2

Kinks in more complicated models

The Z_2 and sine-Gordon kinks discussed in the last chapter are not representative of kinks in models where non-Abelian symmetries are present. Kinks in such models have more degrees of freedom and this introduces degeneracies when imposing boundary conditions, leading to many kink solutions with different internal structures (but the same topology). Indeed, kink-like solutions may exist even when the topological charge is zero. The interactions of kinks in these more complicated models, their formation and evolution, plus their interactions with other particles are very distinct from the kinks of the last chapter.

We choose to focus on kinks in a model that is an example relevant to particle physics and cosmology. The model is the first of many Grand Unified Theories of particle physics that have been proposed [63]. The idea behind grand unification is that Nature really has only one gauge-coupling constant at high energies, and that the disparate values of the strong, weak, and electromagnetic coupling constants observed today are due to symmetry breaking and the renormalization-group running of coupling constants down to low energies. Since there is only one gauge-coupling constant in these models, there is a simple grand unified symmetry group G that is valid at high energies, for example, at the high temperatures present in the very early universe. At lower energies, G is spontaneously broken in stages, eventually leaving only the presently known quantum chromo dynamics (QCD) and electromagnetic symmetries $SU(3)_c \times U(1)_{em}$ of particle physics, with its two different coupling constants. It can be shown [63] that the minimal possibility for G is $SU(5)$. However, since Grand Unified Theories predict proton decay, experimental observation of the longevity of the proton ($\sim 5 \times 10^{33}$ years) leads to constraints on grand unified models. The (non-supersymmetric) $SU(5)$ Grand Unified Theory is ruled out by the current lower limits on the proton's lifetime. Therefore particle-physics model builders consider yet larger groups G, or with an extended scalar field sector, or supersymmetric extensions of $SU(5)$, and other models based on larger groups. Even if the symmetry group is larger than $SU(5)$, it often happens

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that after a series of symmetry breaking, the residual symmetry is $SU(5)$, which then proceeds to break to the current symmetry group. Hence the study of $SU(5)$ symmetry breaking is extremely relevant to particle physics, even if it is not the ultimate grand unified symmetry group.

In this chapter we shall study kinks in a model with $SU(5) \times Z_2$ symmetry though almost all the discussion can be generalized to an $SU(N) \times Z_2$ model for odd values of N [163, 120]. The extra Z_2 symmetry is explained in the next section. Since we only desire to study kinks in a particle-physics motivated model, it would seem simpler to choose a model based on the smaller $SU(3)$ group. However, it can be shown that there is no way to construct a model with just $SU(3)$ symmetry and with the simplest choice of field content, which is one adjoint field. Instead, the model must have the larger $O(8)$ symmetry. Other fields need to be included so as to reduce the $O(8)$ to $SU(3)$, but that introduces additional parameters which make the $SU(3)$ model more messy than the $SU(5)$ model.

Dealing with continuous groups such as $SU(5)$ requires certain background material. The fundamental representation of $SU(N)$ generators is described in Appendix B. A summary of some aspects of the $SU(5)$ model of grand unification is given in Section 5.5.

2.1 $SU(5)$ model

The $SU(5)$ model can be written as[1]

$$L = \mathrm{Tr}(\mathrm{D}_\mu \Phi)^2 - \frac{1}{2}\mathrm{Tr}(X_{\mu\nu}X^{\mu\nu}) - V(\Phi) \tag{2.1}$$

where, in terms of components, Φ is a scalar field (also called a Higgs field) transforming in the adjoint representation of $SU(5)$, that is, $\Phi \to \Phi' = g\Phi g^\dagger$ for $g \in SU(5)$. The gauge field strengths are $X_{\mu\nu} = X^a_{\mu\nu}T^a$ and the $SU(5)$ generators T^a are normalized such that $\mathrm{Tr}(T^aT^b) = \delta^{ab}/2$. The definition of the covariant derivative is

$$\mathrm{D}_\mu = \partial_\mu - \mathrm{ie}X_\mu \tag{2.2}$$

and its action on the adjoint scalar is given by

$$\mathrm{D}_\mu\Phi = \partial_\mu\Phi - \mathrm{ie}[X_\mu, \Phi] \tag{2.3}$$

The gauge field strength is given in terms of the covariant derivative via

$$-\mathrm{ie}X_{\mu\nu} = [\mathrm{D}_\mu, \mathrm{D}_\nu] \tag{2.4}$$

[1] We are using the Einstein summation convention in which repeated group and space-time indices are summed over. So, explicitly, $\Phi = \sum_{a=1}^{24} \Phi^a T^a$. See Appendix B for more details on the $SU(5)$ generators T^a.

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and the potential is the most general quartic in Φ

$$V(\Phi) = -m^2\mathrm{Tr}(\Phi^2) + h[\mathrm{Tr}(\Phi^2)]^2 + \lambda\mathrm{Tr}(\Phi^4) + \gamma\mathrm{Tr}(\Phi^3) - V_0 \tag{2.5}$$

where V_0 is a constant that is chosen so as to set the minimum value of the potential to zero.

