ON HOMOMORPHIC IMAGES OF SPECIAL JORDAN ALGEBRAS

P. M. COHN

1. Introduction. A linear algebra is called a *Jordan algebra* if it satisfies the identities

$$(1) \qquad ab = ba, \quad (a^2b)\,a = a^2(ba).$$

It is well known that a linear algebra S over a field of characteristic different from two is a Jordan algebra if there is an isomorphism $a \to a'$ of the vector-space underlying S into the vector-space of some associative algebra A such that

$$(ab)' = \tfrac{1}{2}(a' \cdot b' + b' \cdot a'),$$

where the dot denotes the multiplication in A. Such an algebra S is called a *special Jordan algebra*. As has been proved by Albert **(1)**, there exist Jordan algebras which are not special and this raises the problem of characterizing the special Jordan algebras within the class of all Jordan algebras. In particular one may ask: Do the special Jordan algebras satisfy any identity which is not a consequence of (1)? This raises the further question whether the class of special Jordan algebras can be defined by identities alone. We are concerned in this note with finding an answer to this second question.[1]

A class $\mathfrak{C}$ of abstract algebras can be defined by identities if and only if it is closed under the operations of taking subalgebras, direct unions and homomorphic images **(5)**. We call such a class a *variety* of algebras, following P. Hall. It follows from the definition that the Jordan algebras form a variety, $\mathfrak{J}$ say. Further, denote by $\mathfrak{S}$ the class of special Jordan algebras. Then it is easily shown that $\mathfrak{S}$ is closed under the operations of taking subalgebras and direct unions **(6)**, but, as we shall prove in §6, $\mathfrak{S}$ is not closed under the operation of taking homomorphic images and is therefore *not* a variety. The class of all homomorphic images of special Jordan algebras is again a variety, $\mathfrak{T}$ say; clearly $\mathfrak{T}$ is the "smallest" variety including all the special Jordan algebras and we have the trivial relation

$$(2) \qquad \mathfrak{S} \subseteq \mathfrak{T} \subseteq \mathfrak{J}.$$

We shall prove in §5 that every algebra in $\mathfrak{T}$ on at most 2 generators is also in $\mathfrak{S}$, and give a criterion for deciding when a 3-generator algebra in $\mathfrak{T}$ is in $\mathfrak{S}$. With this criterion it is easy to construct algebras which are in $\mathfrak{T}$ but not in $\mathfrak{S}$;

Received December 4, 1952; in revised form July 29, 1953.

[1]The relation between Jordan algebras and special Jordan algebras has been completely described by Albert **(2)** in the special case of finite-dimensional semisimple algebras over a field of characteristic zero. We attack the problem from the other end by looking for partial results for the whole class of special Jordan algebras. Therefore the present work does not overlap Albert's.

we give an example in §6. Thus we obtain a Jordan algebra which is a homomorphic image of a special Jordan algebra, but which is not itself special. In §7 the peculiar difficulties of extending the methods of this note to algebras on more than 3 generators are briefly discussed, and it is shown that they cannot be extended without serious modification.

We note that the restriction on the field is essential, since over a field of characteristic two the Jordan product (in the form $xy + yx$) reduces to the Lie product: $xy - yx$, and it is well known that the Lie algebras derived in this way from associative algebras form a variety whatever the number of generators, so that $\mathfrak{S} = \mathfrak{T}$ in this case.[2]

2. $\mathfrak{T}$-algebras. In the whole of this note all algebras will be taken over a fixed but arbitrary field of characteristic $\neq 2$, so that we shall not refer to it explicitly unless necessary.

Let A be an associative algebra and define the *Jordan product* of two elements $x, y \in A$ by

$$\langle x, y\rangle = \tfrac{1}{2}(xy + yx).$$

The set A may be regarded as an algebra with respect to addition and Jordan multiplication and it will then be denoted by $\langle A\rangle$. If U is any subspace of A, we denote by $\langle U\rangle$ the subalgebra of $\langle A\rangle$ generated by U.

Now let A_0 be the free associative algebra on the free generators $x_\lambda (\lambda \in \Lambda)$. We denote by Φ the subspace spanned by the x's and refer to A_0 as the *free associative algebra on* Φ. It is clear that A_0 is uniquely determined by Φ, the x's being a basis of the space Φ. We denote by J_0 the algebra $\langle\Phi\rangle$, so that J_0 is a special Jordan algebra. The elements of J_0 are just the Jordan polynomials in the x's **(8)** and may be called the *Jordan elements* of A_0.

