 2.1 implies that $J_X(\pi) \gg 1$ for all $\pi \in \hat{\mathbb{X}}_{\text{gen}}$ with $C(\pi) < X$. Moreover, the family

$$\mathcal{F}_X^{\text{gen}} := \{\text{generic automorphic representations } \pi \text{ of } \text{PGL}_r(\mathbb{Z}) \text{ with } C(\pi) < X\}$$

has $\ell(\pi)^{-1}$-weighted cardinality $\asymp X^{r-1}$. This is essentially contained in the proof of [19, Theorem 9]. Hence, J_X can be realised as a smoothened characteristic function of the $\ell(\pi)^{-1}$-weighted family $\mathcal{F}_X^{\text{gen}}$.

Consequently, we have an immediate corollary of Theorem 2.1.

Corollary 2.2. *Let π_0 be as in Theorem 2.1. Then*

$$\sum_{\substack{C(\pi) < X \\ \hat{\mathbb{X}} \ni \pi \text{ cuspidal}}} \frac{|L(1/2, \pi \otimes \pi_0)|^2}{L(1, \pi, \text{Ad})} \ll_{\pi_0} X^{r-1} \log X$$

as X tends to infinity.

This is the sharpest possible (Lindelöf on average) second-moment estimate of the cuspidal Rankin–Selberg central L-values.

3. The Fourier expansion of maximal Eisenstein series

We recall some useful information about the Fourier expansion of maximal Eisenstein series. The computation is essentially done in [24], but we extract the relevant computation for completeness.

Let f_s be a holomorphic section in the generalised principal series $\mathcal{I}_{r-1,1}(s)$ such that f_s is constructed from a Schwartz–Bruhat function $\Phi \in \mathcal{S}(\mathbb{A}^r)$, as described in Section 2.4. Let $\text{Eis}(f_s)$ be the Eisenstein series attached to f_s.

We want to understand the Fourier expansion of $\text{Eis}(f_s)$. It is a straightforward calculation using the Bruhat decomposition. We sketch out the essential details for completeness. Let $\mathfrak{R}(s)$ be sufficiently large. We temporarily allow ψ to be a possibly degenerate character of N. Then

$$\int_{N(\mathbb{Q})\backslash N(\mathbb{A})} \text{Eis}(f_s)(ng)\overline{\psi(n)}dn = \sum_{\gamma \in P(\mathbb{Q})\backslash G(\mathbb{Q})} \int_{N(\mathbb{Q})\backslash N(\mathbb{A})} f_s(\gamma ng)\overline{\psi(n)}dn. \tag{3.1}$$

We start with the Bruhat decomposition of $G(\mathbb{Q})$ with respect to $P(\mathbb{Q})$. The Bruhat cells are indexed by a subset of the Weyl group, namely, the subset of Weyl elements w such that $w\alpha > 0$ for all simple roots α other than α_0 that determines P.

Lemma 3.1. *We define the Weyl elements*

$$w_i := \begin{pmatrix} I_{i-1} & & \\ & & I_{r-i} \\ & 1 & \end{pmatrix}, \quad 1 \leq i \leq r.$$

Also let N_i be the subgroup of N of the form

$$N_i := \left\{ n := \begin{pmatrix} I_{i-1} & & \\ & 1 & x \\ & & I_{r-i} \end{pmatrix} \mid x := (x_1, \ldots, x_{r-i}) \right\}.$$

Then

$$G(\mathbb{Q}) = \bigsqcup_{i=1}^{r} P(\mathbb{Q})w_i N_i(\mathbb{Q}),$$

where the union is disjoint.

Proof. Any $\gamma \in G(\mathbb{Q})$ has the bottom row of the form $(0, \ldots, 0, d, *, \ldots, *)$, where $d \neq 0$ and occurs at the ith position for some $1 \leq i \leq r$. There exists an element $x \in N_i$ such that $\gamma = d\gamma'x$ with γ' having bottom row of the form $(0, \ldots, 0, 1, 0, \ldots, 0)$ with 1 at the ith position. It can readily be checked that $\gamma' w_i^{-1} \in P(\mathbb{Q})$. Clearly, the union is disjoint. $\square$

Using Lemma 3.1 and the left-$P(\mathbb{Q})$ invariance of f_s, we can rewrite the right-hand side of equation (3.1) as

$$\sum_{i=1}^{r} \sum_{\gamma \in N_i(\mathbb{Q})} \int_{N(\mathbb{Q})\backslash N(\mathbb{A})} f_s(w_i \gamma ng)\overline{\psi(n)}dn.$$

Note that $N_i := N \cap w_i^{-1} N^t w_i$. Hence, $N = N_i \bar{N}_i$, where $\bar{N}_i := N \cap w_i^{-1} N w_i$. It can be checked that

$$\bar{N}_i = \{n \in N \mid e_i n = e_i\},$$

where $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ with 1 at the ith place.

We write an element $n \in N$ as $n_1 n_2$ with $n_1 \in N_i$ and $n_2 \in \bar{N}_i$. Unfolding the $N_i(\mathbb{Q})$ sum, we obtain that the right-hand side of equation (3.1) is equal to

$$\sum_{i=1}^{r} \int_{\bar{N}_i(\mathbb{Q})\backslash \bar{N}_i(\mathbb{A})} \overline{\psi(n_2)} \int_{N_i(\mathbb{A})} f_s(w_i n_1 n_2 g)\overline{\psi(n_1)}dn_1 dn_2.$$

There exist $n_2' \in \bar{N}_i$ and $n_1' \in N_i$ such that $n_2'n_1' = n_1n_2$ and and $n_2'' \in N$ such that $n_2''w_i = w_in_2'$. Appealing to the left-$N(\mathbb{A})$ invariance of f_s, we conclude that the above expression is

$$\sum_{i=1}^{r} \int_{\bar{N}_i(\mathbb{Q})\backslash\bar{N}_i(\mathbb{A})} \overline{\psi(n_2)} \int_{N_i(\mathbb{A})} f_s(w_in_1'g)\overline{\psi(n_1)}dn_1 dn_2.$$

We check that if $e_in_1 = (0, 1, x)$ for some $x \in \mathbb{A}^{r-i}$, then $n_1' = (0, 1, xu)$ for some upper triangular unipotent matrix u in $\mathrm{GL}_{r-i}(\mathbb{A})$. Also, $\psi(n_1)$ is equal to $\psi(n_1')$. Thus, making the change of variables $xu \mapsto x$, we obtain that

$$\int_{N(\mathbb{Q})\backslash N(\mathbb{A})} \mathrm{Eis}(f_s)(ng)\overline{\psi(n)}dn = \sum_{i=1}^{r} \int_{\bar{N}_i(\mathbb{Q})\backslash\bar{N}_i(\mathbb{A})} \overline{\psi(n')}dn' \int_{N_i(\mathbb{A})} f_s(w_ing)\overline{\psi(n)}dn. \qquad (3.2)$$

Clearly, if ψ is nondegenerate, the above is zero. In particular, if ψ is of the form $\psi_{\tilde{q}}$ for some $\tilde{q} := (q_j)_j \in \mathbb{Q}^{r-1}$, then the ith summand, for $i < r$, on the right-hand side of equation (3.2) does not identically vanish only if $q_j = 0$ for all $j \neq i$. For $i = r$, the same happens only if $\tilde{q} = 0$, in which case the summand is equal to $f_s(g)$. For $q \in \mathbb{Q}$, we denote $(0, \ldots, 0, q, 0, \ldots, 0)$, where q is at the ith place, by $i(q)$.

We define (again on a right half plane, and extend by meromorphic continuation) twisted intertwining operators on $\mathcal{I}_{r-1,1}(s) \ni f_s$ attached to the Weyl element w_i by

$$M_i^q f_s(g) := \int_{N_i(\mathbb{A})} f_s(wng)\overline{\psi_{i(q)}(n)}dn. \qquad (3.3)$$

Thus we obtain the following Fourier expansion of $\mathrm{Eis}(f_s)(g)$.

Lemma 3.2. *Let f_s and $\mathrm{Eis}(f_s)(g)$ be as above. Then*

$$\mathrm{Eis}(f_s)(g) = f_s(g) + \sum_{i=1}^{r-1} \sum_{q \in \mathbb{Q}} M_i^q f_s(g).$$

The terms f_s and $M_i^0 f_s$ are the constant terms of $\mathrm{Eis}(f_s)(g)$.

Let us write $g \in G(\mathbb{A})$ in its Iwasawa coordinates $g = nak$, where $a := \begin{pmatrix} a(y) & \\ & 1 \end{pmatrix}$. Then for $i < r$, we have

$$M_i^q f_s(g) = \int_{N_i(\mathbb{A})} f_s(w_in'nak)\overline{\psi_{i(q)}(n)}dn',$$

which is defined for $\mathfrak{R}(s)$ large enough and can be meromorphically continued.

We work exactly as before to compute the above integral. We write $n = n_1n_2$ with $n_1 \in N_i$ and $n_2 \in \bar{N}_i$ and make the change of variables $n' \mapsto n'n_1^{-1}$. Then we write $w_in'n_2 = n_2'w_in''$ for some $n_2' \in N$ and n'' that is related to n' as before, and we make the change of variables $n'' \mapsto n'$. We use the left $N(\mathbb{A})$-invariance of f_s and the fact that $\psi(n') = \psi(n'')$.

Finally, we make the change of variables $n' \mapsto an'a^{-1}$ and use the transformation property of f_s as in equation (2.3) to obtain

$$M_i^q f_s(g) = \psi_{i(q)}(n_1) \prod_{j=1}^{i-1} |y_j|^{sj} \prod_{j=i}^{r-1} |y_j|^{(1-s)(r-j)} \int_{N_i(\mathbb{A})} f_s(w_ink)\overline{\psi_{i(q)}(ana^{-1})}dn. \qquad (3.4)$$

We first study the integral on the right-hand side of equation (3.4) for $q = 0$. We use the construction of f_s using $\Phi \in \mathcal{S}(\mathbb{A}^r)$ as in Section 2.4. We also parametrise n so that $e_i n = (0, 1, x)$ with $x \in \mathbb{A}^{r-i}$ and make the change of variables $x \mapsto x/t$ to write the integral as

$$\int_{\mathbb{A}^{r-i}} \int_{\mathbb{A}^\times} (k.\Phi)(0, t, x)|t|^{rs-r+i} d^\times t \, dx.$$

Here $(k.\Phi)(x) := \Phi(xk)$. Using Tate's functional equation (see [7, Proposition 3.1.6]), we can rewrite the above as

$$\int_{\mathbb{A}^\times} \widehat{(k.\Phi)}^i (te_i)|t|^{r-i+1-rs} d^\times t,$$

where the partial Fourier transform $\hat{\Phi}^i$ is defined by

$$\hat{\Phi}^i(x_1, \ldots, x_r) := \int_{\mathbb{A}^{r-i+1}} \Phi(x_1, \ldots, x_{i-1}, u_1, \ldots, u_{r-i+1})\psi(x_i u_1 + \cdots + x_r u_{r-i+1}) du.$$

In particular, it can be seen that

$$M_1 f_s(g) = f_{1-s, \hat{\Phi}}(wg^{-t}) =: \tilde{f}_s(g), \tag{3.5}$$

where w is the long Weyl element. It can be checked that $\tilde{f}$ lies in the principal series $\mathcal{I}_{1,r-1}(1-s)$ arising from the associate parabolic $\tilde{P}$.

Now, for $q \neq 0$, the integral on the right-hand side of equation (3.4) gives rise to a degenerate Whittaker function. Parametrising $n \in N_i(\mathbb{A})$ as in Lemma 3.1, one can see that $\psi_{i(q)}(ana^{-1}) = \psi_0(qy_i x_1)$: that is, the value $\psi_{i(q)}(ana^{-1})$ depends on a only through y_i. We define

$$W^i_{f_s}(qy_i, k) := \int_{N_i(\mathbb{A})} f_s(w_i nk)\overline{\psi_{i(q)}(ana^{-1})} dn.$$

Again, the above is defined for $\Re(s)$ sufficiently large and can be extended analytically to all of $\mathbb{C}$, and it can be shown that $W^i_{f_s}(t, k)$ decays rapidly as $t \to \infty$. We prove these claims in Lemma 6.3 (although these results are implicitly done in [17]). In particular, we have

$$M^q_i f_s(g) = W^i_{f_s}(qy_i, k)\psi_0(qx_{i,i+1})$$

for $q \neq 0$.

We summarise the above results and rewrite Lemma 3.2 in the following proposition to record the Fourier expansion of a maximal Eisenstein series.