The model in Eq. (2.1) does not have any topological kinks because there are no broken discrete symmetries. In particular, the Z_2 symmetry under $\Phi \to -\Phi$ is absent owing to the cubic term in Eq. (2.5). Note that $\Phi \to -\Phi$ is not achievable by an $SU(5)$ transformation. To show this, consider $\mathrm{Tr}(\Phi^3)$. This is invariant under any $SU(5)$ transformation, but not under $\Phi \to -\Phi$. However, if $\gamma = 0$, there are topological kinks connecting the two vacua related by $\Phi \to -\Phi$. For non-zero but small γ, these kinks are almost topological. In our analysis in this chapter we set $\gamma = 0$, in which case the symmetry of the model is $SU(5) \times Z_2$. The philosophy underlying grand unification does not forbid discrete symmetry factors since such factors do not entail additional gauge-coupling constants. Indeed, model builders often set $\gamma = 0$ for simplicity. Now a non-zero vacuum expectation value of Φ breaks the discrete Z_2 factor leading to topological kinks.

2.2 $SU(5) \times Z_2$ symmetry breaking and topological kinks

The potential in Eq. (2.5) has a (degenerate) global minimum at

$$\Phi_0 = \frac{\eta}{2\sqrt{15}}\mathrm{diag}(2, 2, 2, -3, -3) \tag{2.6}$$

where $\eta = m/\sqrt{\lambda'}$ provided

$$\lambda \geq 0, \qquad \lambda' \equiv h + \frac{7}{30}\lambda \geq 0 \tag{2.7}$$

For the global minimum to have $V(\Phi_0) = 0$, in Eq. (2.5) we set

$$V_0 = -\frac{\lambda'}{4}\eta^4 \tag{2.8}$$

As discussed in Section 1.10, if we transform Φ_0 by any element of $SU(5) \times Z_2$, the transformed Φ_0 is still at a minimum of the potential. However, Φ_0 is left unmoved by transformations belonging to

$$G_{321} \equiv \frac{[SU(3) \times SU(2) \times U(1)]}{Z_3 \times Z_2} \tag{2.9}$$

where $SU(3)$ acts on the upper-left 3×3 block of Φ_0, $SU(2)$ on the lower-right 2×2 block, and $U(1)$ is generated by Φ_0 itself. Hence, G_{321} is the unbroken symmetry group.

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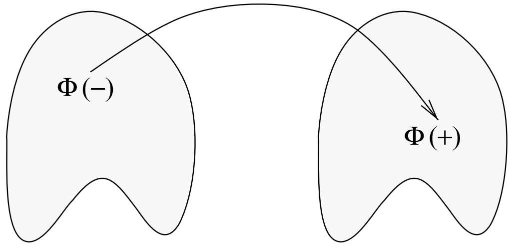

Figure 2.1 The vacuum manifold of the $SU(5) \times Z_2$ model consists of two disconnected 12-dimensional copies. Kink solutions correspond to paths that originate in one piece at $x = -\infty$, denoted by $\Phi(-)$, leave the vacuum manifold, and end in the other disconnected piece at $x = +\infty$. Topological considerations specify that $\Phi(+)$ has to lie in the disconnected piece on the right, but not where it should be located within this piece.

$SU(5)$ has 24 generators while the unbroken group, G_{321}, has a total of 12 generators, namely, 8 of $SU(3)$, 3 of $SU(2)$, and 1 of $U(1)$. Therefore the vacuum manifold is $24 - 12 = 12$ dimensional but in two disconnected pieces as depicted in Fig. 2.1 because of the Z_2 factor. Kink solutions occur if the boundary conditions lie in different disconnected pieces. However, if we start at some point on the vacuum manifold at $x = -\infty$, say $\Phi(-\infty) = \Phi_-$, we have a choice of boundary conditions for Φ_+, the vacuum expectation value of Φ at $x = +\infty$ (compare with the Z_2 case where the path had to go from definite initial to definite final values of Φ).

We will narrow down the possible choices for Φ_+ very shortly. First we point out that the gauge fields can be set to zero in finding kink solutions [163]. To see this explicitly, the only linear term in the gauge field is $ie\mathrm{Tr}(X^i[\Phi, \partial_i\Phi])$. However, our solution for Φ satisfies $[\Phi, \partial_i\Phi] = 0$ [120] and so the variation vanishes to linear order in gauge field fluctuations. A closer look also reveals that the quadratic terms of perturbations in the gauge fields contribute positively to the energy of the kink solutions and so the gauge fields do not cause an instability of the solutions [163]. Hence we set

$$X_\mu = 0 \tag{2.10}$$

As we now show, the boundary conditions that lead to static solutions of the equations of motion are rather special [120].

Theorem: A static solution can exist only if $[\Phi_+, \Phi_-] = 0$.

We only give a sketch of the proof here since it is of a technical nature. The essential idea is that if $\Phi_k(x)$ is a static solution, then the energy should be extremized by it. By considering perturbations of the kind $U(x)\Phi_k U^\dagger(x)$ where $U(x)$ is an infinitesimal rotation of $SU(5)$, one finds that the energy can be extremized only if

$[\Phi_k, \partial_x \Phi_k] = 0$ for all x. Now at large x, we have $\Phi_k \to \Phi_+$. In this region $\partial_x \Phi_k$ has terms that are proportional to Φ_- as well, even if these are exponentially small, since $\Phi(x)$ is an analytic function. Hence, a static solution requires $[\Phi_+, \Phi_-] = 0$.