It is easily proved that every special Jordan algebra is a homomorphic image of J_0, for a suitable Λ, and hence every homomorphic image of a special Jordan algebra is a homomorphic image of J_0. Let us call an algebra a *$\mathfrak{T}$-algebra*, if it is a homomorphic image of a special Jordan algebra, i.e., if it is in $\mathfrak{T}$. Further we shall say that a $\mathfrak{T}$-algebra is *special* if it is a special Jordan algebra. Then we can state the result as

THEOREM 2.1.[3] *Every $\mathfrak{T}$-algebra is a homomorphic image of J_0, for a suitable* Λ.

The theorem can also be expressed by saying that J_0 is the free algebra of the variety $\mathfrak{T}$.

Let T be any $\mathfrak{T}$-algebra. By Theorem 2.1, $T \cong J_0/\mathfrak{a}$, where $\mathfrak{a}$ is an ideal of J_0; therefore T is defined up to isomorphism by J_0 and $\mathfrak{a}$. We require a criterion for deciding when T is special. Since $J_0 \subseteq A_0$, $\mathfrak{a}$ is contained in A_0, but of course it is not in general an ideal of A_0. We denote by $\{\mathfrak{a}\}$ the ideal of A_0 generated by the set $\mathfrak{a}$. Then we have

[2]This follows from the Birkhoff-Witt embedding theorem **(4; 10)**.

[3]Theorem 2.1 is practically a restatement, for the case of special Jordan algebras, of Theorem 1 in **(6)**.

THEOREM 2.2. *Let* $\mathfrak{a}$ *be an ideal of* J_0. *Then* $J_0/\mathfrak{a}$ *is a special Jordan algebra if and only if* $\{\mathfrak{a}\} \cap J_0 \subseteq \mathfrak{a}$.

We note that since $\mathfrak{a} \subseteq J_0$ and $\mathfrak{a} \subseteq \{\mathfrak{a}\}$, we have in any case $\mathfrak{a} \subseteq \{\mathfrak{a}\} \cap J_0$, so that the theorem asserts in fact equality in case $J_0/\mathfrak{a}$ is special.

Proof. Suppose $\mathfrak{a}$ is special. Then, by definition, $J_0/\mathfrak{a}$ can be embedded in an associative algebra, A say. Denote the natural homomorphism of J_0 onto $J_0/\mathfrak{a}$ by $u \to \bar{u}$, then it is clear that the elements $\bar{x}_\lambda$ generate $J_0/\mathfrak{a}$. Now consider the mapping $\phi : x_\lambda \to \bar{x}_\lambda$. Since A_0 is free associative on the x_λ, we can extend ϕ to a homomorphism of A_0 into A, which we again denote by ϕ. Let $\mathfrak{b}$ be the kernel of the homomorphism ϕ, then $\mathfrak{b}$ is an ideal of A_0. Clearly ϕ induces a homomorphism of J_0 onto the Jordan algebra generated by the $\bar{x}_\lambda$, which is just $J_0/\mathfrak{a}$ by definition, and since ϕ maps x_λ into $\bar{x}_\lambda$, it follows that ϕ coincides on J_0 with the natural homomorphism onto $J_0/\mathfrak{a}$. We shall prove

$$\text{(i)} \quad \{\mathfrak{a}\} \subseteq \mathfrak{b}, \qquad \text{(ii)} \quad \mathfrak{b} \cap J_0 \subseteq \mathfrak{a}.$$

From this it then follows that $\{\mathfrak{a}\} \cap J_0 \subseteq \mathfrak{a}$.

To prove (i) let $u \in \mathfrak{a}$, then $\bar{u} = 0$ by definition, and since ϕ coincides with the natural homomorphism on J_0, we have $u^\phi = 0$, which means that $u \in \mathfrak{b}$. Hence $\mathfrak{a} \subseteq \mathfrak{b}$, and since $\mathfrak{b}$ is an ideal of A_0 it follows that $\{\mathfrak{a}\} \subseteq \mathfrak{b}$.

To prove (ii) we take u in $\mathfrak{b} \cap J_0$, then, since $u \in \mathfrak{b}$, $u^\phi = 0$, and because $u \in J_0$ this means that u lies in the kernel of the natural homomorphism, i.e. $u \in \mathfrak{a}$. This proves (ii), and hence $\{\mathfrak{a}\} \cap J_0 \subseteq \mathfrak{a}$.

Conversely, suppose that $\{\mathfrak{a}\} \cap J_0 \subseteq \mathfrak{a}$; then we have equality by the remark which precedes the proof. We consider the associative algebra $A = A_0/\{\mathfrak{a}\}$. Denote by $\bar{x}_\lambda$ the image of x_λ under the natural homomorphism of A_0 onto $A_0/\{\mathfrak{a}\}$ and let J be the subalgebra of $\langle A \rangle$ generated by the $\bar{x}_\lambda$. Then J is the homomorphic image of J_0 under the natural homomorphism $A_0 \to A_0/\{\mathfrak{a}\}$ and therefore $J = J_0/J_0 \cap \{\mathfrak{a}\}$, which equals $J_0/\mathfrak{a}$ by hypothesis. Therefore $J_0/\mathfrak{a} = J$ is a special Jordan algebra, and this completes the proof.