Proposition 3.3. *Let $f_s \in \mathcal{I}_{r-1,1}(s)$ be a holomorphic section and $\mathrm{Eis}(f_s)(g)$ be the corresponding maximal Eisenstein series. Let $g = n(x)\begin{pmatrix} a(y) & \\ & 1 \end{pmatrix} k$ be its Iwasawa decomposition. Then*

$$\mathrm{Eis}(f_s)(g) = f_s(g) + \sum_{i=1}^{r-1} \prod_{j=1}^{i-1} |y_j|^{sj} \prod_{j=i}^{r-1} |y_j|^{(1-s)(r-j)} \left[M^0_i f_s(k) + \sum_{q \in \mathbb{Q}^\times} W^i_{f_s}(qy_i, k)\psi_0(qx_{i,i+1}) \right].$$

The terms containing M^0_i are the constant terms of $\mathrm{Eis}(f_s)(g)$, and the terms containing $W^i_{f_s}$ are holomorphic in s. The above sum converges absolutely and uniformly on compacta.

4. Proof of the main theorem

4.1. Choices of the local components

We start by choosing various vectors and auxiliary test functions. Let $\pi_0 \in \hat{\mathbb{X}}_{\mathrm{gen}}$ be the fixed cuspidal representation as in Theorem 2.1. Let $\phi_0 \in \pi_0$ with Whittaker function $W_0 = \bigotimes_{p \leq \infty} W_{0,p}$, such that

$$W_{0,p} \text{ are unramified for } p < \infty \text{ with } W_{0,p}(1) = 1.$$

Here and elsewhere in the paper, the index set $\{p \leq \infty\}$ denotes the set of places $\{\infty\} \cup \{p \text{ prime in } \mathbb{Z}\}$.
   We choose $W_{0,\infty} \in \pi_{0,\infty}$ so that

$$\|W_{0,\infty}\|_{\pi_{0,\infty}} = 1, \text{ and } W_{0,\infty}\left[\begin{pmatrix} \cdot \\ & 1 \end{pmatrix}\right] \in C_c^\infty(N_{r-1}(\mathbb{R})\backslash\mathrm{GL}_{r-1}(\mathbb{R}), \psi_\infty)^{O_{r-1}(\mathbb{R})},$$

whose existence is guaranteed by the theory of Kirillov model. We choose $\mathcal{S}(\mathbb{A}^r) \ni \Phi = \bigotimes_{p \leq \infty} \Phi_p$ with

$$\Phi_p := \mathrm{char}(\mathbb{Z}_p^r) = \hat{\Phi}_p \text{ for } p < \infty,$$

and for $\tau > 0$ sufficiently small but fixed

$$\Phi_\infty \in C_c^\infty(B_\tau(0, \ldots, 0, 1)),$$

such that Φ_∞ is nonnegative and has values sufficiently concentrated near 1. Here B_τ denotes the ball of radius τ. Thus Φ_∞ can be thought as a smoothened characteristic function of $B_\tau(0, \ldots, 0, 1)$.
   Let $f_s := f_{s,\Phi} \in \mathcal{I}_{r-1,1}(s)$ be associated to Φ according to equation (2.2). The choice of Φ_∞ ensures that there exist sufficiently small $\tau_1, \tau_2 > 0$ (depending on τ and Φ_∞) such that $f_{1/2,\infty}\left[\begin{pmatrix} I_{r-1} \\ c & 1 \end{pmatrix}\right]$ is supported on $|c| \leq \tau_1$ and has values $\asymp 1$ for $|c| \leq \tau_2$. This implies that

$$\int_{\mathbb{R}^{r-1}} |f_{1/2,\infty}|^2\left[\begin{pmatrix} I_{r-1} \\ c & 1 \end{pmatrix}\right] dc \asymp 1,$$

with an absolute implied constant. We renormalise Φ_∞ by an absolute constant so that the above integral is 1.

4.2. Computation of the spectral side

Let $\mathrm{Eis}(f_s) := \mathrm{Eis}(f_{s,\Phi})$ be the maximal Eisenstein series attached to $f_{s,\Phi} \in \mathcal{I}_{r-1,1}(s)$, which is defined in Section 4.1. Let $X > 1$ be a large number tending to infinity and

$$A(\mathbb{A}) \ni x := (x_p)_p, \quad x_\infty := \mathrm{diag}(X, \ldots, X, 1) \in A(\mathbb{R}) \text{ and } x_p = 1 \text{ for all } p < \infty.$$

Our point of departure is the following period, which we write in two different ways:

$$\int_{\mathbb{X}} |\phi_0(g)|^2 |\mathrm{Eis}(f_s)(gx)|^2 dg = \langle \phi_0 \mathrm{Eis}(f_s)(.x), \phi_0 \mathrm{Eis}(f_s)(.x)\rangle. \tag{4.1}$$

We use the Parseval relation in equation (2.5) and the notations in Section 2.6 to write the right-hand side of equation (4.1) as

$$\int_{\hat{\mathbb{X}}} \sum_{\phi \in \mathcal{B}(\pi)} |\Psi(f_s(.x), \phi_0, \bar{\phi})|^2 d\mu_{\mathrm{aut}}(\pi), \tag{4.2}$$

where $\mathcal{B}(\pi)$ is an orthonormal basis of π.

Lemma 4.1. *Let $\pi \in \hat{\mathbb{X}} \setminus \hat{\mathbb{X}}_{\text{gen}}$ be a nongeneric representation. Then $\Psi(f_s, \phi_0, \bar{\phi}) = 0$ for all $\phi \in \pi$ and $s \in \mathbb{C}$.*

Proof. For $\mathfrak{R}(s)$ sufficiently large, we have (see Section 2.6)

$$\Psi(f_s, \phi_0, \bar{\phi}) = \int_{P(\mathbb{Q}) \backslash G(\mathbb{A})} \phi_0(g) \overline{\phi(g)} f_s(g) dg.$$

We follow the computation of [11, p.104–105]. We use the Fourier expansion

$$\phi_0(g) = \sum_{\gamma \in N(\mathbb{Q}) \backslash P(\mathbb{Q})} W_0(\gamma g)$$

and the left $P(\mathbb{Q})$-invariance of f_s and unfold over $P(\mathbb{Q})$ to get

$$\Psi(f_s, \phi_0, \bar{\phi}) = \int_{N(\mathbb{Q}) \backslash G(\mathbb{A})} W_0(g) \overline{\phi(g)} f_s(g) dg.$$

We fold the last integral over $N(\mathbb{A})$ and use the left N-equivariance of W_0 and the left N-invariance of f_s to obtain

$$\Psi(f_s, \phi_0, \bar{\phi}) = \int_{N(\mathbb{A}) \backslash G(\mathbb{A})} W_0(g) f_s(g) \int_{N(\mathbb{Q}) \backslash N(\mathbb{A})} \overline{\phi(ng)} \psi(n) dn dg.$$

By definition, the inner integral vanishes as ϕ is nongeneric. Finally, by analytic continuation of Ψ, we extend the result for all $s \in \mathbb{C}$. □

Thus Lemma 4.1 allows us to reduce the integral in equation (4.2) only over $\hat{\mathbb{X}}_{\text{gen}}$. Once we restrict to $\pi \in \hat{\mathbb{X}}_{\text{gen}}$, we can use the Eulerian property of the zeta integral Ψ as in Section 2.6. If $\phi \in \pi$ with $\|\phi\|_\pi = 1$ and Whittaker function $W_\phi = \bigotimes_{p \le \infty} W_p$ such that W_p is unramified and $W_p(1) = 1$ for $p < \infty$, then by Schur's lemma, we have

$$\|\phi\|_\pi^2 = \ell(\pi) \|W_\infty\|_{\pi_\infty}^2, \tag{4.3}$$

where $\ell(\pi)$ only depends on the nonarchimedean data of π. A standard Rankin–Selberg computation[2] yields that $\ell(\pi) \asymp L(1, \pi, \text{Ad})$ for a cuspidal π. In fact, in our case, $\ell(\pi)$ is equal to $L(1, \pi, \text{Ad})$ up to a positive constant dependent only on n.

Another standard computation [11, Theorem 3.3] shows that

$$\Psi_p(f_{s,p}, W_{0,p}, \overline{W_p}) = L_p(s, \pi_0 \otimes \bar{\pi}), \quad p < \infty, \tag{4.4}$$

where f_s is as chosen in Section 4.1 and $L_p(s, .)$ denotes the unramified p-adic Euler factor of $L(s, .)$. Thus by meromorphic continuation, we have

$$\Psi(f_s, \phi_0, \bar{\phi}) = L(s, \pi_0 \otimes \bar{\pi}) \Psi_\infty(f_{s,\infty}, W_{0,\infty}, \overline{W_\infty})$$

for all $s \in \mathbb{C}$ whenever both sides of the above are defined. Using the equation above, and recalling the harmonic weight in equation (4.3) and the spectral weight in equation (2.6), we obtain that equation (4.2) is equal to

$$\int_{\hat{\mathbb{X}}_{\text{gen}}} \frac{|L(s, \pi_0 \otimes \bar{\pi})|^2}{\ell(\pi)} J(f_{s,\infty}(.x_\infty) W_{0,\infty}, \pi_\infty) d\mu_{\text{aut}}(\pi).$$

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[2]See [1, eq. (3.11)] and the computation there for the spherical case; the general case follows similarly.

We appeal to the holomorphicity of the zeta integrals to specify $s = 1/2$ and define a normalised spectral weight $J_X(\pi_\infty)$ by

$$J_X(\pi) := J_X(\pi_\infty) := X^{r-1} J(f_{1/2,\infty}(.x_\infty)W_{0,\infty}, \pi_\infty). \tag{4.5}$$

Thus we write the main equation of our proof:

$$\int_{\hat{\mathbb{X}}_{\text{gen}}} \frac{|L(1/2, \pi_0 \otimes \bar{\pi})|^2}{\ell(\pi)} J_X(\pi) d\mu_{\text{aut}}(\pi) = X^{r-1} \langle |\phi_0|^2, |\text{Eis}(f_{1/2})(.x)|^2 \rangle. \tag{4.6}$$

4.3. Computation of the period side

Again recall the choices of the local factors in Section 4.1. We write $f_s = \bigotimes_{p \leq \infty} f_{s,p}$; then for $k_p \in K_p$,

$$f_{s,p}(k_p) = \int_{\mathbb{Q}_p^\times} \Phi_p(te_r k_p)|\det(tk_p)|^s d^\times t.$$

Here we fix Haar measures $d^\times t$ on $\mathbb{Q}_p^\times$ and (respectively, dt on $\mathbb{Q}_p$) such that $\text{vol}(\mathbb{Z}_p^\times) = 1$ (respectively, $\text{vol}(\mathbb{Z}_p) = 1$).

First, we record that $f_{s,p}$ is an unramified vector in $\mathcal{I}_{r-1,1}(s)_p$ and $\tilde{f}_{s,p}$ is an unramified vector in $\mathcal{I}_{1,r-1}(1-s)_p$, which is a generalised principal series attached to the opposite parabolic of P. Note that $te_r k_p \in \mathbb{Z}_p^r$ if and only if $t \in \mathbb{Z}_p$. Thus

$$f_{s,p}(k_p) = \sum_{m=0}^\infty p^{-mrs} = (1 - p^{-rs})^{-1},$$

for $\mathfrak{R}(s) > 0$. Similarly, using equation (3.5), we have

$$\tilde{f}_{s,p}(k_p) = f_{1-s,\hat{\Phi}_p}(wk_p^{-t}) = (1 - p^{-r(1-s)})^{-1}$$

for $\mathfrak{R}(s) < 1$. Thus for g in the fundamental domain of $\mathbb{X}$, we can write

$$f_s(g) = \zeta(rs)f_{s,\infty}(g_\infty) \tag{4.7}$$

and

$$\tilde{f}_s(g) = \zeta(r - rs)\tilde{f}_{s,\infty}(g_\infty), \tag{4.8}$$

for all $s \in \mathbb{C}$, which can be achieved by meromorphic continuation.