The theorem immediately narrows down the possibilities that we need to consider when trying to construct kink solutions. If we fix

$$\Phi_- = \Phi_0 = \frac{\eta}{2\sqrt{15}} \mathrm{diag}(2, 2, 2, -3, -3) \tag{2.11}$$

Φ_+ can take on the following three values

$$\begin{aligned}
\Phi_+^{(0)} &= -\frac{\eta}{2\sqrt{15}} \mathrm{diag}(2, 2, 2, -3, -3) \\
\Phi_+^{(1)} &= -\frac{\eta}{2\sqrt{15}} \mathrm{diag}(2, 2, -3, 2, -3) \\
\Phi_+^{(2)} &= -\frac{\eta}{2\sqrt{15}} \mathrm{diag}(2, -3, -3, 2, 2)
\end{aligned} \tag{2.12}$$

One can also rotate these three choices by elements of the unbroken group G_{321-} that leaves Φ_- invariant and obtain three disjoint classes of possible values of Φ_+. The three choices given above are representatives of their classes.

The kink solution for any of the three boundary conditions is of the form

$$\phi_k^{(q)} = F_+^{(q)}(x)\mathbf{M}_+^{(q)} + F_-^{(q)}(x)\mathbf{M}_-^{(q)} + g^{(q)}(x)\mathbf{M}^{(q)} \tag{2.13}$$

where $q = 0, 1, 2$ labels the solution class,

$$\mathbf{M}_+^{(q)} = \frac{\Phi_+^{(q)} + \Phi_-^{(q)}}{2}, \qquad \mathbf{M}_-^{(q)} = \frac{\Phi_+^{(q)} - \Phi_-^{(q)}}{2} \tag{2.14}$$

and $\mathbf{M}^{(q)}$ will be specified below.

The boundary conditions for $F_\pm^{(q)}$ are

$$F_-^{(q)}(\mp\infty) = \mp 1, \qquad F_+^{(q)}(\mp\infty) = +1, \qquad g^{(q)}(\mp\infty) = 0 \tag{2.15}$$

The formulae for $\mathbf{M}_\pm^{(q)}$ and $\mathbf{M}^{(q)}$ can now be explicitly written using Eq. (2.12) in (2.14)

$$\mathbf{M}_+^{(q)} = \eta \frac{5}{4\sqrt{15}} \mathrm{diag}(\mathbf{0}_{3-q}, \mathbf{1}_q, -\mathbf{1}_q, \mathbf{0}_{2-q}) \tag{2.16}$$

$$\mathbf{M}_-^{(q)} = \eta \frac{1}{4\sqrt{15}} \mathrm{diag}(-4\mathbf{1}_{3-q}, \mathbf{1}_q, \mathbf{1}_q, 6\mathbf{1}_{2-q}) \tag{2.17}$$

$$\mathbf{M}^{(q)} = \mu\, \mathrm{diag}(q(2-q)\mathbf{1}_{3-q}, -(2-q)(3-q)\mathbf{1}_{2q}, q(3-q)\mathbf{1}_{2-q}) \tag{2.18}$$

with the normalization μ given by

$$\mu = \eta[2q(2-q)(3-q)(12-5q)]^{-1/2} \tag{2.19}$$

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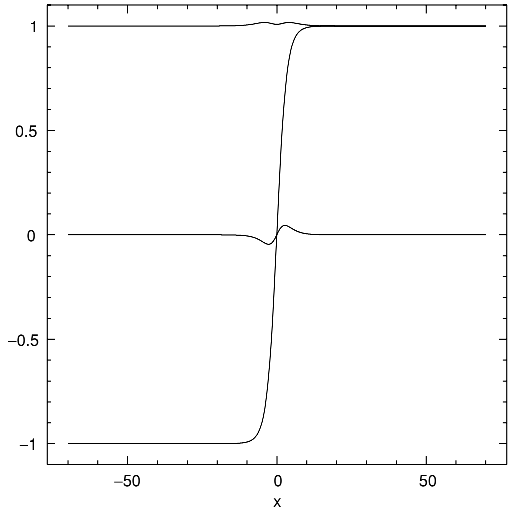

Figure 2.2 The profile functions $F_+^{(1)}(x)$ (nearly 1 throughout), $F_-^{(1)}(x)$ (shaped like a tanh function), and $g^{(1)}(x)$ (nearly zero) for the $q = 1$ topological kink with parameters $h = -3/70$, $\lambda = 1$, and $\eta = 1$.

If $q = 0$ or $q = 2$ we set $\mu = 0$. We have used $\mathbf{0}_k$ and $\mathbf{1}_k$ to denote the $k \times k$ zero and unit matrices respectively. Note that the matrices $\mathbf{M}_\pm^{(q)}$ are relatively orthogonal

$$\mathrm{Tr}(\mathbf{M}_+^{(q)}\mathbf{M}_-^{(q)}) = 0 \tag{2.20}$$

but are not normalized to $\eta^2/2$.