The concept of an associative algebra and the special Jordan algebras embedded in it can be generalized to the case of an abstract algebra A and subsets of A which are closed under certain combinations of the operators of A **(6)**. Both the theorems of this section can be extended without difficulty to this general case.

3. The reversal operator. Let A_0 again be the free associative algebra on the space with the basis x_λ ($\lambda \in \Lambda$). We define the *reversal operator* j on A_0 as the linear mapping of A_0 into itself given by

$$x^j_\lambda = x_\lambda, \qquad \lambda \in \Lambda,$$
$$(uv)^j = v^j \, . \, u^j, \qquad u, v \in A_0.$$

These equations together with linearity define j completely. It is clear that j is an involution; in fact j is essentially the fundamental involution on the universal associative enveloping algebra of J_0 **(7)**. If $u \in A_0$, then u^j will be called the *reverse* of u, and u is said to be *reversible*, if $u^j = u$. The set of all reversible elements of A_0 forms a subalgebra of $\langle A_0 \rangle$, as is easily verified.

We define a second linear mapping $u \to u^*$ in A_0 by the rule:

$$u^* = \tfrac{1}{2}(u + u^j). \tag{3}$$

If $u \in A_0$, then u^* is reversible, and $u^* = u$ if and only if u is reversible.

LEMMA 3.1. *If $u, v \in A_0$, then*

$$\langle u^*, v \rangle^* = \langle u^*, v^* \rangle. \tag{4}$$

The lemma is proved by a straightforward calculation:

$$\langle u^*, v \rangle^* = \tfrac{1}{2}\{\langle u^*, v \rangle + \langle u^*, v^j \rangle\} = \tfrac{1}{2}\langle u^*, v + v^j \rangle = \langle u^*, v^* \rangle.$$

Since the right-hand side of (4) is symmetric in u and v, we also have

$$\langle u, v^* \rangle^* = \langle u^*, v^* \rangle. \tag{5}$$

The formula (5) bears a remarkable resemblance to Baker's formula for commutators in A_0. If $u \to u^\dagger$ denotes the operation of forming left-normed commutators in the free generators of A_0 and $[u, v] = uv - vu$, then Baker's formula **(3; 9)** states

$$(uv^\dagger)^\dagger = [u^\dagger, v^\dagger] \qquad \textit{for all } u, v \in A_0.$$

This operation † leaves a homogeneous element of A_0 unchanged except for a scalar factor, if and only if it is a Lie element (i.e. a sum of commutators in the x's; see e.g. **(9)**). As we shall see in §4, the operation * defined by (3) leaves an element of A_0 involving less than four generators unchanged if and only if it is a Jordan element.

Consider again the free associative algebra A_0 on the x's. Any given element w of A_0 is obtained from the x's by the operations of A_0 and we refer to w, considered as a function of the x's, as an *associative polynomial* in the x's. If w belongs to J_0 it can be formed from the x's by the operations of J_0, and as such it is called a *Jordan polynomial* **(8)**. The usual definitions of degree and homogeneity in the x's taken together or in any one of them can then be extended to associative and Jordan polynomials. If $f(\xi_1, \ldots, \xi_n)$ is a (Jordan or associative) polynomial in n variables, we define the *reverse* of f to be the polynomial f^j, where

$$f^j(x_1, \ldots, x_n) = [f(x_1, \ldots, x_n)]^j.$$

Since the x's are free generators, this defines f^j as a polynomial. We say that f is *reversible*, if $f^j = f$.

As long as we restrict ourselves to polynomials with the free generators x_λ as arguments there is no need for the new terminology, but we shall want to use it in the case where the arguments are not x's, but certain other elements in A_0. Thus, if $u_1, u_2, \ldots, u_n$ are given elements of A_0 and f is a reversible polynomial, then the element $f(u_1, \ldots, u_n)$ of A_0 is not necessarily reversible since the u's need not be so. To give an example, if $u = x_1 x_2$, then the element $x_1x_2x_3 + x_3x_1x_2$ is not reversible, but by writing it as $ux_3 + x_3u$ we can express it as a reversible polynomial in u and x_3.