Let $\mathfrak{R}(s)$ be sufficiently small. From Proposition 3.3, we can see that among the constant terms of $\text{Eis}(f_{1/2+s})(g)$, the terms that do not lie in $L^2(\mathbb{X})$ are $f_{1/2+s}(g)$ and $\tilde{f}_{1/2+s}(g)$. Similarly, one checks that the constant terms of

$$\overline{\text{Eis}(f_{1/2})}\text{Eis}(f_{1/2+s}) - \overline{f_{1/2}}f_{1/2+s} - \overline{\tilde{f}_{1/2}}\tilde{f}_{1/2+s}$$

are integrable in $L^2(\mathbb{X})$. Inspired by this, we define a regularised Eisenstein series of the form

$$\tilde{E}_s := \overline{\text{Eis}(f_{1/2})}\text{Eis}(f_{1/2+s}) - \text{Eis}(\overline{f_{1/2}}f_{1/2+s}) - \text{Eis}(\overline{\tilde{f}_{1/2}}\tilde{f}_{1/2+s}). \tag{4.9}$$

Proof of Theorem 2.1. Recall equations (4.6) and (4.9). We write the inner product on the right-hand side of equation (4.6) as

$$\lim_{s \to 0} \langle |\phi_0|^2, \tilde{E}_s(.x) \rangle + \lim_{s \to 0} \left[\langle |\phi_0|^2, \operatorname{Eis}(\overline{f_{1/2}} f_{1/2+s})(.x) \rangle + \langle |\phi_0|^2, \operatorname{Eis}(\overline{\tilde{f}_{1/2}} \tilde{f}_{1/2+s})(.x) \rangle \right].$$

The second term is the degenerate term as in equation (5.1). From equation (5.4), Proposition 5.2 and Lemma 5.1, we obtain that the second term above is

$$rL(1, \pi_0, \operatorname{Ad}) \frac{\zeta(r/2)^2}{\zeta(r)} \log X + O_{\pi_0}(1).$$

On the other hand, we write the first term above, which is the regularised term, as

$$\lim_{s \to 0} \int_{\mathbb{X}} |\phi_0|^2 (gx^{-1}) \overline{\tilde{E}_s(g)} dg,$$

and bound this by

$$\| \phi_0 \|^2_{L^\infty(\mathbb{X})} \int_{\mathbb{X}} |\tilde{E}_s(g)| dg.$$

From Proposition 6.1, we know that the last integral is convergent for s being sufficiently small, and $\tilde{E}_s$ is holomorphic in a sufficiently small neighbourhood of $s = 0$. Thus, using Cauchy's residue theorem, we can write the above limit as

$$\int_{|s|=\epsilon} \frac{1}{s} \int_{\mathbb{X}} |\phi_0|^2 (gx^{-1}) \overline{\tilde{E}_s(g)} dg \frac{ds}{2\pi i}$$

for some arbitrary small but fixed $\epsilon > 0$. Applying Proposition 6.1 once again, we confirm the above integral is $O_{\phi_0, \epsilon}(1)$.

Now nonnegativity and the first property of the spectral weight $J_X(\pi)$ follow from the definition in equation (4.5). The second property follows from Proposition 7.1. Finally, the third property follows from equation (2.7) and Lemma 5.1. □

5. Analysis of the degenerate terms in the period side

In this section, we analyse the degenerate terms

$$\lim_{s \to 0} \left[\langle |\phi_0|^2, \operatorname{Eis}(\overline{f_{1/2}} f_{1/2+s})(.x) \rangle + \langle |\phi_0|^2, \operatorname{Eis}(\overline{\tilde{f}_{1/2}} \tilde{f}_{1/2+s})(.x) \rangle \right]. \tag{5.1}$$

Note that $\overline{f_{1/2}} f_{1/2+s} \in \mathcal{I}_{r-1,1}(1+s)$ is such that its local component $\overline{f_{1/2,p}} f_{1/2+s,p}$ is unramified for $p < \infty$. Thus by the uniqueness of the spherical vector, $\overline{f_{1/2,p}} f_{1/2+s,p} \in \mathcal{I}_{r-1,1}(1+s)_p$ is a multiple of the unramified vector

$$g \mapsto \int_{\mathbb{Q}_p^\times} \Phi_p(te_r g) |\det(tg)|^{1+s} d^\times t.$$

Comparing the values of the functions at the identity as before, we check that the multiple is

$$\frac{(1 - p^{-r/2})^{-1}(1 - p^{-r/2-rs})^{-1}}{(1 - p^{-r-rs})^{-1}} \quad \text{for } p < \infty.$$

We compute the first term inside the limit in equation (5.1) for $\mathfrak{R}(s)$ large. Doing a similar computation as in Section 2.6 and using equation (4.4), we obtain

$$\langle |\phi_0|^2, \mathrm{Eis}(\overline{\tilde{f}_{1/2} f_{1/2+s}})(.x)\rangle = \frac{\zeta(r/2)\zeta(r/2+rs)}{\zeta(r+rs)} L(1+s, \pi_0 \otimes \tilde{\pi}_0)$$
$$\Psi_\infty(\overline{f_{1/2,\infty}} f_{1/2+s,\infty}(.x_\infty), W_{0,\infty}, \overline{W_{0,\infty}}). \quad (5.2)$$

Finally, we meromorphically continue the above to the whole complex plane.

Similarly, we compute the second term inside the limit in equation (5.1). Note that in this case, $\tilde{f}$ lies in $\mathcal{I}_{1,r-1}$ associated to the parabolic $\tilde{P}$. Working as in Section 2.6, we obtain that

$$\langle |\phi_0|^2, \mathrm{Eis}(\overline{\tilde{f}_{1/2}} \tilde{f}_{1/2+s})(.x)\rangle = \int_{\tilde{P}(\mathbb{Q})\backslash G(\mathbb{A})} |\phi_0|^2(g)\overline{\tilde{f}_{1/2}} \tilde{f}_{1/2+s}(gx)dg.$$

We recall the definition of $\tilde{f}$ in equation (3.5) and make the change of variables $g \mapsto wg^{-t}$ to obtain that the above is equal to

$$\int_{P(\mathbb{Q})\backslash G(\mathbb{A})} |\tilde{\phi}_0|^2(g)\overline{\hat{f}_{1/2}} \hat{f}_{1/2-s}(gx^{-1})dg,$$

where $\tilde{\phi}_0(g) := \phi_0(wg^{-t})$, which lies in the contragredient representation $\tilde{\pi}_0$. Note that $\hat{\Phi}_p = \Phi_p$ for $p < \infty$. Thus, doing a calculation similar to that preceding equation (5.2), we obtain

$$\langle |\phi_0|^2, \mathrm{Eis}(\overline{\tilde{f}_{1/2}} \tilde{f}_{1/2+s})(.x)\rangle = \frac{\zeta(r/2)\zeta(r/2-rs)}{\zeta(r-rs)} L(1-s, \pi_0 \otimes \tilde{\pi}_0)$$
$$\Psi_\infty(\overline{\hat{f}_{1/2,\infty}} \hat{f}_{1/2-s,\infty}(.x_\infty^{-1}), \tilde{W}_{0,\infty}, \overline{\tilde{W}_{0,\infty}}). \quad (5.3)$$

Recalling the definition of the contragredient $\tilde{W}_0$ and making the change of variables $g_\infty \to wg_\infty^{-t}$ in the definition of the zeta integral Ψ_∞, we also have

$$\Psi_\infty(\overline{\hat{f}_{1/2,\infty}} \hat{f}_{1/2-s,\infty}(.x_\infty^{-1}), \tilde{W}_{0,\infty}, \overline{\tilde{W}_{0,\infty}}) = \Psi_\infty(\overline{\tilde{f}_{1/2,\infty}} \tilde{f}_{1/2+s,\infty}(.x_\infty), W_{0,\infty}, \overline{W_{0,\infty}}).$$

In the following Lemma 5.1, we prove the archimedean factors Ψ_∞ on the right-hand side of equations (5.2) and (5.3) are equal for $s = 0$. We first record that

$$\Psi_\infty(|h|^2, W_{0,\infty}, \overline{W_{0,\infty}}) = \|hW_{0,\infty}\|^2_{L^2(N(\mathbb{R})\backslash G(\mathbb{R}))},$$

where h is either $f_{1/2,\infty}(.x_\infty)$ or $\tilde{f}_{1/2,\infty}(.x_\infty)$.

Lemma 5.1. *Recall the choices of the local components in Section 4.1. We have*

$$\|W_{0,\infty} f_{1/2,\infty}(.x_\infty)\|^2 = \|W_{0,\infty} \tilde{f}_{1/2,\infty}(.x_\infty)\|^2 = \|W_{0,\infty} f_{1/2,\infty}\|^2 = 1,$$

where all the norms are taken in $L^2(N(\mathbb{R})\backslash G(\mathbb{R}))$.

Proof. To ease the notations, we drop ∞ from the subscripts in this proof.

First recall that

$$\tilde{f}_{1/2}(g) = \hat{f}_{1/2}(wg^{-t}),$$

which implies, by a change of variable $g \mapsto wg^{-t}$, that

$$\|W_0 \tilde{f}_{1/2}(.x)\|^2 = \int_{N(\mathbb{R})\backslash G(\mathbb{R})} |W_0(g)|^2 |\hat{f}_{1/2}(wg^{-t}x^{-1})|^2 dg = \int_{N(\mathbb{R})\backslash G(\mathbb{R})} |\tilde{W}_0(g)|^2 |\hat{f}_{1/2}(gx^{-1})|^2 dg.$$

We make the change of variables $g \mapsto gx$ and then employ the Whittaker–Plancherel formula as in equation (2.7) to write the above as

$$\|\tilde{\pi}_0(x)\tilde{W}_0 \hat{f}_{1/2}\|^2 = \int_{\widehat{G(\mathbb{R})}} \sum_{W \in \mathcal{B}(\sigma)} |\Psi(\hat{f}_{1/2}, \tilde{\pi}_0(x)\tilde{W}_0, W)|^2 d\mu_{\text{loc}}(\sigma).$$

We use the $\text{GL}(r) \times \text{GL}(r)$ local functional equation as in equation (2.4) and the unitarity of the gamma factor at $1/2$ to obtain that

$$|\Psi(\hat{f}_{1/2}, \tilde{\pi}_0(x)\tilde{W}_0, W)|^2 = |\Psi(f_{1/2}, \pi_0(x^{-1})W_0, \tilde{W})|^2.$$

Consequently, applying the Whittaker–Plancherel again with the orthonormal basis $\mathcal{B}(\sigma) := \{\sigma(x)\tilde{W}\}$, we obtain

$$\|W_0 \tilde{f}_{1/2}(.x)\|^2 = \|\tilde{\pi}_0(x)\tilde{W}_0 \hat{f}_{1/2}\|^2 = \|\pi_0(x^{-1})W_0 f_{1/2}\|^2 = \|W_0 f_{1/2}(.x)\|^2,$$

which proves the first equality.

Thus now it is enough to prove that

$$\|W_0 f_{1/2}\|^2 = \|W_0 f_{1/2}(.x)\|^2.$$

We use Bruhat coordinates to write

$$\int_{N(\mathbb{R})\backslash G(\mathbb{R})} |W_0|^2(g) |f_{1/2}|^2 \left[g \begin{pmatrix} X\mathrm{I}_{r-1} & \\ & 1 \end{pmatrix} \right] dg$$
$$= \int_{N_{r-1}(\mathbb{R})\backslash \text{GL}_{r-1}(\mathbb{R})} \int_{\mathbb{R}^{r-1}} |W_0|^2 \left[\begin{pmatrix} h & \\ c & 1 \end{pmatrix} \right] |f_{1/2}|^2 \left[\begin{pmatrix} hX & \\ cX & 1 \end{pmatrix} \right] dc \frac{dh}{|\det(h)|}.$$

Using the transformation property of $f_{1/2}$ as in equation (2.3) and making the change of variables $c \mapsto c/X$, we obtain that the above is equal to

$$\int_{\mathbb{R}^{r-1}} |f_{1/2}|^2 \left[\begin{pmatrix} \mathrm{I}_{r-1} & \\ c & 1 \end{pmatrix} \right] \int_{N_{r-1}(\mathbb{R})\backslash \text{GL}_{r-1}(\mathbb{R})} |W_0|^2 \left[\begin{pmatrix} h & \\ & 1 \end{pmatrix} \begin{pmatrix} \mathrm{I}_{r-1} & \\ c/X & 1 \end{pmatrix} \right] dh dc.$$

Using the G-invariance of the inner product in the Whittaker model as in Section 2.5, we conclude that the inner integral above is equal to

$$\int_{N_{r-1}(\mathbb{R})\backslash \text{GL}_{r-1}(\mathbb{R})} |W_0|^2 \left[\begin{pmatrix} h & \\ & 1 \end{pmatrix} \right] dh = \int_{N_{r-1}(\mathbb{R})\backslash \text{GL}_{r-1}(\mathbb{R})} |W_0|^2 \left[\begin{pmatrix} h & \\ & 1 \end{pmatrix} \begin{pmatrix} \mathrm{I}_{r-1} & \\ c & 1 \end{pmatrix} \right] dh.$$

Thus, reverse engineering the above manipulation with Bruhat coordinates (that is, taking $X = 1$), we conclude the proof of the first two equalities.