Now we discuss the three kink solutions in the $SU(5) \times Z_2$ model. For $q = 0$, the solution is that of a Z_2 kink that has been embedded in the $SU(5) \times Z_2$ model. The explicit solution is

$$F_+^{(0)}(x) = 0, \qquad F_-^{(0)}(x) = -\tanh\left(\frac{x}{w}\right), \qquad g^{(0)}(x) = 0 \tag{2.21}$$

where $w = \sqrt{2}/m$. For $q = 1$, the profile functions have been evaluated numerically and are shown in Fig. 2.2. Approximate analytic solutions can also be found in [120]. For $q = 2$ the solution has also been found numerically. Here we describe an approximate solution which is exact if

$$\frac{h}{\lambda} = -\frac{3}{20} \tag{2.22}$$

i.e. $\lambda' = \lambda/12$. With this particular choice

$$F_+^{(2)}(x) = 1, \qquad F_-^{(2)}(x) = \tanh\left(\frac{x}{w}\right), \qquad g^{(2)}(x) = 0 \tag{2.23}$$

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where $w = \sqrt{2}/m$. This is also an approximate solution for $h/\lambda \approx -3/20$. The energy of the approximate solution can be used to estimate the mass of the $q = 2$ kink

$$M^{(2)} \approx \frac{M^{(0)}}{6}\left\{\frac{1}{6}\left[1+\frac{5\lambda}{12\lambda'}\right]\right\}^{1/2} \equiv M^{(0)}\frac{\sqrt{p}}{6} \tag{2.24}$$

where $M^{(2)}$ denotes the mass of the $q = 2$ kink, and $M^{(0)} = 2\sqrt{2}m^3/3\lambda'$. The expression for the energy is exact for $h/\lambda = -3/20$.

It can be shown for a range of parameters that the $q = 2$ kink solution is perturbatively stable. Numerical evaluations of the energy find that the $q = 2$ kink is lighter than the $q = 0, 1$ kinks for all values of p. Equation (2.24) shows the $q = 2$ kink is lighter than the $q = 0$ kink for a large range of parameters. This can be understood qualitatively by noting that only one component of Φ changes sign in the $q = 2$ kink, while 3 and 5 components change sign in the $q = 1$ and $q = 0$ kinks respectively.

2.3 Non-topological $SU(5) \times Z_2$ kinks

An interesting point to note is that the ansatz in Eq. (2.13) is valid even if $\Phi_{\pm}^{(q)}$ are not in distinct topological sectors. These imply the existence of non-topological kink solutions in the model [120]. If we include a subscript NT to denote "non-topological" and T to denote "topological," we have

$$\Phi_{\mathrm{NT}k}^{(q)} = F_{+}^{(q)}(x)\mathbf{M}_{\mathrm{NT}+}^{(q)} + F_{-}^{(q)}(x)\mathbf{M}_{\mathrm{NT}-}^{(q)} + g^{(q)}(x)\mathbf{M}_{\mathrm{NT}}^{(q)} \tag{2.25}$$

where the $\mathbf{M}_{\mathrm{NT}\pm}$ matrices are still defined by Eq. (2.14) with the non-topological values of $\Phi_{\pm}$. $\mathbf{M}_{\mathrm{NT}}$ is still given by Eq. (2.18). To consider a non-topological domain wall, we simply want to consider Φ_{+} to be in the same discrete sector as Φ_{-}. If $\Phi_{\mathrm{T}+}$ denotes a boundary condition for a topological kink, a possible boundary condition for a non-topological kink is: $\Phi_{\mathrm{NT}+} = -\Phi_{\mathrm{T}+}$. Then we find

$$\mathbf{M}_{\mathrm{NT}+}^{(q)} = \mathbf{M}_{\mathrm{T}-}^{(q)}, \qquad \mathbf{M}_{\mathrm{NT}-}^{(q)} = \mathbf{M}_{\mathrm{T}+}^{(q)}, \qquad \mathbf{M}_{\mathrm{NT}}^{(q)} = \mathbf{M}_{\mathrm{T}}^{(q)} \tag{2.26}$$

Hence

$$\Phi_{\mathrm{NT}k}^{(q)} = F_{-}^{(q)}(x)\mathbf{M}_{\mathrm{T}+}^{(q)} + F_{+}^{(q)}(x)\mathbf{M}_{\mathrm{T}-}^{(q)} + g^{(q)}(x)\mathbf{M}_{\mathrm{T}}^{(q)} \tag{2.27}$$

To get $F_{\mp}^{(q)}$ for the non-topological kink we have to solve the topological $F_{\pm}^{(q)}$ equation of motion but with the boundary conditions for $F_{\mp}^{(q)}$ (see Eq. (2.15)). To obtain $g^{(q)}$ for the non-topological kink, we need to interchange $F_{+}^{(q)}$ and $F_{-}^{(q)}$ in the topological equation of motion. The boundary conditions for $g^{(q)}$ are unchanged. Generally the non-topological solutions, when they exist, are unstable. However,

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Table 2.1 *The space of three topological kinks in the $SU(5)$ model.* G_{321} is the group $SU(3) \times SU(2) \times U(1)$. The dimensionality of the space of each type of kink is also given.