LEMMA 3.2. *Let* $\mathfrak{b}$ *be an ideal of* A_0 *with a set of reversible elements* u_ι $(\iota \in \mathrm{I})$ *as generators. If* w *is a reversible element of* $\mathfrak{b}$ *then* w *can be written as a reversible associative polynomial in the* u'*s and* x'*s which is linear homogeneous in the* u'*s.*

Proof. Any element w of b can be written as $f(u, x)$, where f is a polynomial in the u's and x's in which each term has degree at least one in the u's. If any term of f contains more than one factor u, we express the surplus factors in terms of the x's, and in this way obtain w as a polynomial $g(u, x)$ which is linear homogeneous in the u's. The reverse polynomial $g^j(u, x)$ is again homogeneous linear in the u's and $\{g(u, x)\}^j = g^j(u^j, x) = g^j(u, x)$, since the u's are reversible. Using the fact that w is reversible we have

$$w = w^* = \tfrac{1}{2}\{g(u, x) + g^j(u, x)\},$$

and this is the required representation of w as a reversible associative polynomial which is linear homogeneous in the u's.

To illustrate the point of the lemma, let $u = x^2$, then $w = ux$ is the value of the associative polynomial $f(\xi, \eta) = \xi\eta$ for $\xi = u$, $\eta = x$, and f is not reversible, but w can also be written as $\frac{1}{2}(ux + xu)$ and this expresses w as a reversible associative polynomial which is linear homogeneous in u.

4. The connection between reversible and Jordan elements.

Let R be the set of reversible elements of A_0. Then R is a subalgebra of $\langle A_0 \rangle$, and since R contains the generators x_λ, it contains J_0. Thus every Jordan element of A_0 is reversible.[4]

The question naturally arises under what circumstances J_0 equals R. We shall prove now that this is the case when the number of free generators of A_0 is less than four and only then. To obtain the best possible results it is convenient to suppose that the index-set Λ is totally ordered; so in order to avoid unnecessary detail we shall from now on take the free generators of A_0 to be well-ordered and write them as $x_1, x_2, \ldots$ without specifying the index-set unless this is relevant.

THEOREM 4.1. *Every reversible element of* A_0 *can be expressed as a Jordan polynomial in the generators* $x_1, x_2, \ldots$ *and the elements*

$$(x_{i_1}x_{i_2}x_{i_3}x_{i_4})^* \qquad i_1 < i_2 < i_3 < i_4;\ i = 1, 2, \ldots . \tag{6}$$

The expression (6) will be called a *tetrad*, more precisely the tetrad defined by

$$x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4}.$$

[4]Since A_0 is the universal associative enveloping algebra of J_0, this is a special case of a result by Jacobson and Rickart **(8)**.

We shall refer to (6) as a tetrad even if the indices are not ascending, but we insist on their being distinct.

We prove Theorem 4.1 by induction on the degree of the element, considered as an associative polynomial in the x's. If the reversible element w has degree n, the terms of highest degree will be of the form

$$\sum a_{i_1 \ldots i_n} (x_{i_1} \ldots x_{i_n})^*. \tag{7}$$

Let us denote the subalgebra of $\langle A_0 \rangle$ generated by the x's and by the elements (6) by S for the moment; it will be enough to prove (under the induction hypothesis) that

$$(x_{i_1} \ldots x_{i_n})^* \in S.$$

For brevity we shall write this as a congruence

$$(x_{i_1} \ldots x_{i_n})^* \equiv 0, \tag{8}$$

where the modulus S is always understood. If $n = 1$, (8) holds by definition of S. Now let $n > 1$. By the induction hypothesis,

$$(x_{i_2} \ldots x_{i_n})^* \equiv 0.$$

Because S is a subalgebra of $\langle A_0 \rangle$, it follows that

$$\langle x_{i_1}, (x_{i_2} \ldots x_{i_n})^* \rangle \equiv 0,$$

and so by applying Lemma 3.1, we get

$$\tfrac{1}{2}\{ (x_{i_1} x_{i_2} \ldots x_{i_n})^* + (x_{i_2} \ldots x_{i_n} x_{i_1})^* \}$$
$$= \langle x_{i_1}, x_{i_2} \ldots x_{i_n} \rangle^* = \langle x_{i_1}, (x_{i_2} \ldots x_{i_n})^* \rangle \equiv 0.$$

Thus

$$(x_{i_1} \ldots x_{i_n})^* \equiv - (x_{i_2} \ldots x_{i_n} x_{i_1})^*. \tag{9}$$

The congruence (9) states that the left-hand side of (8) changes sign (mod S) under a cyclic permutation of the suffixes. If n is odd then by repeating this operation n times we get

$$(x_{i_1} \ldots x_{i_n})^* \equiv - (x_{i_1} \ldots x_{i_n})^*,$$

and hence (8). If n is even we apply Lemma 3.1 with

$$u = x_{i_1} x_{i_2}, \quad v = x_{i_3} \ldots x_{i_n}$$

and obtain similarly

$$0 \equiv \langle (x_{i_1} x_{i_2})^*, \quad x_{i_3} \ldots x_{i_n} \rangle^*$$
$$\equiv \tfrac{1}{4}\{ (x_{i_1} x_{i_2} x_{i_3} \ldots x_{i_n})^* + (x_{i_2} x_{i_1} x_{i_3} \ldots x_{i_n})^* + (x_{i_3} \ldots x_{i_n} x_{i_1} x_{i_2})^*$$
$$+ (x_{i_3} \ldots x_{i_n} x_{i_2} x_{i_1})^* \}.$$