From the above proof, we also obtain that

$$\|W_0 f_{1/2}\|^2 = \|W_0\|_{\pi_0}^2 \int_{\mathbb{R}^{r-1}} |f_{1/2}|^2 \left[\begin{pmatrix} \mathrm{I}_{r-1} & \\ c & 1 \end{pmatrix} \right] dc.$$

We deduce the last equality recalling the normalisations of W_0 and $f_{1/2}$. □

It is known that (see [11, Theorem 4.2]) if π_0 is cuspidal, then $L(s, \pi_0 \otimes \tilde{\pi}_0)$ has a simple pole at $s = 1$ with residue $L(1, \pi_0, \mathrm{Ad})$. Let us write

$$L(1 + s, \pi_0 \otimes \tilde{\pi}_0) = \frac{L(1, \pi_0, \mathrm{Ad})}{s} + O_{\pi_0}(1),$$

as $s \to 0$. Thus, using equations (5.2) and (5.3) and Lemma 5.1, we can evaluate the limit in equation (5.1) as

$$\lim_{s \to 0}\left[\langle|\phi_0|^2, \mathrm{Eis}(\overline{f_{1/2}}f_{1/2+s})(.x)\rangle + \langle|\phi_0|^2, \mathrm{Eis}(\overline{\tilde{f}_{1/2}}\tilde{f}_{1/2+s})(.x)\rangle\right] =$$

$$L(1, \pi_0, \mathrm{Ad})\frac{\zeta(r/2)^2}{\zeta(r)}\Psi'(f_{1/2,\infty}, W_{0,\infty}) + O_{\pi_0,\Phi}(1), \quad (5.4)$$

where $\Psi'(f_{1/2,\infty}, W_{0,\infty})$ is defined as

$$\partial_{s=0}\left(\Psi_\infty(\overline{f_{1/2,\infty}}f_{1/2+s,\infty}(.x_\infty), W_{0,\infty}, \overline{W_{0,\infty}}) - \Psi_\infty(\overline{\tilde{f}_{1/2,\infty}}\tilde{f}_{1/2+s,\infty}(.x_\infty), W_{0,\infty}, \overline{W_{0,\infty}})\right).$$

Here and elsewhere in the paper, we write $\partial_{s=0}$ as an abbreviation of $\frac{\partial}{\partial s}\big|_{s=0}$.

Proposition 5.2. *We have*

$$\Psi'(f_{1/2,\infty}, W_{0,\infty}) = r \log X + O_{W_{0,\infty},\Phi_\infty}(1)$$

as X tends to infinity.

Proposition 5.2 follows immediately from the following Lemma 5.3, Lemma 5.5 and Lemma 5.1. Again, to ease the notations, we drop ∞ subscripts from the proofs of the next two lemmata.

Lemma 5.3. *We have*

$$\partial_{s=0}\Psi_\infty(\overline{\tilde{f}_{1/2,\infty}}\tilde{f}_{1/2+s,\infty}(.x_\infty), W_{0,\infty}, \overline{W_{0,\infty}}) = -\log X\|W_{0,\infty}f_{1/2,\infty}\|^2 + O_{W_{0,\infty},\Phi_\infty}(1)$$

as X tends to infinity.

Proof. We start by make the change of variables $g \mapsto wg^{-t}$ in the zeta integral to write

$$\Psi(\overline{\tilde{f}_{1/2}}\tilde{f}_{1/2+s}(.x), W_0, \overline{W_0}) = \Psi(\overline{\hat{f}_{1/2}}\hat{f}_{1/2-s}(.x^{-1}), \tilde{W}_0, \overline{\tilde{W}_0}).$$

We use Bruhat coordinates as in the proof of Lemma 5.1 to write the above zeta integral as

$$\int_{N_{r-1}(\mathbb{R})\backslash \mathrm{GL}_{r-1}(\mathbb{R})}\int_{\mathbb{R}^{r-1}}|\tilde{W}_0|^2\left[\begin{pmatrix}h & \\ c & 1\end{pmatrix}\right]\overline{\hat{f}_{1/2}}\hat{f}_{1/2-s}\left[\begin{pmatrix}h/X & \\ c/X & 1\end{pmatrix}\right]dc\frac{dh}{|\det(h)|}.$$

Again, as in the proof of Lemma 5.1, we use the transformation property of $\overline{\hat{f}_{1/2}}\hat{f}_{1/2-s}$ as in equation (2.3) and make the change of variables $c \mapsto cX$ to obtain that the above is equal to

$$X^{(r-1)s}\int_{\mathbb{R}^{r-1}}\overline{\hat{f}_{1/2}}\hat{f}_{1/2-s}\left[\begin{pmatrix}\mathrm{I}_{r-1} & \\ c & 1\end{pmatrix}\right]\int_{N_{r-1}(\mathbb{R})\backslash \mathrm{GL}_{r-1}(\mathbb{R})}|\tilde{W}_0|^2\left[\begin{pmatrix}h & \\ cX & 1\end{pmatrix}\right]|\det(h)|^{-s}dhdc.$$

Differentiating at $s = 0$, we obtain that the above is equal to

$$(r-1)\log X \int_{\mathbb{R}^{r-1}} |\hat{f}_{1/2}|^2 \left[\begin{pmatrix} I_{r-1} & \\ c & 1 \end{pmatrix}\right] \int_{N_{r-1}(\mathbb{R})\backslash \mathrm{GL}_{r-1}(\mathbb{R})} |\tilde{W}_0|^2 \left[\begin{pmatrix} h & \\ cX & 1 \end{pmatrix}\right] dh\, dc$$

$$+ \int_{\mathbb{R}^{r-1}} \overline{\hat{f}_{1/2}} \partial_{s=0}\hat{f}_{1/2-s}\left[\begin{pmatrix} I_{r-1} & \\ c & 1 \end{pmatrix}\right] \int_{N_{r-1}(\mathbb{R})\backslash \mathrm{GL}_{r-1}(\mathbb{R})} |\tilde{W}_0|^2 \left[\begin{pmatrix} h & \\ cX & 1 \end{pmatrix}\right] dh\, dc$$

$$- \int_{\mathbb{R}^{r-1}} |\hat{f}_{1/2}|^2 \left[\begin{pmatrix} I_{r-1} & \\ c & 1 \end{pmatrix}\right] \int_{N_{r-1}(\mathbb{R})\backslash \mathrm{GL}_{r-1}(\mathbb{R})} |\tilde{W}_0|^2 \left[\begin{pmatrix} h & \\ cX & 1 \end{pmatrix}\right] \log|\det(h)|\, dh\, dc. \qquad (5.5)$$

The first summand in equation (5.5) is easy to understand. Using the invariance of the unitary product exactly as in the proof of Lemma 5.1, we can yield that the first summand is equal to

$$(r-1)\log X \|\tilde{W}_0\|_{\tilde{\pi}_0} \|f_{1/2}\|^2.$$

From the Whittaker–Plancherel expansion in equation (2.7), the $\mathrm{GL}(r) \times \mathrm{GL}(r-1)$ local functional equation (see [11, Proposition 3.2]), and the unitarity of the γ-factor, as in the proof of Lemma 5.1, one also gets that $\|\tilde{W}_0\|_{\tilde{\pi}_0} = \|W_0\|_{\pi_0}$.

We claim that the second summand in equation (5.5) is of bounded size. Note that again the invariance of the unitary inner product implies that the inner integral is equal to $\|\tilde{W}_0\|_{\tilde{\pi}_0}^2$. Thus, using Cauchy's integral formula, we can write the second summand as

$$\|\tilde{W}_0\|_{\tilde{\pi}_0}^2 \int_{|s|=\epsilon} \frac{1}{s^2} \int_{\mathbb{R}^{r-1}} \overline{\hat{f}_{1/2}}\hat{f}_{1/2-s}\left[\begin{pmatrix} I_{r-1} & \\ c & 1 \end{pmatrix}\right] dc\, \frac{ds}{\pi i}$$

for some sufficiently small $\epsilon > 0$. To show that the above integrals converge, we start with the Iwasawa decomposition of $\begin{pmatrix} I_{r-1} & \\ c & 1 \end{pmatrix}$. One can check by induction or otherwise that there exists a $\tilde{z}(c) \in \mathbb{R}^\times$ so that

$$\begin{pmatrix} I_{r-1} & \\ c & 1 \end{pmatrix} = \tilde{z}(c)\tilde{n}(c)\begin{pmatrix} \tilde{a}(c) & \\ & 1 \end{pmatrix}\tilde{k}(c); \quad \tilde{n}(c) \in N(\mathbb{R}),\ \tilde{k}(c) \in K_\infty,$$

$$\tilde{a}(c) := \mathrm{diag}(a_1(c),\ldots,a_{r-1}(c)); \quad a_i(c) := \frac{\sqrt{1+c_1^2+\cdots+c_{i-1}^2}}{\sqrt{1+c_1^2+\cdots+c_i^2}\sqrt{1+|c|^2}}. \qquad (5.6)$$

Thus, using the transformation property in equation (2.3), we get

$$\overline{\hat{f}_{1/2}}\hat{f}_{1/2-s}\left[\begin{pmatrix} I_{r-1} & \\ c & 1 \end{pmatrix}\right] \leq (1+|c|^2)^{-r/2(1+\Re(s))} \|\hat{f}_{1/2}\hat{f}_{1/2-s}\|_{L^\infty(K_\infty)}.$$

Thus the second summand of equation (5.5) is bounded by

$$\ll_{\epsilon,f,W_0} \int_{\mathbb{R}^{r-1}} (1+|c|^2)^{-r/2(1-\epsilon)}\, dc.$$

The above integral is convergent for sufficiently small ϵ.

We now focus on the third summand in equation (5.5). In the inner integral, we use Iwasawa coordinates for $h = ak$, move the K-integral outside and make the change of variables $c \mapsto ck$ to rewrite it as

$$-\int_{O_{r-1}(\mathbb{R})}\int_{\mathbb{R}^{r-1}}|\hat{f}_{1/2}|^2\left[\begin{pmatrix}I_{r-1}\\ck&1\end{pmatrix}\right]$$
$$\int_{A_{r-1}(\mathbb{R})}|\tilde{W}_0|^2\left[\begin{pmatrix}a\\&1\end{pmatrix}\begin{pmatrix}I_{r-1}\\cX&1\end{pmatrix}\begin{pmatrix}k\\&1\end{pmatrix}\right]\log|\det(a)|\frac{da}{\delta(a)}dcdk.$$

We use the Iwasawa decomposition of $\begin{pmatrix}I_{r-1}\\cX&1\end{pmatrix}$ as in equation (5.6) to write it as $\tilde{n}(cX)\begin{pmatrix}\tilde{a}(cX)\\&1\end{pmatrix}\tilde{k}(cX)$. Then, using the left $N(\mathbb{R})$-invariance of $|\tilde{W}_0|^2$ and changing variable $a \mapsto a \times \tilde{a}(cX)^{-1}$, we obtain that the above quantity is equal to

$$\int_{O_{r-1}(\mathbb{R})}\int_{\mathbb{R}^{r-1}}|\hat{f}_{1/2}|^2\left[\begin{pmatrix}I_{r-1}\\ck&1\end{pmatrix}\right]\log|\det(\tilde{a}(cX))|\delta(\tilde{a}(cX))$$
$$\int_{A_{r-1}(\mathbb{R})}|\tilde{W}_0|^2\left[\begin{pmatrix}a\\&1\end{pmatrix}\tilde{k}(cX)\begin{pmatrix}k\\&1\end{pmatrix}\right]\frac{da}{\delta(a)}dcdk$$
$$-\int_{O_{r-1}(\mathbb{R})}\int_{\mathbb{R}^{r-1}}|\hat{f}_{1/2}|^2\left[\begin{pmatrix}I_{r-1}\\ck&1\end{pmatrix}\right]\delta(\tilde{a}(cX))$$
$$\int_{A_{r-1}(\mathbb{R})}|\tilde{W}_0|^2\left[\begin{pmatrix}a\\&1\end{pmatrix}\tilde{k}(cX)\begin{pmatrix}k\\&1\end{pmatrix}\right]\log|\det(a)|\frac{da}{\delta(a)}dcdk.$$

We write the above as $A - B$, where A denotes the first term and B denotes the second term above. To analyse A, we reverse-engineer the above process: make the change of variables $a \mapsto a \times \tilde{a}(cX)$, use the left $N(\mathbb{R})$-invariance of $|\tilde{W}_0|^2$ and make the change of variables $c \mapsto ck^{-1}$ to obtain

$$A = \int_{\mathbb{R}^{r-1}}|\hat{f}_{1/2}|^2\left[\begin{pmatrix}I_{r-1}\\c&1\end{pmatrix}\right]\int_{O_{r-1}(\mathbb{R})}\log|\det(a(ck^{-1}X))|$$
$$\int_{A_{r-1}(\mathbb{R})}|\tilde{W}_0|^2\left[\begin{pmatrix}ak\\&1\end{pmatrix}\begin{pmatrix}I_{r-1}\\cX&1\end{pmatrix}\right]\frac{da}{\delta(a)}dkdc.$$