Kink	Space	Dimensionality
$q = 0$	G_{321}/G_{321}	0
$q = 1$	$G_{321}/[SU(2) \times U(1)^3]$	6
$q = 2$	$G_{321}/[SU(2)^2 \times U(1)^2]$	4

the possibility that some of them may be locally stable for certain potentials cannot be excluded.

2.4 Space of $SU(5) \times Z_2$ kinks

The kink solutions discussed in Section 2.1 can be transformed into other degenerate solutions using the $SU(5)$ transformations. Hence, each solution is representative of a space of solutions. We now discuss the space associated with each of these solutions.

If we denote a kink solution in the $SU(5) \times Z_2$ model by $\Phi_{\mathrm{k}}^{(q)}$, another solution is

$$\phi_{\mathrm{k}}^{(q)h} = h\phi_{\mathrm{k}}^{(q)}h^{\dagger}, \quad h \in G_{321-} \tag{2.28}$$

where G_{321-} is the unbroken group whose elements leave Φ_- unchanged.[2] The reason $\Phi_{\mathrm{k}}^{(q)h}$ also describes a solution is that the rotation h does not change the energy of the field configuration, $\Phi_{\mathrm{k}}^{(q)}$. Therefore $\Phi_{\mathrm{k}}^{(q)h}$ has the same energy and the same topology as $\Phi_{\mathrm{k}}^{(q)}$, and hence it describes another kink solution.

Of the elements of G_{321-}, there are some that act trivially on $\Phi_{\mathrm{k}}^{(q)}$ and for these h, $\Phi_{\mathrm{k}}^{(q)h}$ is not distinct from $\Phi_{\mathrm{k}}^{(q)}$. These elements form a subgroup of G_{321-} that we call K_q. Therefore the space of kinks can be labeled by elements of the coset space G_{321-}/K_q. Since we are given the forms of the kink solutions in Eq. (2.13), it is not hard to work out K_q. For example, for the $q = 2$ kink, K_q is given by the $SU(5)$ elements that commute with both G_{321-} and G_{321+} and so $K_q = SU(2)^2 \times U(1)^2$. Once we have determined K_q the dimensionality of the coset space G_{321-}/K_q is determined as the dimensionality of G_{321-}, which is 12, minus the dimensionality of K_q, which is 12, 6, and 8 for $q = 0$, 1, and 2 respectively.

The three classes of kink solutions labeled by the index q in the $SU(5) \times Z_2$ model have different spaces as shown in Table 2.1.

[2] We could also have included elements that change $\Phi_+^{(q)}$ as well as Φ_-. These would simply be global rotations of the entire solution and would be the same for every type of defect.

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The dimensionality of the space of a given type of kink solution also corresponds to the dimensionality of the space of boundary conditions Φ_+ for which that type of kink solution is obtained. As an example, there is only one value of Φ_+, namely $\Phi_+ = -\Phi_-$, that gives rise to the $q = 0$ kink. While for the $q = 1$ kink, one can choose Φ_+ to be any value from a 6-dimensional space. This means that, in any process where boundary conditions are chosen at random, the probabilities of getting the correct boundary conditions for a $q = 0$ or a $q = 2$ kink are of measure zero, since the space of boundary conditions for the $q = 1$ kink is two dimensions greater than that for the $q = 2$ kink. In any random process, the $q = 1$ kink is always obtained. Since this kink is unstable, it then decays into the $q = 2$ kink. Therefore the production of $q = 2$ kinks is a two-step process in this system. We will see further evidence of this two-step process in Chapter 6.

2.5 S_n kinks

The $SU(5) \times Z_2$ model discussed above shows novel features because of the large non-Abelian symmetry. It is possible to see some of the richness of the model by going to a simpler model where the continuous non-Abelian symmetries are replaced by discrete non-Abelian symmetries (also see [92] for a similar model). If we truncate the $SU(5) \times Z_2$ model to just the diagonal degrees of freedom of Φ, we get a model that is symmetric only under permutations of the diagonal entries and the overall Z_2. Hence the symmetry group is $S_5 \times Z_2$, where S_5 is the permutation group of five objects. The model now has four real scalar fields, one for each diagonal generator of $SU(5)$. With this truncation we can write

$$\Phi \to f_1\lambda_3 + f_2\lambda_8 + f_3\tau_3 + f_4 Y \tag{2.29}$$

where the f_i are functions of space and time, and the generators λ_3, λ_8, τ_3, and Y are defined in Appendix B. Inserting this form of Φ into the $SU(5) \times Z_2$ Lagrangian in Eq. (2.1) we get

$$L = \frac{1}{2}\sum_{i=1}^{4}(\partial_\mu f_i)^2 + V(f_1, f_2, f_3, f_4) \tag{2.30}$$

and

$$\begin{aligned} V = &-\frac{m^2}{2}\sum_{i=1}^{4} f_i^2 + \frac{h}{4}\left(\sum_{i=1}^{4} f_i^2\right)^2 + \frac{\lambda}{8}\sum_{a=1}^{3} f_a^4 + \frac{\lambda}{4}\left[\frac{7}{30}f_4^4 + f_1^2 f_2^2\right] \\ &+ \frac{\lambda}{20}\left[4\left(f_1^2 + f_2^2\right) + 9f_3^2\right]f_4^2 + \frac{\lambda}{\sqrt{5}}f_2 f_4\left(f_1^2 - \frac{f_2^2}{3}\right) + \frac{m^2}{4}\eta^2 \end{aligned} \tag{2.31}$$