By (9) we can apply two cyclic interchanges to each of the third and fourth terms without affecting their values (mod S). This will change them into the first and second term respectively, so that

$$\tfrac{1}{2}\{(x_{i_1}\dots x_{i_n})^* + (x_{i_2}x_{i_1}x_{i_3}\dots x_{i_n})^*\} \equiv 0.$$

Multiplying by two and transposing, we obtain

$$(x_{i_1}x_{i_2}x_{i_3}\dots x_{i_n})^* \equiv -(x_{i_2}x_{i_1}x_{i_3}\dots x_{i_n})^*. \tag{10}$$

By (9) and (10) the left-hand side of (8) changes sign (mod S) if we apply the permutations $(1\,2\dots n)$ or $(1\,2)$ to the indices. These two permutations are odd (since n is even) and they generate the symmetric group on n letters, so it follows that if we apply any permutation to the indices on the left-hand side of (8), this has the effect of multiplying it by ± 1 (mod S) according to whether the permutation is even or odd. In particular if two indices are equal, the expression is $\equiv 0$ and (8) follows.

To complete the proof we distinguish three cases:

(i) $n = 2$. Then

$$(x_{i_1}x_{i_2})^* = \langle x_{i_1}, x_{i_2}\rangle \equiv 0.$$

(ii) $n = 4$. We need only consider the case where the indices are distinct. By at most changing the sign we can arrange the indices in any given tetrad in ascending order and then it is $\equiv 0$ by the definition of S.

(iii) $n \geqslant 6$. We apply Lemma 3.1 with

$$u = x_{i_1}x_{i_2}x_{i_3}x_{i_4}, \quad v = x_{i_5}\dots x_{i_n}.$$

This gives

$$\tfrac{1}{4}\{(x_{i_1}\dots x_{i_4}x_{i_5}\dots x_{i_n})^* + (x_{i_4}\dots x_{i_1}x_{i_5}\dots x_{i_n})^* + (x_{i_5}\dots x_{i_n}x_{i_1}\dots x_{i_4})^* + (x_{i_5}\dots x_{i_n}x_{i_4}\dots x_{i_1})^*\} \equiv 0.$$

Each of the second, third, and fourth terms differs from the first by an even permutation and hence we obtain

$$(x_{i_1}x_{i_2}\dots x_{i_n})^* \equiv 0,$$

i.e. (8). Hence the expression (7) is in S and by subtracting it from w we can reduce the degree to $n - 1$. By induction it then follows that $w \in S$ and this completes the proof.

Since any tetrad of the form (6) must contain four distinct x's, there can be no such tetrads when the number of free generators is less than four. Hence we deduce

THEOREM 4.2. *If the number of free generators is less than four then every reversible element is in* J_0.

The following scholium, noted already in **(7)**, without proof, shows that Theorem 4.1 is best possible.

SCHOLIUM 4.3. *If the number of free generators of* A_0 *is* $\geqslant 4$ *and if* $i_1 < i_2 < i_3 < i_4$, *then*

$$(x_{i_1}x_{i_2}x_{i_3}x_{i_4})^* \notin J_0.$$

In the proof we can clearly confine ourselves to the case where the number of free generators of A_0 is just 4. Thus we have to prove that $(x_1x_2x_3x_4)^* \notin J_0$, and this will follow if we can prove it in any homomorphic image of A_0. Let A be the associative algebra on x_1, x_2, x_3, x_4 with the relations[5]

$$x_ix_j + x_jx_i = 0 \qquad i, j = 1, 2, 3, 4.$$

Then A has a basis consisting of the elements

$$x_{i_1} \ldots x_{i_r}, \qquad i_1 < \ldots < i_r, \quad r = 1, 2, 3, 4.$$

The Jordan algebra generated by the x's, J say, is spanned by x_1, x_2, x_3, x_4. But $(x_1x_2x_3x_4)^* = x_1x_2x_3x_4 \notin J$ and the scholium follows.