But $\det(\tilde{a}(cX)) = (1 + X^2|c|^2)^{-r/2} = \det(a(ck^{-1}X))$ for all $k \in O_{r-1}(\mathbb{R})$. Using that, we can move the integral over $O_{r-1}(\mathbb{R})$ to couple with the integral over $A_{r-1}(\mathbb{R})$ to obtain an integral over $N_{r-1}(\mathbb{R})\backslash GL_{r-1}(\mathbb{R})$. Then, once again appealing to the invariance of the unitary product, we obtain

$$A = \|\tilde{W}_0\|_{\tilde{\pi}_0}^2\int_{\mathbb{R}^{r-1}}|\hat{f}_{1/2}|^2\left[\begin{pmatrix}I_{r-1}\\c&1\end{pmatrix}\right]\log|\det(\tilde{a}(cX))|dc.$$

Note that

$$\log|\det(\tilde{a}(cX))| = -\frac{r}{2}\log(1 + X^2|c|^2)$$
$$= -\frac{r}{2}\log(1 + X^2) + O(\log(1 + |c|^2)) = -r\log X + O_\epsilon((1 + |c|)^\epsilon).$$

Using the Iwasawa decomposition and transformation property of $\hat{f}_{1/2}$ as in equation (2.3), similar to the second case, we obtain

$$A + r \log X \|\tilde{W}_0\|_{\tilde{\pi}_0}^2 \int_{\mathbb{R}^{r-1}} |\hat{f}_{1/2}|^2 \left[\begin{pmatrix} I_{r-1} \\ c & 1 \end{pmatrix} \right] dc$$

$$\ll_{\tilde{W}_0, \epsilon} \|\hat{f}_{1/2}\|_{L^\infty(K_\infty)}^2 \int_{\mathbb{R}^{r-1}} (1 + |c|^2)^{-r/2+\epsilon} dc \ll_{\Phi, \tilde{W}_0} 1.$$

Working as in the proof of Lemma 5.1, we check that

$$\|\tilde{W}_0\|_{\tilde{\pi}_0}^2 \int_{\mathbb{R}^{r-1}} |\hat{f}_{1/2}|^2 \left[\begin{pmatrix} I_{r-1} \\ c & 1 \end{pmatrix} \right] dc = \|W_0 f_{1/2}\|^2.$$

Thus we obtain

$$A = -r \log X \|W_0 f_{1/2}\|^2 + O_{W_0, f}(1). \tag{5.7}$$

Now we prove that B is of bounded size. To prove that, we first claim that

$$\int_{A_{r-1}(\mathbb{R})} |\tilde{W}_0|^2 \left[\begin{pmatrix} a \\ & 1 \end{pmatrix} \tilde{k}(cX) \begin{pmatrix} k \\ & 1 \end{pmatrix} \right] \log |\det(a)| \frac{da}{\delta(a)} \ll_{\tilde{W}_0} 1$$

uniformly in c. We assume the claim. Now note that

$$\delta(\tilde{a}(cX)) = \frac{(1 + X^2|c|^2)^{r/2-1}}{\prod_{i=1}^{r-2}(1 + c_1^2 X^2 + \cdots + c_i^2 X^2)} \ll \frac{X^{r-2}(1 + |c|^2)^{r/2-1}}{\prod_{i=1}^{r-2}(1 + c_i^2 X^2)}.$$

We use the Iwasawa decomposition and work as before. Using transformation of $\hat{f}_{1/2}$ as in equation (2.3), we thus get

$$B \ll_{W_0, f} \int_{O_{r-1}(\mathbb{R})} \int_{\mathbb{R}^{r-1}} (1 + |ck|^2)^{-r/2} \frac{X^{r-2}(1 + |c|^2)^{r/2-1}}{\prod_{i=1}^{r-2}(1 + c_i^2 X^2)} dc \, dk$$

$$\ll \int_{\mathbb{R}^{r-1}} \prod_{i=1}^{r-1} (1 + c_i^2)^{-1} dc,$$

which we obtain by noting that $|ck| = |c|$ for $k \in O_{r-1}(\mathbb{R})$ and making the change of variables $c_i \mapsto c_i/X$ for $i \leq r - 2$. It is easy to see that the above integral is convergent, which yields that

$$B = O_{W_0, f}(1).$$

Now, to prove the claim above, let $\omega := \tilde{k}(cX) \begin{pmatrix} k \\ & 1 \end{pmatrix} \in K_\infty$ implicitly depending on cX. Note that from Lemma 7.2, we get that

$$\tilde{\pi}_0(\omega) \tilde{W}_0 \left[\begin{pmatrix} a(y) \\ & 1 \end{pmatrix} \right] \ll_{\epsilon, M, W_0} \delta^{1/2-\epsilon}(a(y)) |\det(a(y))|^{1/2-\vartheta_0-\epsilon} \prod_{i=1}^{r-1} \min(1, |y_i|^{-M}).$$

Thus we obtain

$$\int_{A_{r-1}(\mathbb{R})} |\tilde{\pi}_0(\omega) \tilde{W}_0|^2 \left[\begin{pmatrix} a \\ & 1 \end{pmatrix} \right] \log |\det(a)| \frac{da}{\delta(a)}$$

$$\ll_{W_0, \eta, M} \int_{(\mathbb{R}^\times)^{r-1}} \prod_{i=1}^{r-1} \min(1, |y_i|^{-M}) |\det(a(y))|^{1-2\vartheta_0-\epsilon} (|\det(a(y))|^\epsilon + |\det(a(y))|^{-\epsilon}) \prod_i d^\times y_i.$$

Employing the bound of ϑ_0 from the statement of Theorem 2.1, we check that the above integral is convergent for large enough M and sufficiently small $\epsilon > 0$, which yields the claim.   □

Remark 5.4. In the very last estimate of the proof of Lemma 5.3, we can only prove that the integral of the Whittaker function is of bounded size. It is not clear to us if or how one can improve the estimate to be a constant plus a power-saving error term. This would potentially explicate the constant term of the asymptotic expansion in Theorem 2.1 with a power-saving error term; see Remark 1.4.

Lemma 5.5. *We have*

$$\partial_{s=0}\Psi_\infty(\overline{f_{1/2,\infty}}f_{1/2+s,\infty}(.x_\infty), W_{0,\infty}, \overline{W_{0,\infty}}) = (r-1)\log X \|W_{0,\infty}f_{1/2,\infty}\|^2 + O_{W_{0,\infty},\Phi_\infty}(1)$$

as X tends to infinity.

Proof. The proof of this lemma is very similar to (and easier than) the proof of Lemma 5.3. We first write $\Psi(\overline{f_{1/2}}f_{1/2+s}(.x), W_0, \overline{W_0})$ as

$$X^{(r-1)s}\int_{N_{r-1}(\mathbb{R})\backslash \mathrm{GL}_{r-1}(\mathbb{R})}\int_{\mathbb{R}^{r-1}}|W_0|^2\left[\begin{pmatrix} h & \\ c/X & 1 \end{pmatrix}\right]\overline{f_{1/2}}f_{1/2+s}\left[\begin{pmatrix} \mathrm{I}_{r-1} & \\ c & 1 \end{pmatrix}\right]dc|\det(h)|^s dh.$$

Note that the $s = 0$ derivative in the the statement of this lemma can be computed exactly the same as we did in the calculation of equation (5.5) of Lemma 5.3 and can be seen equal to

$$(r-1)\log X \int_{\mathbb{R}^{r-1}}|f_{1/2}|^2\left[\begin{pmatrix} \mathrm{I}_{r-1} & \\ c & 1 \end{pmatrix}\right]\int_{N_{r-1}(\mathbb{R})\backslash \mathrm{GL}_{r-1}(\mathbb{R})}|W_0|^2\left[\begin{pmatrix} h & \\ c/X & 1 \end{pmatrix}\right]dhdc$$

$$+\int_{\mathbb{R}^{r-1}}\overline{f_{1/2}}\partial_{s=0}f_{1/2+s}\left[\begin{pmatrix} \mathrm{I}_{r-1} & \\ c & 1 \end{pmatrix}\right]\int_{N_{r-1}(\mathbb{R})\backslash \mathrm{GL}_{r-1}(\mathbb{R})}|W_0|^2\left[\begin{pmatrix} h & \\ c/X & 1 \end{pmatrix}\right]dhdc$$

$$+\int_{\mathbb{R}^{r-1}}|f_{1/2}|^2\left[\begin{pmatrix} \mathrm{I}_{r-1} & \\ c & 1 \end{pmatrix}\right]\int_{N_{r-1}(\mathbb{R})\backslash \mathrm{GL}_{r-1}(\mathbb{R})}|W_0|^2\left[\begin{pmatrix} h & \\ c/X & 1 \end{pmatrix}\right]\log|\det(h)|dhdc. \quad (5.8)$$

Exactly as in the proof of Lemma 5.3, we can check (for example, changing $\hat{f}_{1/2-s}$ to $f_{1/2+s}$ and $\tilde{W}_0$ to W_0) that the first and second summands in equation (5.8) are

$$(r-1)\log X \|W_0 f_{1/2}\|^2$$

and $O_{W_0,f}(1)$, respectively. We claim that the third summand in equation (5.8) is also $O_{W_0,f}(1)$, which yields the lemma.

From the relation between f and Φ from equation (2.2), we write

$$f_{1/2}\left[\begin{pmatrix} \mathrm{I}_{r-1} & \\ c & 1 \end{pmatrix}\right] = \int_{\mathbb{R}^\times}\Phi(t(c,1))|t|^{r/2}d^\times t.$$

Recall the choice of Φ in Section 4.1. Support of Φ being on $B_\tau(0,\ldots,0,1)$ implies that in the above integral, $t \asymp 1$ and hence $c \ll 1$. Below, we show that

$$\int_{N_{r-1}(\mathbb{R})\backslash \mathrm{GL}_{r-1}(\mathbb{R})}|W_0|^2\left[\begin{pmatrix} h & \\ c/X & 1 \end{pmatrix}\right]\log|\det(h)|dh \ll_{W_0} 1,$$

which clearly implies our claim above.

We write $h = ak$ in Iwasawa coordinates and let $\omega := \begin{pmatrix} k \\ c/X & 1 \end{pmatrix}$. Note that as $k \in O_{r-1}(\mathbb{R})$ and $c/X \ll 1$, there exists a fixed compact set $\Omega \in G(\mathbb{R})$ such that $\omega \in \Omega$ for all relevant c and k. Thus it is enough to show that

$$\int_{A_{r-1}(\mathbb{R})} |\pi_0(\omega) W_0|^2 \left[\begin{pmatrix} a \\ & 1 \end{pmatrix} \right] \log |\det(a)| \frac{da}{\delta(a)} \ll_{W_0, \Omega} 1.$$

This can be done similarly as we did at the end of the proof of Lemma 5.3. $\qquad\square$

6. Analysis of the regularised term in the period side

Let $s \in \mathbb{C}$ with sufficiently small $\Re(s)$. Recall the regularised Eisenstein series $\tilde{E}_s$ from equation (4.9). The main proposition of this section is the following.

Proposition 6.1. $\tilde{E}_s$ *is holomorphic in a sufficiently small neighbourhood of $s = 0$ and is integrable on $\mathbb{X}$.*

Note from the definition in equation (4.9) that $\tilde{E}_s$ is holomorphic in a punctured neighbourhood of $s = 0$. Thus it is enough to prove that $\tilde{E}_s$ is holomorphic at $s = 0$. Recall the description of the poles of the maximal Eisenstein series in Section 2.4. We know, in particular, $\overline{\mathrm{Eis}(f_{1/2})}\mathrm{Eis}(f_{1/2+s})$ is holomorphic at $s = 0$, and we thus only need to show the following.

Lemma 6.2. *For fixed $g \in \mathbb{X}$,*

$$\mathrm{Eis}(\overline{f_{1/2}} f_{1/2+s})(g) + \mathrm{Eis}(\overline{\tilde{f}_{1/2}} \tilde{f}_{1/2+s})(g)$$

is holomorphic at $s = 0$.

Proof. Our argument is to show that the residues R and $\tilde{R}$ (which are independent of g) at the simple poles at $s = 0$ of $\mathrm{Eis}(\overline{f_{1/2}} f_{1/2+s})$ and $\mathrm{Eis}(\overline{\tilde{f}_{1/2}} \tilde{f}_{1/2+s})$, respectively, cancel each other.