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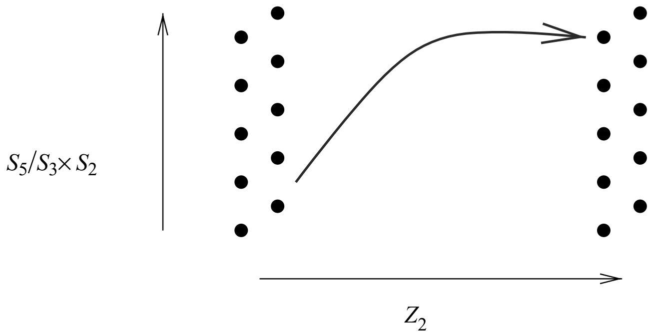

Figure 2.3 The vacuum manifold for the $S_5 \times Z_2$ model contains two sets of ten points related by the Z_2 symmetry. Kink solutions exist that interpolate between vacua related by Z_2 transformations and also between vacua within one set of ten points. The former correspond to the topological kinks in $SU(5) \times Z_2$ and the latter to the non-topological kinks in that model.

This model has the desired $S_5 \times Z_2$ symmetry because it is invariant under permutations of the diagonal elements of Φ, that is, under permutations of various linear combinations of f_i. The Z_2 symmetry is under $f_i \to -f_i$ for every i.

Symmetry breaking proceeds as in the $SU(5) \times Z_2$ case. The $S_5 \times Z_2$ symmetry is broken by a vacuum expectation value along the Y direction i.e. $f_4 \neq 0$. The residual symmetry group consists of permutations in the $SU(3)$ and $SU(2)$ blocks. Therefore the unbroken symmetry group is $H = S_3 \times S_2$. There are $5! \times 2 = 240$ elements of $S_5 \times Z_2$ and $3! \times 2! = 12$ elements of H. Therefore the vacuum manifold consists of $240/12 = 20$ distinct points. Ten of these points are related to the other ten by the non-trivial element of Z_2 as shown in Fig. 2.3. If we fix the boundary condition at $x = -\infty$, then a Z_2 kink can be obtained with ten different boundary conditions at $x = +\infty$. These ten solutions must somehow correspond to the kink solutions that we have already found in the $SU(5) \times Z_2$ case. Counting all the possible different diagonal possibilities for Φ_+ in the $SU(5) \times Z_2$ model we see that there are three $q = 2$ kinks, six $q = 1$ kinks, and one $q = 0$ kink, making a total of ten kinks. In the $S_5 \times Z_2$ model there are ten more (one of these is the trivial solution) kinks that do not involve the Z_2 transformation (change of sign) in going from Φ_- to Φ_+. These are the ten remnants of the non-topological kinks described in Section 2.3.

2.6 Symmetries within kinks

The symmetry groups outside the kink, $G_{321\pm}$, are isomorphic (see Fig. 2.4). However, the fields transform differently under the elements of these groups. As a result, there is a "clash of symmetries" [43] inside the kink, and the unbroken symmetry

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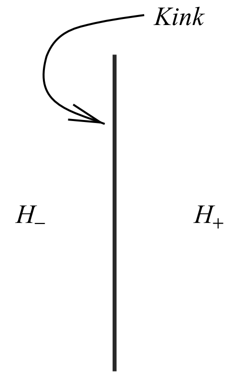

Figure 2.4 A kink and the symmetries outside denoted by $H_\pm$. The groups H_+ and H_- are isomorphic but their action on fields may not necessarily be identical.

group within the kink is generally *smaller* than that outside. This does not happen in the case of the Z_2 kink in which the symmetry outside is trivial while inside it is Z_2 (since the field vanishes). We now examine the clash of symmetries in the case of the $SU(5) \times Z_2$ $q = 2$ kink.

The general form of $\Phi_{\rm k}^{(2)}$ is given in Eq. (2.13) with the profile functions in Eq. (2.23). Then

$$\Phi_{\rm k}^{(2)}(x = 0) = M_+^{(2)} \propto \text{diag}(0, 1, 1, -1, -1) \tag{2.32}$$

The symmetries within the kink are given by the elements of $SU(5) \times Z_2$ that leave $M_+^{(2)}$ invariant. Hence the internal symmetry group consists of two $SU(2)$ factors, one for each block proportional to the 2×2 identity, and two $U(1)$ factors since all diagonal elements of $SU(5)$ commute with $M_+^{(2)}$. Therefore the symmetry group inside the $SU(5) \times Z_2$ kink is $[SU(2)]^2 \times [U(1)]^2$. This is *smaller* than the $SU(3) \times SU(2) \times U(1)$ symmetry group outside the kink.[3]

The conclusion that the symmetry inside a kink is smaller than that outside holds quite generally [164]. Classically this would imply that there are more massless particles outside the kink than inside it. However, when quantum effects are taken into account this classical picture can change because the fundamental states in the outside region may consist of confined groups of particles ("mesons" and "hadrons") that are very massive [51]. If a particle carries non-Abelian charge of a symmetry that is unbroken outside the wall but broken inside to an Abelian subgroup, it may cost less energy for the particle to live on the wall. This is because it may be