5. The embedding of 2- and 3-generator algebras. The theorems of §4 enable us to find necessary and sufficient conditions for $\mathfrak{T}$-algebras on at most 3 generators to be special, and to prove that every 2-generator $\mathfrak{T}$-algebra is special. Of course the case of a 1-generator $\mathfrak{T}$-algebra is trivial, since any Jordan product of degree n in a single generator x is just x^n. This follows by induction from the formula

$$\langle x^i, x^j \rangle = x^{i+j}.$$

THEOREM 5.1. *Let A_0 be the free associative algebra on the free generators x, y, z, and J_0 the special Jordan-algebra on x, y, z as before. If $u_1, u_2, \ldots$ are any elements of J_0 and $\mathfrak{a}$ is the ideal of J_0 generated by them, then $J_0/\mathfrak{a}$ is special if and only if*

$$(u_ixyz)^* \in \mathfrak{a} \qquad i = 1, 2, \ldots . \tag{11}$$

Proof. By Theorem 2.2, $J_0/\mathfrak{a}$ is special if and only if $\{\mathfrak{a}\} \cap J_0 \subseteq \mathfrak{a}$. Since the number of free generators is 3, J_0 consists just of the reversible elements of A_0 (Theorem 4.2), and so $J_0/\mathfrak{a}$ is special if and only if every reversible element of $\{\mathfrak{a}\}$ is in $\mathfrak{a}$. It is clear that $(u_ixyz)^*$ is reversible and belongs to $\{\mathfrak{a}\}$, hence condition (11) is necessary. Conversely, suppose that (11) is satisfied and let w be any reversible element in $\{\mathfrak{a}\}$. By Lemma 3.2, w can be written as a reversible associative polynomial f in the u's and x, y, z which is linear homogeneous in the u's. We now regard $x, y, z, u_1, u_2, \ldots$ as independent. Because f is reversible, it can by Theorem 4.1 be expressed as a Jordan polynomial ϕ in $x, y, z, u_1, u_2, \ldots$ and tetrads involving these variables. Since f is linear in the u's, so is ϕ and therefore no tetrad can involve more than one u, but it must involve at least one, since the four arguments of a tetrad are distinct. By a permutation of the arguments any such tetrad can be reduced to the form $(u_i xyz)^*$ plus a Jordan polynomial in u_i, x, y, z. By hypothesis $(u_i xyz)^* \in \mathfrak{a}$, hence every term of ϕ has been reduced to a Jordan product with at least one factor in $\mathfrak{a}$, and it follows that ϕ itself is in $\mathfrak{a}$. Since $\phi = f = w$, and w is any reversible element of $\{\mathfrak{a}\}$, this shows that $\{\mathfrak{a}\} \cap J_0 \subseteq \mathfrak{a}$ if (11) holds and this completes the proof.

[5]Thus A is the Grassmann algebra on the x's. I am indebted to one of the referees for the idea of this proof.

https://doi.org/10.4153/CJM-1954-026-9 Published online by Cambridge University Press

In the next section we shall construct an ideal which fails to satisfy this criterion and therefore defines a non-special $\mathfrak{T}$-algebra. First we deduce what happens in the case of two generators.

THEOREM 5.2. *Any $\mathfrak{T}$-algebra on two generators is special.*

Proof. We have to show that if A_0 is free associative on x, y and J_0 the special Jordan algebra on x, y, then every homomorphic image of J_0 is special. Let $\mathfrak{a}$ be any ideal of J_0 and $u_1, u_2, \ldots$ a set of generators of $\mathfrak{a}$. Following the proof of Theorem 5.1 we can express any reversible element w of $\{\mathfrak{a}\}$ as a Jordan polynomial in $x, y, u_1, u_2, \ldots$ and the tetrads in these variables. Moreover, this polynomial may be taken to be linear homogeneous in the u's, therefore no tetrads can occur, since now any tetrad must involve at least two distinct u's. Thus w can be written as a Jordan polynomial in $x, y, u_1, u_2, \ldots$ which is linear homogeneous in the u's and it follows that $w \in \mathfrak{a}$. Hence $\{\mathfrak{a}\} \cap J_0 \subseteq \mathfrak{a}$; this shows that $J_0/\mathfrak{a}$ is special and the proof is complete.

6. Example of a non-special $\mathfrak{T}$-algebra.

In this section we construct an ideal of J_0 which does not satisfy the conditions of Theorem 5.1 and therefore defines a quotient algebra of J_0 which is not special.

We take the free generators of A_0 to be x, y, z and consider the element

$$u = \langle x, x\rangle - \langle y, y\rangle \qquad (= x^2 - y^2).$$

Let $\mathfrak{a}$ be the ideal of J_0 generated by u. We shall show that $(uxyz)^* \notin \mathfrak{a}$. Then $J_0/\mathfrak{a}$ will be non-special by Theorem 5.1.