Let ϕ_0 be the cusp form as we have chosen in Section 4.1. From equation (5.2), we get

$$\begin{aligned} R\|\phi_0\|_2^2 &= \mathrm{Res}_{s=0} \langle |\phi_0|^2, \mathrm{Eis}(\overline{f_{1/2}} f_{1/2+s}) \rangle \\ &= \frac{\zeta(r/2)^2}{\zeta(r)} L(1, \pi_0, \mathrm{Ad}) \Psi_\infty(|f_{1/2,\infty}|^2, W_{0,\infty}, \overline{W_{0,\infty}}). \end{aligned}$$

Similarly, from equation (5.3), we get

$$\begin{aligned} \tilde{R}\|\phi_0\|_2^2 &= \mathrm{Res}_{s=0} \langle |\phi_0|^2, \mathrm{Eis}(\overline{\tilde{f}_{1/2}} \tilde{f}_{1/2+s}) \rangle \\ &= -\frac{\zeta(r/2)^2}{\zeta(r)} L(1, \pi_0, \mathrm{Ad}) \Psi_\infty(|\tilde{f}_{1/2,\infty}|^2, W_{0,\infty}, \overline{W_{0,\infty}}). \end{aligned}$$

From Lemma 5.1 with $x_\infty = 1$ (and the equation preceding Lemma 5.1), we conclude that the Ψ_∞ factors in the above expressions of R and $\tilde{R}$ are equal. $\qquad\square$

Now we prove some preparatory lemmata to prove the integrability of $\tilde{E}_s$ on $\mathbb{X}$. We actually show that $\tilde{E}_s$ is integrable in the Siegel domain $\mathbb{S}$ as in equation (2.1), which contains $\mathbb{X}$. Let $g \in \mathbb{S}$ with $g = (g_\infty, k_f)$, where $g_\infty = n_\infty \begin{pmatrix} a(y_\infty) \\ & 1 \end{pmatrix} k_\infty \in G(\mathbb{R})$ in Iwasawa coordinates and $k_f \in K_f := \prod_{p < \infty} K_p$. As $g \in \mathbb{S}$, we have $y_{j,\infty} \gg 1$. We recall the quantities in Proposition 3.3 from Section 3.

Lemma 6.3. *Suppose that $i < r$. Let $s \in \mathbb{C}$ be away from a pole of $M_i^0 f_s$ with $|\Re(s)| < 2$. Then*

$$\|M_i^0 f_s\|_{L^\infty(K)} \ll 1.$$

Further, let $\mathbb{R}^\times \times K_f \ni (y_\infty, 1) =: y$. Then for all $k \in K$ and s with $|\Re(s)| < 2$,

$$\sum_{q \in \mathbb{Q}^\times} W_{f_s}^i(qy, k) \ll_N |y_\infty|^{-N},$$

where the sum in the left-hand side converges absolutely.

Proof. In this proof, we assume that $\Phi \in \mathcal{S}(\mathbb{A}^r)$ is an arbitrary Schwartz function. We get that for $k \in K$,

$$M_i^0 f_s(k) = \int_{\mathbb{A}^{r-i}} \int_{\mathbb{A}^\times} (k.\Phi)(0, t, x)|t|^{rs-r+i} d^\times t \ll \int_{|x|, |t| \ll 1} |t|^{rs-r+i} d^\times t dx \ll_{K, \Re(s)} 1,$$

if $\Re(s)$ is sufficiently large. On the other hand, using the Tate functional equation and working similarly, we obtain

$$M_i^0 f_s(k) = \int_{\mathbb{A}^\times} \widehat{(k.\Phi)}^i(te_i)|t|^{r-i+1-rs} \ll_{K, \Re(s)} 1,$$

if $\Re(s)$ is sufficiently negative. Using the Phragmén–Lindelöf convexity principle and the compactness of K, we deduce the first claim.

Let $z \in \mathbb{A}$ and $k \in K$. Following a similar computation after equation (3.4) in Section 3, we get that

$$W_{f_s}^i(z, k) = \int_{\mathbb{A}^{r-i}} \int_{\mathbb{A}^\times} (k.\Phi)(0, t, x)\overline{\psi_0(zx_1/t)}|t|^{rs-r+i} d^\times t dx.$$

This converges absolutely if $\Re(s)$ is sufficiently large.

We first concentrate on the x_1 integral. In the archimedean component of this integral, we integrate by parts with respect to the $x_{1,\infty}$ variable. This yields that the archimedean integral is bounded by $\ll_N |z_\infty|^{-N}|t_\infty|^N$ for all large N.

In the p-adic component, we note that compact support of Φ_p forces $x_{1,p}$ to vary over a compact space. This implies that the p-adic integral vanishes unless $|z_p/t_p| \ll 1$. However, the support condition of Φ_p ensures that $|t_p| \ll 1$, which in turn restricts z_p to be of bounded size.

Thus we can analytically continue the integral representation of $W_{f_s}^i$ to $\Re(s)$ sufficiently negative, but fixed. Altogether, estimating the integrals as before, we obtain if $\Re(s) \geq -2$; then for sufficiently large N, we have

$$W_{f_s}^i(z, k) \ll_{K,N} |z_\infty|^{-N} \prod_{p < \infty} \mathrm{char}_{|z_p| \ll 1}.$$

Thus for $q \in \mathbb{Q}^\times$ and y as in the statement of this lemma, we have

$$W_{f_s}^i(qy, k) \ll_{K,N} |y_\infty q_\infty|^{-N},$$

if the denominator of q is bounded; otherwise, the above is zero. Thus the sum over $q \in \mathbb{Q}^\times$ is absolutely convergent for sufficiently large N. We conclude using the compactness of K. □

Lemma 6.4. *Let $g \in \mathbb{S}$ and $s \in \mathbb{C}$ with sufficiently small $\Re(s)$. Then*

$$\overline{\mathrm{Eis}(f_{1/2})(g)}\mathrm{Eis}(f_{1/2+s})(g) - \overline{f_{1/2}(g)}f_{1/2+s}(g) - \overline{\tilde{f}_{1/2}(g)}\tilde{f}_{1/2+s}(g) \ll \delta^{1-\eta}\left[\begin{pmatrix} a(y_\infty) & \\ & 1 \end{pmatrix}\right]$$

for some $\eta > 0$.

Proof. Recall that

$$\delta\left[\begin{pmatrix} a(y_\infty) & \\ & 1 \end{pmatrix}\right] = \prod_{j=1}^{r-1} |y_{j,\infty}|^{j(r-j)}.$$

Note that $y_{j,\infty} \gg 1$ as $g \in \mathbb{S}$. Thus it is enough the show that the exponents of $|y_{j,\infty}|$ arising in the left-hand side in the expression in the lemma are less than $j(r-j)$.

We recall equation (3.4) and for $1 < i < r$ write

$$H_s^i(g) := \sum_{q \in \mathbb{Q}} M_i^q f_{1/2+s}(g)$$

$$= M_i^0 f_{1/2+s}(g) + \sum_{q \in \mathbb{Q}^\times} \psi_{i(q)}(n_1) \prod_{j=1}^{i-1} |y_j|^{(1/2+s)j} \prod_{j=i}^{r-1} |y_j|^{(1/2-s)(r-j)} W_{f_{1/2+s}}^i(qy_i, k),$$

where n_1 is a unipotent element as in Section 2.4. Using Lemma 6.3, we obtain that

$$H_s^i(g) \ll \prod_{j=1}^{i-1} |y_{j,\infty}|^{(1/2+\Re(s))j} \prod_{j=i}^{r-1} |y_{j,\infty}|^{(1/2-\Re(s))(r-j)}.$$

On the other hand, we similarly obtain

$$H_s^1(g) := \sum_{q \in \mathbb{Q}^\times} M_1^q f_{1/2+s}(g) \ll_{K,N} \prod_{j=1}^{r-1} |y_{j,\infty}|^{(1/2-\Re(s))(r-j)-N\delta_{j=1}}.$$

We also record that

$$f_{1/2+s}(g) \ll \prod_{j=1}^{r-1} |y_{j,\infty}|^{(1/2+\Re(s))j}$$

and

$$\tilde{f}_{1/2+s}(g) \ll \prod_{j=1}^{r-1} |y_{j,\infty}|^{(1/2-\Re(s))(r-j)}.$$

We use Lemma 3.2 to rewrite

$$\mathrm{Eis}(f_{1/2+s})(g) = f_{1/2+s}(g) + \tilde{f}_{1/2+s}(g) + H_s^1(g) + \sum_{1 < i < r} H_s^i(g).$$

After multiplying $\overline{\mathrm{Eis}(f_{1/2})}$ and $\mathrm{Eis}(f_{1/2+s})$ using the above expression and subtracting the terms $\overline{f_{1/2}} f_{1/2+s}$, and $\overline{\tilde{f}_{1/2}} \tilde{f}_{1/2+s}$, we are left with the following type of terms whose bounds are given below:

$$\overline{f_{1/2}} \tilde{f}_{1/2+s}(g) \ll \prod_{j=1}^{r-1} |y_{j,\infty}|^{r/2-(r-j)\Re(s)}.$$

If we replace the left-hand side above with $\overline{\tilde{f}_{1/2}} f_{1/2+s}(g)$, then a similar inequality holds, with the exponent in the right-hand side being $r/2 + j\Re(s)$. In any case, for sufficiently small $\Re(s)$ and $r \geq 3$, we have

$$r/2 + r|\Re(s)| < j(r-j), \quad 1 \leq j < r.$$

A similar estimate can be done for $\overline{f_{1/2}}H_s^1(g)$. Next we check that

$$\overline{H_0^1(g)}\tilde{f}_{1/2+s} \ll \prod_{j=1}^{r-1} |y_{j,\infty}|^{(1-\Re(s))(r-j)-N\delta_{j=1}}.$$

A similar estimate can be obtained if we replace $\tilde{f}_{1/2+s}$ by H_s^1 on the left-hand side above. Clearly, for sufficiently small $\Re(s)$, we have

$$(1 - \Re(s))(r - j) - N\delta_{j=1} < j(r - j).$$

Finally, for $1 < i < r$, the exponent of $y_{j,\infty}$ of $\overline{H_0^i}$ is $\le (r-2)/2$. On the other hand, the same of G_s is $\le (1/2 + |\Re(s)|)(r - 1)$ for G_s being one of $f_{1/2+s}$, $\tilde{f}_{1/2+s}$, or H_s^i with $i < r$. So the exponent of $y_{j,\infty}$ of the product $\overline{H_0^i}G_s$ for $1 < i < r$ is

$$\le (r - 2 + r - 1)/2 + (r - 1)|\Re(s)| < j(r - j)$$

for sufficiently small $\Re(s)$.

Similarly, one estimates remaining terms of the form $\overline{H_0^1}f_{1/2+s}$, $\overline{G_0}H_s^i$ and $\overline{\tilde{f}_{1/2}}H_s^1$, which we leave for the reader. Hence we conclude the proof. $\qquad\square$

Lemma 6.5. *Let $g \in \mathbb{S}$ and $s \in \mathbb{C}$ with sufficiently small $\Re(s)$. Then*

$$\mathrm{Eis}(\overline{f_{1/2}}f_{1/2+s})(g) - \overline{f_{1/2}(g)}f_{1/2+s}(g) \ll \delta^{1-\eta}\left[\begin{pmatrix} a(y_\infty) & \\ & 1 \end{pmatrix}\right]$$

and also

$$\mathrm{Eis}(\overline{\tilde{f}_{1/2}}\tilde{f}_{1/2+s})(g) - \overline{\tilde{f}_{1/2}(g)}\tilde{f}_{1/2+s}(g) \ll \delta^{1-\eta}\left[\begin{pmatrix} a(y_\infty) & \\ & 1 \end{pmatrix}\right]$$

for some $\eta > 0$.

Proof. We take a very similar path as in the proof of Lemma 6.4. Let $s \in \mathbb{C}$ be away from the poles of the relevant Eisenstein series and $\Re(s)$ be sufficiently small.

First note that $\overline{\tilde{f}_{1/2}}\tilde{f}_{1/2+s} \in \mathcal{I}_{1,r-1}(1 - s)$. We use the functional equation of the Eisenstein series [11, Proposition 2.1]: there exists $\tilde{F}_s \in \mathcal{I}_{r-1,1}(s)$ such that

$$\mathrm{Eis}(\overline{\tilde{f}_{1/2}}\tilde{f}_{1/2+s}) = \mathrm{Eis}(\tilde{F}_s).$$

In fact, $\tilde{F}_s$ is the preimage of $\overline{\tilde{f}_{1/2}}\tilde{f}_{1/2+s}$ under the standard intertwiner from $\mathcal{I}_{1,r-1}(1 - s)$ to $\mathcal{I}_{r-1,1}(s)$: that is,

$$\overline{\tilde{f}_{1/2}}\tilde{f}_{1/2+s} = M_1^0\tilde{F}_s,$$

and in particular, $\tilde{F}_s$ is holomorphic in a sufficiently small neighbourhood of $s = 0$.