[3] As in Section 2.4 we could have found the symmetry group inside the kink by finding those transformations in G_{321-} that are also contained in G_{321+}.

unconfined inside the wall where it only carries Abelian charge, while it can only exist as a heavy meson or a hadron outside the wall.[4]

2.7 Interactions of static kinks in non-Abelian models

The interaction potential between kinks found in Section 1.8 is easily generalized to kinks in non-Abelian field theories. Following the procedure discussed in that section, the force in the $SU(5) \times Z_2$ case is

$$F = \frac{\mathrm{d}P}{\mathrm{d}t} = \left[-\mathrm{Tr}(\dot{\Phi}^2) - \mathrm{Tr}(\Phi'^2) + V(\Phi) \right]_{x_1}^{x_2} \tag{2.33}$$

where $-a - R$ and $-a + R$ are defined in Fig. 1.3. Evaluation of F yields an exponentially small interaction force whose sign depends on $\mathrm{Tr}(Q_1 Q_2)$ [121] where Q_1 and Q_2 are the topological charges of the kinks. If the Higgs field at $x = -\infty$ is Φ_-, between the two kinks is Φ_0, and is Φ_+ at $x = +\infty$, then $Q_1 \propto \Phi_0 - \Phi_-$ and $Q_2 \propto \Phi_+ - \Phi_0$ (see Eq. (1.8)).

What is most interesting about the interaction is that a kink and an antikink can repel. Here one needs to be careful about the meaning of an "antikink." An antikink should have a topological charge that is opposite to that of a kink. That is, a kink and its antikink together should be in the trivial topological sector. But this condition still leaves open several different kinds of antikinks for a given kink. To be specific consider a kink-antikink pair, where the Higgs field across the kink changes from $\Phi(-\infty) \propto +(2, 2, 2, -3, -3)$ to $\Phi(0) \propto -(2, -3, -3, 2, 2)$. (Here we suppress the normalization factor and the "diag" for convenience of writing.) There can be two types of antikinks to the right of this kink. In the first type (called Type I) the Higgs field can go from $\Phi(0) \propto -(2, -3, -3, 2, 2)$ to $\Phi(+\infty) \propto +(2, 2, 2, -3, -3)$, which is the same as the value of the Higgs field at $x = -\infty$ and thus reverts the change in the Higgs across the kink. In the second type (Type II), the Higgs field can go from $\Phi(0) \propto -(2, -3, -3, 2, 2)$ to $\Phi(+\infty) \propto +(-3, 2, 2, -3, 2)$. Now the Higgs at $x = +\infty$ is not the same as the Higgs at $x = -\infty$, but the two asymptotic field values are in the same topological sector.

By evaluating $\mathrm{Tr}(Q_1 Q_2)$, where Q_1 and Q_2 are the charge matrices of the two kinks, it is easy to check that the force between a kink and its Type I antikink is attractive, but the force between a kink and its Type II antikink is repulsive. The $q = 2$ kinks can have charge matrices $Q^{(i)}$ that we list up to a proportionality factor

$$Q^{(1)} = (-4, 1, 1, 1, 1), \qquad Q^{(2)} = (1, -4, 1, 1, 1), \qquad Q^{(3)} = (1, 1, -4, 1, 1),$$
$$Q^{(4)} = (1, 1, 1, -4, 1), \qquad Q^{(5)} = (1, 1, 1, 1, -4) \tag{2.34}$$

[4] Localization of particles to the interior of defects has led to the construction of cosmological scenarios where our observed universe is a three-dimensional defect or "brane" embedded in a higher dimensional space-time.

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Stable antikinks have the same charges but with a minus sign. Then, one can take a kink with one of the five charges listed above and it repels an antikink that has the -4 occurring in a different entry because $\mathrm{Tr}(Q_1 Q_2) > 0$. Hence, there are combinations of kinks and antikinks for which the interaction is repulsive. Further, in a statistical system a kink is most likely to have a Type II antikink as a neighbor and such a kink-antikink pair cannot annihilate since the force is repulsive.

The result that the force between two kinks is proportional to the trace of the product of the charges extends to other solitons (e.g. magnetic monopoles) as well. In this way, the forces between certain monopoles with equivalent magnetic charge can be attractive whereas normally we would think that like magnetic charges repel, and between certain monopoles and antimonopoles can be repulsive.

2.8 Kink lattices

In this section we describe the possibility of forming stable lattices of domain walls in one spatial dimension and the consequences in higher dimensions. Our discussion is in the context of the $S_5 \times Z_2$ model though similar structures have been seen in other field theory models as well [92, 43].