Suppose that $w = (uxyz)^* \in \mathfrak{a}$. Then w can be written as a Jordan polynomial $\phi(u, x, y, z)$ and by an argument similar to that used in proving Lemma 3.2 we may suppose ϕ to be linear homogeneous in u. Let $\phi_n(u, x, y, z)$ be the sum of the terms of ϕ which are homogeneous of degree n in the last three arguments. Then

$$(uxyz)^* = \sum_n \phi_n(u, x, y, z).$$

Since A_0 is free, we may equate the homogeneous terms of degree n in x, y, z and this gives (because ϕ is linear homogeneous in u, and u is homogeneous in x, y, z)

$$(12) \qquad \begin{aligned} (uxyz)^* &= \phi_3(u, x, y, z), \\ \phi_n(u, x, y, z) &= 0 \qquad \text{for } n \neq 3. \end{aligned}$$

Now we decompose ϕ into parts which are homogeneous in each of its arguments: Let

$$\phi(u, x, y, z) = \sum \phi_{ijk}(u, x, y, z),$$

where ϕ_{ijk} is homogeneous of degree i, j, k respectively in the second, third and fourth argument. This process of picking out the homogeneous terms of a Jordan polynomial ϕ gives again a Jordan polynomial ϕ_{ijk} because a homogeneous Jordan polynomial, when considered as an associative polynomial, is still homogeneous.

We consider the homogeneous terms of degree 3, 1, 1 respectively in x, y, z in (12). Equating such terms, we get

$$(x^3yz)^* = \phi_{111}(x^2, x, y, z). \tag{13}$$

Similarly, by equating the terms of degree 1, 3, 1 in x, y, z we get

$$(y^2xyz)^* = \phi_{111}(y^2, x, y, z), \tag{14}$$

while ϕ_{ijk} with $(i, j, k) \neq (1, 1, 1)$ does not contribute to the result. Let us write

$$f(u, x, y, z) = (uxyz)^* - \phi_{111}(u, x, y, z),$$

where we regard u, x, y, z as independent. The associative polynomial f is linear homogeneous in each of its four arguments. Since ϕ_{111} is a Jordan polynomial, f is reversible; and it vanishes for $u = y^2 - x^2$, but does not vanish identically, because mod J_0 it is congruent to $(uxyz)^*$. From the vanishing of f for $u = x^2 - y^2$ we get

$$f(x^2, x, y, z) = f(y^2, x, y, z) = 0, \tag{15}$$

which is just another way of expressing the equations (13) and (14). If we treat u again as a fourth free variable, then $f(u, x, y, z)$, as a reversible linear homogeneous polynomial in u, x, y, z, must be a linear combination of the elements

$$\begin{aligned} v_1 &= (uxyz)^*, & v_2 &= (uxzy)^*, & v_3 &= (uyxz)^*, & v_4 &= (uyzx)^*, \\ v_5 &= (uzxy)^*, & v_6 &= (uzyx)^*, & v_7 &= (xuyz)^*, & v_8 &= (xuzy)^*, \\ v_9 &= (yuxz)^*, & v_{10} &= (yuzx)^*, & v_{11} &= (zuxy)^*, & v_{12} &= (zuyx)^*. \end{aligned}$$

For these twelve expressions form a basis for all the reversible elements which are linear homogeneous in u, x, y, z. Let $f(u, x, y, z) = \sum a_i v_i$, where the a's are in the underlying field. Then the equation $f(x^2, x, y, z) = 0$ implies that

$$a_1 + a_7 = a_2 + a_8 = a_9 + a_{11} = 0; \quad a_3 = a_4 = a_5 = a_6 = a_{10} = a_{12} = 0;$$

and $f(y^2, x, y, z) = 0$ implies that

$$a_3 + a_9 = a_4 + a_{10} = a_7 + a_{12} = 0; \quad a_1 = a_2 = a_5 = a_6 = a_8 = a_{11} = 0.$$

These equations together show that all the coefficients a_i vanish and hence $f(u, x, y, z)$ must vanish identically, which is a contradiction. Therefore $(uxyz)^* \notin \mathfrak{a}$ and so $J_0/\mathfrak{a}$ is not special. Expressing this result in terms of $\mathfrak{T}$-algebras we have

SCHOLIUM 6.1. *If J is the $\mathfrak{T}$-algebra*[6] *on three generators x, y, z with the single defining relation $x^2 = y^2$, then J is non-special, i.e., J cannot be embedded in an associative algebra.*

It is of interest to note that although the number of generators must be at least three (by Theorem 5.2), the defining relation involves only two of them. This means that the non-special $\mathfrak{T}$-algebra J of Scholium 6.1 is actually the

[6]The multiplication in J is here denoted by juxtaposition.

free product of two special Jordan algebras, namely the $\mathfrak{T}$-algebra on x and y with the single defining relation $x^2 = y^2$ and the free $\mathfrak{T}$-algebra on z.