From Lemma 3.2, we get that

$$\mathrm{Eis}(\overline{\tilde{f}_{1/2}}\tilde{f}_{1/2+s}) - \overline{\tilde{f}_{1/2}(g)}\tilde{f}_{1/2+s}(g) = \mathrm{Eis}(\tilde{F}_s)(g) - M_1^0\tilde{F}_s(g)$$

$$= \tilde{F}_s(g) + \sum_{q\in\mathbb{Q}^\times} M_1^q\tilde{F}_s(g) + \sum_{1<i<r}\sum_{q\in\mathbb{Q}} M_i^q\tilde{F}_s(g).$$

We now bound each summand above similarly to the proof of Lemma 6.4. As $\tilde{F}_s \in \mathcal{I}_{r-1,1}(s)$, we obtain that

$$\tilde{F}_s(g) \ll \left| \tilde{F}_s \left[\begin{pmatrix} a(y_\infty) & \\ & 1 \end{pmatrix} \right] \right| \ll \prod_{j=1}^{r-1} |y_{j,\infty}|^{j\Re(s)}.$$

Here we applied $\|\tilde{F}_s\|_{L^\infty(K)} \ll 1$, which can be deduced similarly to the proof of Lemma 6.3 and applying holomorphicity of $\tilde{F}_s$ for small s.

On the other hand, once again recalling equation (3.4), we obtain for $1 < i < r$ that

$$\sum_{q \in \mathbb{Q}} M_i^q \tilde{F}_s(g) \ll M_i^0 \tilde{F}_s \left[\begin{pmatrix} a(y_\infty) & \\ & 1 \end{pmatrix} \right]$$

$$+ \prod_{j=1}^{i-1} |y_{j,\infty}|^{j\Re(s)} \prod_{j=i}^{r-1} |y_{j,\infty}|^{(1-\Re(s))(r-j)} \sum_{q \in \mathbb{Q}^\times} \|W_{\tilde{F}_s}^i (qy_i, .)\|_{L^\infty(K)}$$

$$\ll \prod_{j=1}^{i-1} |y_{j,\infty}|^{j\Re(s)} \prod_{j=i}^{r-1} |y_{j,\infty}|^{(1-\Re(s))(r-j)}.$$

In the last estimate above, we used that

$$\sum_{q \in \mathbb{Q}^\times} \|W_{\tilde{F}_s}^i (qy_i, .)\|_{L^\infty(K)} \ll_N |y_{i,\infty}|^{-N},$$

which can be deduced similarly to the proof of Lemma 6.3. Similarly, we deduce that

$$\sum_{q \in \mathbb{Q}^\times} M_1^q \tilde{F}_s(g) \ll_N \prod_{j=1}^{r-1} |y_{j,\infty}|^{(1-\Re(s))(r-j)-N\delta_{j=1}}.$$

In each case, the exponent of $y_{j,\infty}$ is strictly smaller than $j(r-j)$, which concludes the proof for the second assertion for sufficiently small $\Re(s)$. The first assertion can be proved similarly (and more easily), which we leave for the reader. $\square$

Proof of Proposition 6.1. In Lemma 6.2, we have already proved the holomorphicity of $\tilde{E}_s$ at $s = 0$. From Lemma 6.4 and Lemma 6.5, we conclude by the triangle inequality that for sufficiently small $\Re(s)$ and $g \in \mathbb{S}$,

$$\tilde{E}_s(g) \ll \delta^{1-\eta} \left[\begin{pmatrix} a(y_\infty) & \\ & 1 \end{pmatrix} \right]$$

for some $\eta > 0$. Thus

$$\int_{\mathbb{X}} |\tilde{E}_s|(g) dg \ll \int_{y_{j,\infty} \gg 1} \delta^{-\eta} \left[\begin{pmatrix} a(y_\infty) & \\ & 1 \end{pmatrix} \right] \prod_j d^\times y_{j,\infty}.$$

The last integral is convergent, and we conclude. $\square$

7. Analysis of the spectral side

Recall the spectral weight $J_X(\pi_\infty)$ from equation (4.5), the choices of the local components from Section 4.1, and the ϑ_0-temperedness assumption on $\pi_{0,\infty}$ from the statement of Theorem 2.1. In this section, we prove the remaining second property of the spectral weight as described in Theorem 2.1. That

is, we show that $J_X(\pi_\infty)$ is uniformly bounded away from zero if π_∞ is ϑ-tempered with $\vartheta + \vartheta_0 < 1/2$ and $C(\pi_\infty) < X$.

Proposition 7.1. *Let $\pi \in \mathbb{X}_{\mathrm{gen}}$ be such that π_∞ is ϑ-tempered with $\vartheta + \vartheta_0 < 1/2$. Let π_0 be the cuspidal automorphic representation as in Theorem 2.1. Then*

$$J_X(\pi_\infty) \gg 1, \quad if \, C(\pi_\infty) < X,$$

where the implied constant possibly depends on $W_{0,\infty}, \Phi_\infty$.

For the rest of this section, to ease notations, we drop the ∞-subscript everywhere.

We recall the notations and definition of the Sobolev norm $\mathcal{S}_d$ as in [27, §2.3.2], [19, §3.9]. Let $\{H\}$ be a basis of $\mathrm{Lie}(G(\mathbb{R}))$. We define a Laplacian on $G(\mathbb{R})$ by

$$\mathcal{D} := 1 - \sum_H H^2, \tag{7.1}$$

which is positive definite and self-adjoint on any unitary representation ξ of $G(\mathbb{R})$. For any $v \in \xi$, we define the dth Sobolev norm of v by

$$\mathcal{S}_d(v) := \|\mathcal{D}^d v\|_\xi.$$

We refer to [27, §2.4] for a collection of useful properties of the Sobolev norm.

Let $W \in \pi$ be a unit vector such that in the Kirillov model W is given by

$$W\left[\begin{pmatrix} g & \\ & 1 \end{pmatrix}\right] := W_0\left[\begin{pmatrix} g & \\ & 1 \end{pmatrix}\right]. \tag{7.2}$$

Note that such a choice is valid due to the choice of W_0 in Section 4.1. In fact, $W \in \pi$ is an analytic newvector in the sense of Section 2.9.

Lemma 7.2. *Let W_0 be as in Section 4.1. Let $A_{r-1}(\mathbb{R})O_{r-1}(\mathbb{R}) \ni h = ak$ as before. If $c \ll 1$, then*

$$W_0\left[\begin{pmatrix} h & \\ c/X & 1 \end{pmatrix}\right] \ll_{\eta,\pi_0} |\det(a)|^{-\vartheta_0} \delta^{1/2-\eta}\left[\begin{pmatrix} a & \\ & 1 \end{pmatrix}\right] \min(1, a_{r-1}^{-M}) \prod_{i=1}^{r-2} \min(1, (a_i/a_{i+1})^{-M})$$

for any $\eta > 0$.

This lemma is proved in [19, Lemma 5.2] for π_0 being a tempered representation. Here we modify the proof to accommodate the ϑ_0-tempered case.

Proof. Let $W_0' := \pi_0\left[\begin{pmatrix} k & \\ c/X & 1 \end{pmatrix}\right] W_0$. Note that $k \in O_{r-1}(\mathbb{R})$ and $|c| \ll 1$ vary over compact sets. Hence it is enough to show that

$$W_0'\left[\begin{pmatrix} a & \\ & 1 \end{pmatrix}\right] \ll_{\eta,M,\pi_0} |\det(a)|^{-\vartheta_0} \delta^{1/2-\eta}\left[\begin{pmatrix} a & \\ & 1 \end{pmatrix}\right] \min(1, a_{r-1}^{-M}) \prod_{i=1}^{r-2} \min(1, (a_i/a_{i+1})^{-M}).$$

We take a very similar path as in the proof of [19, Lemma 5.2].

We define $W_1 := d\pi_0(Y^M)(W_0')$, where Y is a Lie algebra element such that

$$d\pi_0(Y)W_0'\left[\begin{pmatrix} a & \\ & 1 \end{pmatrix}\right] = a_{r-1}W_0'\left[\begin{pmatrix} a & \\ & 1 \end{pmatrix}\right].$$

Thus it is enough to prove that

$$W_1\left[\begin{pmatrix} a \\ & 1 \end{pmatrix}\right] \ll_{\eta,M,\pi_0} |\det(a)|^{-\vartheta_0} \delta^{1/2-\eta}\left[\begin{pmatrix} a \\ & 1 \end{pmatrix}\right] \prod_{i=1}^{r-2} \min(1, (a_i/a_{i+1})^{-M}). \tag{7.3}$$

We use the Dixmier–Malliavin Lemma (see [13]) to find finitely many $\alpha_i \in C_c^\infty(G(\mathbb{R}))$ and $W_i \in \pi_0^\infty$ such that

$$W_1 = \sum_i \pi_0(\alpha_i) W_i.$$

Thus to prove equation (7.3), it is enough to show equation (7.3) with W_1 replaced by $\pi_0(\alpha_i) W_i =: W_2$ for each i.

Let $\sigma \in \mathbb{R}$. We use the Whittaker–Plancherel formula to expand

$$|\det(a)|^{-\sigma} W_2\left[\begin{pmatrix} a \\ & 1 \end{pmatrix}\right] = \int_{\widehat{\mathrm{GL}_{r-1}(\mathbb{R})}} \sum_{W' \in \mathcal{B}(\pi')} W'(a) Z_{W_2,\sigma}(W') d\mu_{\mathrm{loc}}(\pi'), \tag{7.4}$$

which is valid for σ in some left half plane. Here

$$Z_{W_2,\sigma}(W') := \int_{N_{r-1}(\mathbb{R}) \backslash \mathrm{GL}_{r-1}(\mathbb{R})} W_2\left[\begin{pmatrix} h \\ & 1 \end{pmatrix}\right] \overline{W'(h)} |\det(h)|^{-\sigma} dh$$
$$= \gamma(1/2 - \sigma, \pi_0 \otimes \overline{\pi'})^{-1} \omega_{\pi'}(-1)^{r-1} \int_{N_{r-1}(\mathbb{R}) \backslash \mathrm{GL}_{r-1}(\mathbb{R})} \tilde{W}_2\left[\begin{pmatrix} h \\ & 1 \end{pmatrix}\right] \overline{\tilde{W}'(h)} |\det(h)|^{\sigma} dh.$$

In the last line, we have used the $\mathrm{GL}(r) \times \mathrm{GL}(r-1)$ local functional equation. Here $\gamma(.)$ denotes the local gamma factor and $\omega_{\pi'}$ denotes the central character of π'. Finally, $\tilde{W}$ denotes the contragredient of W defined by $\tilde{W}(g) := W(wg^{-t})$, where w is the long Weyl element of the respective group. Let $\tilde{\alpha}_i(g) := \alpha_i(g^{-t})$. Let N^* be the unipotent radical of upper triangular matrices attached to the partition $r = (r-1) + 1$. Recalling that $W_2 = \pi_0(\alpha_i) W_i$, we can write

$$\tilde{W}_2\left[\begin{pmatrix} h \\ & 1 \end{pmatrix}\right] = \int_{G(\mathbb{R})} \tilde{\alpha}_i(g) \tilde{W}_i\left[\begin{pmatrix} h \\ & 1 \end{pmatrix} g\right] dg$$
$$= \int_{N^* \backslash G(\mathbb{R})} \tilde{W}_i\left[\begin{pmatrix} h \\ & 1 \end{pmatrix} g\right] \int_{N^*} \tilde{\alpha}_i(n^* g) \overline{\psi_{e_{r-1}h}(n^*)} dn^* dg,$$

where e_{r-1} is the row vector $(0, \ldots, 0, 1)$. Then we have that

$$Z_{W_2,\sigma}(W') = \gamma(1/2 - \sigma, \pi_0 \otimes \overline{\pi'})^{-1} \omega_{\pi'}(-1)^{r-1}$$
$$\int_{N^* \backslash G(\mathbb{R})} \int_{N_{r-1}(\mathbb{R}) \backslash \mathrm{GL}_{r-1}(\mathbb{R})} \tilde{W}_i\left[\begin{pmatrix} h \\ & 1 \end{pmatrix} g\right] \int_{N^*} \tilde{\alpha}_i(n^* g) \overline{\psi_{e_{r-1}h}(n^*)} dn^* \overline{\tilde{W}'(h)} |\det(h)|^{\sigma} dh dg.$$

We choose an orthonormal basis $\mathcal{B}(\pi')$ consisting of eigenfunctions of the Laplacian $\mathcal{D}'$ on $\mathrm{GL}_{r-1}(\mathbb{R})$, as defined in equation (7.1), and integrate by parts the h-integral L times with respect to $\mathcal{D}'$. We note that $W' \otimes |\det|^{\sigma}$ is also an eigenfunction of $\mathcal{D}'$. We recall a bound of the gamma factor from [19, Lemma 3.1]:

$$\gamma(1/2 - \sigma, \pi_0 \otimes \overline{\pi'}) \ll_{\sigma,\pi_0} C(\pi')^{r\sigma}.$$