We know that Z_2 topology forces a kink to be followed by an antikink. Then we can set up a sequence of kinks and antikinks whose charges are arranged in the following way

$$\ldots Q^{(1)} \bar{Q}^{(5)} Q^{(3)} \bar{Q}^{(1)} Q^{(5)} \bar{Q}^{(3)} \ldots \tag{2.35}$$

where $Q^{(i)}$ and $\bar{Q}^{(i)}$ refer to a kink and an antikink of type i respectively (see Eq. (2.34)). Alternately, this sequence of kinks would be achieved with the following sequence of Higgs field vacuum expectation values (illustrated in Fig. 2.5)

$$\begin{aligned}
\ldots \to -(2,2,2,-3,-3) &\to +(2,-3,-3,2,2) \\
&\to -(-3,2,2,-3,2) \\
&\to +(2,-3,2,2,-3) \\
&\to -(2,2,-3,-3,2) \\
&\to +(-3,-3,2,2,2) \\
&\to -(2,2,2,-3,-3) \to \ldots
\end{aligned} \tag{2.36}$$

The forces between kinks fall off exponentially fast and hence the dominant forces are between nearest neighbors. As discussed in the previous section, the sign of the force between the ith soliton (kink or antikink) and the $(i+1)$th soliton (antikink or kink) is proportional to $\mathrm{Tr}(Q_i Q_{i+1})$ where Q_i is the charge of the ith object. For the sequence above, $\mathrm{Tr}(Q_i Q_{i+1}) > 0$ for every i and neighboring solitons repel each other. In particular, they cannot overlap and annihilate.

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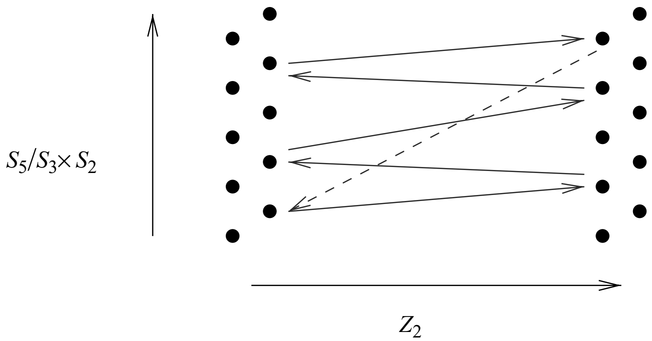

Figure 2.5 In the lattice of kinks of Eq. (2.36), the vacua are arranged sequentially in a pattern so as to return to the starting vacuum only after several transitions between the two discrete (Z_2) sectors.

The sequence of kinks in Eq. (2.35) has a period of six kinks. These six kinks have a net topological charge that vanishes since the last vacuum expectation value in Eq. (2.36) is the same as the first value. Hence we can put the sequence in a periodic box, i.e. compact space. This gives us a finite lattice of kinks.

The sequence described above has the minimum possible period (namely, six). It is easy to construct other sequences with greater periodicity. For example

$$\ldots Q^{(1)}\bar{Q}^{(5)}Q^{(3)}\bar{Q}^{(4)}Q^{(2)}\bar{Q}^{(1)}Q^{(5)}\bar{Q}^{(3)}Q^{(4)}\bar{Q}^{(2)}\ldots \tag{2.37}$$

is a repeating sequence of ten kinks.

The lattice of kinks is a solution in both the $S_5 \times Z_2$ and the $SU(5) \times Z_2$ models. However, it is stable in the former and unstable in the latter. The instability in the $SU(5) \times Z_2$ model occurs because a kink of a given charge, say $Q^{(3)}$, can change with no energy cost into a kink of some other charge, for example $Q^{(1)}$. Then, in the sequence of Eq. (2.35), the third kink changes into $Q^{(1)}$, then annihilates with the antikink with charge $\bar{Q}^{(1)}$ on its right. In this way the lattice can relax into the vacuum. In the $S_5 \times Z_2$ case, however, the degree of freedom that can change the charge of a kink is absent and the lattice is stable.

So far we have been discussing a kink lattice in one periodic dimension. This is equivalent to having a kink lattice in a circular space. Next consider what happens in a plane in two spatial dimensions. A circle in this plane can once again have a kink lattice since neighboring kinks and antikinks repel. However, when extended to the whole plane, the kink lattice must have a nodal point as shown in Fig. 2.6. In three spatial dimensions, the nodal points must extend into nodal curves.[5]

We shall discuss kink lattices further in Chapter 6.

[5] This is very similar to the case where several domain walls terminate on topological strings, except that there are no topological strings in the model.

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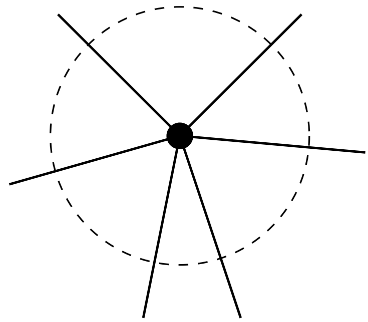

Figure 2.6 A domain wall lattice consisting of six domain walls can be formed in a one-dimensional sub-space (dashed circle) of a two-dimensional plane. This domain wall lattice is stable. Extending it to the two-dimensional plane, the different domain walls converge to a nodal point. This implies that the $S_5 \times Z_2$ model contains domain wall nodes (or junctions) in two dimensions and nodal curves in three spatial dimensions.

2.9 Open questions

1. Discuss all topological and non-topological kink solutions in an $SU(N) \times Z_2$ model where N is even. In [163] the case with odd N is discussed.[6]

[6] However, it is incorrectly stated that the Z_2 symmetry is included in $SU(N)$ when N is even, as can be seen from the $\mathrm{Tr}(\Phi^3)$ argument of Section 2.1.

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