The $\mathfrak{T}$-algebra J described in Scholium 6.1 also provides an example of a 3-generator $\mathfrak{T}$-algebra which cannot be embedded in a 2-generator $\mathfrak{T}$-algebra, for by Theorem 5.2 any such $\mathfrak{T}$-algebra can be embedded in an associative algebra A and this would provide an embedding of J in A.

7. $\mathfrak{T}$-algebras on more than four generators. The conditions of Theorem 5.1 can in principle be applied to any 3-generator $\mathfrak{T}$-algebra with a finite set of defining relations, but this may be a non-trivial problem for some $\mathfrak{T}$-algebras, and it is to be expected that any set of conditions for $\mathfrak{T}$-algebras on four or more generators will be equally if not more difficult to apply, and therefore less useful. Such conditions, if they exist, must be essentially different from the one given in Theorem 5.1. We shall briefly indicate the reason for this fact.

The criterion of Theorem 5.1 depends on the fact that for less than four generators the Jordan elements can be characterized by means of the reversal operator j and the crucial point is the application of Lemma 3.2 which states roughly that a reversible element of A_0 which is expressible as a polynomial of given degree in certain reversible elements of A_0 can also be expressed as a reversible polynomial of the same degree in these elements. More precisely, we can say that * is an idempotent operator permuting the places, such that, if $f(\xi_1, \xi_2, \ldots ; \eta_1, \eta_2, \ldots)(=f(\xi, \eta)$ for short) is any associative polynomial linear homogeneous in the η's, and f^* the polynomial defined by

$$f^*(\xi, \eta) = [f(\xi, \eta)]^*,$$

then

$$f(x, u)^* = f^*(x, u), \tag{16}$$

where the x's are free generators and the u's are reversible elements in the x's.

If there were a criterion similar to that of Theorem 5.1 for 4-generator $\mathfrak{T}$-algebras, with * replaced by a different idempotent operator permuting the places, or even by a series of such idempotent place-permutation operators, one for each set of homogeneous Jordan elements of a given degree,[7] then these operators would satisfy an analogue of Lemma 3.2 with $u_1, u_2, \ldots$ all homogeneous of the same degree in the free generators, and this would require (16) to hold for the new operators (with f homogeneous of the appropriate degree). From this it would follow that, for any Jordan element u homogeneous in x, y, z, the element

$$(uxyz)^*(=\tfrac{1}{2}(uxyz + zyxu))$$

satisfies the criterion, because qua polynomial in x, y, z it is a Jordan element by Theorem 4.2. Hence it could then be expressed as a Jordan polynomial in

[7]Such idempotent place-permutation operators exist in the case of Lie elements for homogeneous elements of any degree (cf. the remark after Lemma 3.1), but they do not give a characterization of Lie algebras which are embeddable in associative algebras because these operators do not satisfy the analogue of (16). We note however that Lie-algebras are embeddable in associative algebras in any case (cf. footnote 2).

u, x, y, z. But this contradicts the example constructed in §6, where $u = x^2 - y^2$. Therefore such idempotent place-permutation operators cannot exist.

I should like to thank Drs. B. H. Neumann and G. Higman for reading the manuscript and suggesting improvements. I also wish to express my gratitude to the Centre National de la Recherche Scientifique and the British Council for a Research Fellowship to France, where I began the work embodied in this note. I am especially grateful to Professor J. Dieudonné for the many instructive discussions I had with him on this and other subjects.

References

1. A. A. Albert, *On a certain algebra of quantum mechanics*, Ann. Math., *35* (1934), 65–73.

2. ———, *A structure theory for Jordan-algebras*, Ann. Math., *48* (1947), 546–567.

3. H. F. Baker, *Alternants and continuous groups*, Proc. Lond. Math. Soc. (2), *3* (1905), 24–47.

4. G. Birkhoff, *Representability of Lie-algebras and Lie-groups by matrices*, Ann. Math., *38* (1937), 526–532.

5. ———, *Universal algebra*, Proc. Can. Math. Congress (Montreal 1946), 310–326.

6. G. Birkhoff and P. M. Whitman, *Representation of Jordan and Lie algebras*, Trans. Am. Math. Soc., *65* (1949), 116–136.

7. F. D. Jacobson and N. Jacobson, *Classification and representation of semisimple Jordan algebras*, Trans. Am. Math. Soc., *65* (1949), 141–169.

8. N. Jacobson and C. E. Rickart, *Jordan homomorphisms of rings*, Trans. Am. Math. Soc., *69* (1950), 479–502.

9. F. Wever, *Invarianten in Lieschen Ringen*, Math. Ann., *120* (1949), 563–580.

10. E. Witt, *Treue Darstellung Liescher Ringe*, J. reine angew. Math., *177* (1937), 152–160.

The University, Manchester