We apply the Cauchy–Schwarz on the above h integral. Then we use the above bound of the gamma factor and unitarity of π_0 to obtain that

$$Z_{W_2,\sigma}(W') \ll C(\pi')^{r\sigma} \lambda_{\tilde{W}'}^{-L} \int_{N^*\backslash G(\mathbb{R})} \left(\int_{N_{r-1}(\mathbb{R})\backslash \mathrm{GL}_{r-1}(\mathbb{R})} \left| \mathcal{D}'^L \left(\int_{N^*} \alpha_i(n^*g)\overline{\psi_{e_{r-1}h}(n^*)}dn^* \right) \right. \right.$$
$$\left. \left. \overline{\tilde{W}'(h)} |\det(h)|^\sigma \right|^{2dh} \right)^{1/2} dg,$$

where $\lambda_{\tilde{W}'}$ is the $\mathcal{D}'$-eigenvalue of $\tilde{W}'$. The above N^*-integral gives rise to a Schwartz function in $e_{r-1}h$, which can be seen integrating by parts several times in the N^*-integral. Thus

$$\mathcal{D}'^L \left(\int_{N^*} \alpha_i(n^*g)\overline{\psi_{e_{r-1}h}(n^*)}dn^* \right) \ll \min(1, |e_{r-1}h|^{-N}).$$

Noting that g varies over a compact set in $G(\mathbb{R})$ modulo N^*, we obtain that

$$Z_{W_2,\sigma}(W') \ll C(\pi')^{r\sigma} \lambda_{\tilde{W}'}^{-L} \left(\int_{N_{r-1}(\mathbb{R})\backslash \mathrm{GL}_{r-1}(\mathbb{R})} \min(1, |e_{r-1}h|^{-N}) |\tilde{W}'(h)|^2 |\det(h)|^{2\sigma} dh \right)^{1/2}.$$

We use [19, Lemma 5.2] on $\tilde{W}'$ (which is in the tempered representation $\tilde{\pi}'$) to check that the above integral is absolutely convergent for any $\sigma > 0$. In particular, from the location of the first pole of $\gamma(1/2 - \sigma, \pi_0 \otimes \overline{\pi}')^{-1}$, we may conclude that one can choose σ in $(0, 1/2 - \vartheta_0)$ in the definition of $Z_{W_2,\sigma}(W')$.

Again, we use [19, Lemma 5.2] to estimate $W'(a)$ in equation (7.4) by

$$\ll \delta^{1/2-\eta}(a) \prod_{i=1}^{r-2} \min(1, (a_i/a_{i+1})^{-M}) \lambda_{W'}^d,$$

where d only depends on M. We choose $\sigma = 1/2 - \vartheta_0 - \eta$ to obtain that

$$W_2 \left[\begin{pmatrix} a & \\ & 1 \end{pmatrix} \right] \ll_{\eta,M,\pi_0} |\det(a)|^{-\vartheta_0} \delta^{1/2-\eta} \left[\begin{pmatrix} a & \\ & 1 \end{pmatrix} \right]$$
$$\prod_{i=1}^{r-2} \min(1, (a_i/a_{i+1})^{-M}) \int_{\widehat{\mathrm{GL}_{r-1}(\mathbb{R})}} C(\pi')^{r\sigma} \sum_{W' \in \mathcal{B}(\pi')} \lambda_{W'}^{d-L} d\mu_{\mathrm{loc}}(\pi').$$

We make L sufficiently large and invoke [19, Lemma 3.3] to conclude that the above sum and integral are absolutely convergent.   □

Lemma 7.3. *Let W be as in equation (7.2) and W_0 be as in Section 4.1. Let V be W or W_0 and ξ be π or π_0, respectively. Also let $A_{r-1}(\mathbb{R})\mathrm{O}_{r-1}(\mathbb{R}) \ni h = ak$, where $a = \mathrm{diag}(a_1, \ldots, a_{r-1})$ and $|c| \ll 1$. Then for any sufficiently small $\eta > 0$,*

$$V \left[\begin{pmatrix} h & \\ c/X & 1 \end{pmatrix} \right] - V \left[\begin{pmatrix} h & \\ & 1 \end{pmatrix} \right] \ll_\eta |\det(a)|^{-\theta} \delta^{1/2-\eta} \left[\begin{pmatrix} a & \\ & 1 \end{pmatrix} \right] \frac{C(\xi)|c|}{X}.$$

Here θ is ϑ or ϑ_0 depending on whether ξ is π or π_0, respectively.

This is essentially the main result of analytic newvectors, proved in [19, Proposition 4.1], but in a more quantitative form. We need to only modify the proof of [19, Proposition 4.1], and we describe that here.

Proof. Let $\sigma \in \mathbb{R}$ be in some left half plane. As in the proof of [19, Proposition 4.1], we write the difference in the lemma as

$$\int_{\widehat{\mathrm{GL}_{r-1}(\mathbb{R})}} \omega_{\bar{\pi}'}((-1)^{r-1}C(\xi)^{-1})C(\xi)^{(r-1)\sigma}\gamma(1/2-\sigma,\xi\otimes\bar{\pi}')^{-1}\sum_{W'\in\mathcal{B}(\pi')}W'(h)|\det(h)|^{\sigma}$$

$$\int_{N_{r-1}(\mathbb{R})\backslash\mathrm{GL}_{r-1}(\mathbb{R})}(e(cw't^{-1}e_1C(\xi)/X)-1)V\left[\begin{pmatrix}C(\xi)&\\&t\end{pmatrix}w\right]\overline{W'(tw')}|\det(t)|^{-\sigma}dtd\mu_{\mathrm{loc}}(\pi'),$$

where w' is the long Weyl element of $\mathrm{GL}(r-1)$. Note that π' is tempered. We now use [19, Lemma 5.2] for tempered representations to estimate

$$W'(h)\ll\delta^{1/2-\eta}(a)S_d(W')$$

for any $\eta>0$ and some $d>0$. We choose $\sigma=1/2-\theta-\eta$ (which is admissible) and proceed as in the proof of [19, Proposition 4.1] to conclude. $\square$

Proof of Proposition 7.1. Recall the definition of J_X from equations (4.5) and (2.6). In the expression of equation (2.6), we choose a basis $\mathcal{B}(\pi)$ containing an analytic newvector W as in equation (7.2). To show the required lower bound of J_X, it is enough to drop all but the term containing W from the sum in equation (2.6) by positivity and show that

$$X^{r-1}|\Psi(f_{1/2}(.x),W_0,\overline{W})|^2\gg 1$$

if $C(\pi)<X$.

First, using equation (2.2) and the choices of the local components as in Section 4.1, we get

$$f_{1/2}\left[\begin{pmatrix}\mathrm{I}_{r-1}&\\c&1\end{pmatrix}\right]=\int_{\mathbb{R}^{\times}}\Phi(t(c,1))|t|^{r/2}d^{\times}t\geq0.$$

The support condition of Φ in Section 4.1 implies that the above vanishes unless $|c|<\tau$. We use Bruhat coordinates and make the change of variables to write $X^{\frac{r-1}{2}}\Psi(f_{1/2}(.x),W_0,\overline{W})$ as

$$\int_{N_{r-1}(\mathbb{R})\backslash\mathrm{GL}_{r-1}(\mathbb{R})}\int_{\mathbb{R}^{r-1}}W_0\left[\begin{pmatrix}h&\\c/X&1\end{pmatrix}\right]\overline{W\left[\begin{pmatrix}h&\\c/X&1\end{pmatrix}\right]}f_{1/2}\left[\begin{pmatrix}\mathrm{I}_{r-1}&\\c&1\end{pmatrix}\right]dc\frac{dh}{|\det(h)|^{1/2}}.$$

We use Lemma 7.3 for W, noting that $|c|<\tau$ and $C(\pi)<X$, to obtain that the above integral is

$$\int_{N_{r-1}(\mathbb{R})\backslash\mathrm{GL}_{r-1}(\mathbb{R})}\overline{W\left[\begin{pmatrix}h&\\&1\end{pmatrix}\right]}\int_{\mathbb{R}^{r-1}}W_0\left[\begin{pmatrix}h&\\c/X&1\end{pmatrix}\right]f_{1/2}\left[\begin{pmatrix}\mathrm{I}_{r-1}&\\c&1\end{pmatrix}\right]dc\frac{dh}{|\det(h)|^{1/2}}$$

$$+O_{\eta}\left(\tau\int_{N_{r-1}(\mathbb{R})\backslash\mathrm{GL}_{r-1}(\mathbb{R})}\delta^{1/2-\eta}\left[\begin{pmatrix}h&\\&1\end{pmatrix}\right]\int_{\mathbb{R}^{r-1}}\left|W_0\left[\begin{pmatrix}h&\\c/X&1\end{pmatrix}\right]f_{1/2}\left[\begin{pmatrix}\mathrm{I}_{r-1}&\\c&1\end{pmatrix}\right]\right|dc\frac{dh}{|\det(h)|^{1/2+\vartheta}}\right).$$
(7.5)

We use Lemma 7.3 for W_0 and the definition of W in the Kirillov model as in equation (7.2) to obtain that the main term of equation (7.5) is equal to

$$
\int_{N_{r-1}(\mathbb{R})\backslash \mathrm{GL}_{r-1}(\mathbb{R})} |W_0|^2 \left[\begin{pmatrix} h \\ & 1 \end{pmatrix} \right] \frac{dh}{|\det(h)|^{1/2}} \int_{\mathbb{R}^{r-1}} f_{1/2} \left[\begin{pmatrix} \mathrm{I}_{r-1} \\ c & 1 \end{pmatrix} \right] dc
$$
$$
+ O_{\pi_0} \left(\frac{1}{X} \int_{N_{r-1}(\mathbb{R})\backslash \mathrm{GL}_{r-1}(\mathbb{R})} |W_0| \left[\begin{pmatrix} h \\ & 1 \end{pmatrix} \right] \delta^{1/2-\eta} \left[\begin{pmatrix} h \\ & 1 \end{pmatrix} \right] \frac{dh}{|\det(h)|^{1/2+\vartheta_0}} \right.
$$
$$
\left. \int_{\mathbb{R}^{r-1}} |f_{1/2}| \left[\begin{pmatrix} \mathrm{I}_{r-1} \\ c & 1 \end{pmatrix} \right] dc \right). \tag{7.6}
$$

From the choice of Φ in Section 4.1, we obtain that

$$
0 \leq \int_{\mathbb{R}^{r-1}} f_{1/2} \left[\begin{pmatrix} \mathrm{I}_{r-1} \\ c & 1 \end{pmatrix} \right] dc \asymp 1.
$$

Also, the choice of W_0 in Section 4.1 ensures that

$$
\int_{N_{r-1}(\mathbb{R})\backslash \mathrm{GL}_{r-1}(\mathbb{R})} |W_0|^2 \left[\begin{pmatrix} h \\ & 1 \end{pmatrix} \right] \frac{dh}{|\det(h)|^{1/2}} \asymp_{\pi_0} 1.
$$

So the main term of equation (7.6) is

$$
\asymp_{\pi_0} \int_{\mathbb{R}^{r-1}} f_{1/2} \left[\begin{pmatrix} \mathrm{I}_{r-1} \\ c & 1 \end{pmatrix} \right] dc.
$$

On the other hand, the error term in equation (7.6) is trivially $\ll_{\pi_0,\tau} X^{-1}$, which follows from the support condition of W_0 as in Section 4.1. In total, we obtain that equation (7.6), which is the main term of equation (7.5), is

$$
\asymp_{\pi_0} \int_{\mathbb{R}^{r-1}} f_{1/2} \left[\begin{pmatrix} \mathrm{I}_{r-1} \\ c & 1 \end{pmatrix} \right] dc + O_{\pi_0,\tau}(1/X).
$$

Now we focus on the error term of equation (7.5). We use Iwasawa coordinates in the integral and use Lemma 7.2 to estimate the error term by

$$
\ll_M \tau \int_{\mathbb{R}^{r-1}} f_{1/2} \left[\begin{pmatrix} \mathrm{I}_{r-1} \\ c & 1 \end{pmatrix} \right] dc \int_{A_{r-1}(\mathbb{R})} |\det(a)|^{1/2-\vartheta-\vartheta_0} \delta^{-2\eta} \left[\begin{pmatrix} h \\ & 1 \end{pmatrix} \right]
$$
$$
\min(1, a_{r-1}^{-M}) \prod_{i=1}^{r-2} \min(1, (a_i/a_{i+1})^{-M}) d^{\times} a.
$$

We recall the assumption that $\vartheta + \vartheta_0 < 1/2$. Hence, the inner integral is convergent for sufficiently small η and large enough M. Thus we obtain that equation (7.5) is

$$
\asymp_{\pi_0} (1 + \tau O_{\pi_0}(1)) \int_{\mathbb{R}^{r-1}} f_{1/2} \left[\begin{pmatrix} \mathrm{I}_{r-1} \\ c & 1 \end{pmatrix} \right] dc + O_{\pi_0,\tau}(1/X).
$$

We conclude that the above is $\gg 1$ by making τ sufficiently small but fixed. $\quad\square$

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