SUMMABILITY METHODS ON MATRIX SPACES

JOSEPHINE MITCHELL

§1. **Introduction.** The matrix spaces under consideration are the four main types of irreducible bounded symmetric domains given by Cartan **(5)**. Let $z = (z_{jk})$ be a matrix of complex numbers, z' its transpose, z^* its conjugate transpose and $I = I^{(n)}$ the identity matrix of order n. Then the first three types are defined by

$$D = [z \mid I - zz^* > 0], \tag{1}$$

where z is an n by m matrix $(n \leqslant m)$, a symmetric or a skew-symmetric matrix of order n **(16)**. The fourth type is the set of complex spheres satisfying

$$|z'z| < 1,\ 1 - 2z^*z + |z'z|^2 > 0,$$

where z is an n by 1 matrix. It is known that each of these domains possesses a distinguished boundary B which in the first three cases is given by

$$B = [u \mid uu^* = I]. \tag{2}$$

(In the case of skew symmetric matrices the distinguished boundary is given by (2) only if n is even.)

In § 2 we consider the following problem for the first type of domain with $m = n$, in which case u is a unitary matrix, the (real) dimension of B is n^2 and of D is $2n^2$. Let $f(u)$ be a real integrable function defined on B and consider the integral operator

$$I(f, z) = \int_B P(z, u) f(u)\, dV, \tag{3}$$

where $P(z, u)$ is the Poisson kernel **(14)**

$$P(z, u) = V^{-1} \det{}^n (I - zu^*)^{-1} (I - zz^*) (I - uz^*)^{-1}, \tag{4}$$

V is the Euclidean volume of B, and dV the Euclidean volume element. It is known that $I(f, z)$ is a harmonic function of z if $I - zz^* > 0$ or $I - zz^* < 0$. (I proved this fact in **(14)** for $z \in D$ but the proof is valid for all z and $u \in B$ for which $\det(I - zu^*) \neq 0$. It is easily proved that $\det(I - zu^*) \neq 0$ for $u \in B$ and all z such that $I - zz^* > 0$ or $I - zz^* < 0$.) Here a harmonic function is a function of class C^2, which satisfies on D the Laplace equation corresponding to the invariant metric of D, that is, the metric invariant with respect to the group of 1 to 1 analytic transformations mapping D onto itself **(14)**. This invariant metric is given by

$$ds^2 = \sigma[(I - zz^*)^{-1} dz (I - z^*z)^{-1} dz^*],$$

Received November 30, 1959. Presented to meetings of American Mathematical Society on January 29, 1960 and January 24, 1961.

https://doi.org/10.4153/CJM-1961-006-9 Published online by Cambridge University Press

where $\sigma(A)$ is the trace of the matrix A and $dz = (dz_{jk})$, and the corresponding Laplace equation is

$$4\sigma[\bar{\partial}(I - z^*z)\partial'(I - zz^*)] = 0, \qquad \partial = (\partial/\partial z_{jk}).$$

It has been proved by Hua and Lowdenslager that given a real function f, continuous on B, there exists a function F, harmonic on D, such that $F(z) \to f(u_0)$ as $z \to u_0 \in B$ radially, that is, along the set ρu_0, $0 \leqslant \rho < 1$ **(6; 9)**. Further, if F is continuous on the closure $\bar{D}$ of D and satisfies certain other conditions due to Lowdenslager on the boundary of D other than B, then F is unique **(8)**. Now for the particular case of the unit circle, $z\bar{z} = 1$, if we merely assume that f is integrable on it, then $I(f, \rho u_0) \to f(u_0)$ as $\rho \to 1 (0 \leqslant \rho < 1)$ **(20)**; this method of approach is known as *Abel–Poisson summability of Fourier series*. We prove this result for matrix spaces. (See note added in proof).

In § 3 we consider for the first type of domain $(n \leqslant m)$ some properties of complete orthonormal systems (CONS) of complex homogeneous polynomials defined on D. The space D is circular with center at $z = 0$, that is, if $z \in D$, then $e^{i\theta}z \in D$ for $0 \leqslant \theta < 2\pi$. Hence any two powers

$$P(z) = \prod_{j,k} z_{jk}^{s_{jk}}, \qquad Q(z) = \prod_{j,k} z_{jk}^{t_{jk}}, \tag{5}$$

s_{jk}, t_{jk} non-negative integers, for which

$$\sum_{j,k} s_{jk} \neq \sum_{j,k} t_{jk}, \tag{6}$$

are orthogonal, that is,

$$(P, Q) = \int_D P(z)\bar{Q}(z)dW = 0 \tag{7}$$

(dW is Euclidean volume element on D) **(6)**. Also if $f \in$ class L^2 on D, which means that f is single-valued and analytic on D and has finite norm $||f|| = [(f, f)]^{\frac{1}{2}}$ **(2)**, then the set $\{P\}$ is complete with respect to functions of class L^2 **(6)**.

Here we refine conditions (6) to show that $(P, Q) \neq 0$ implies

$$\sum_k s_{\nu k} = \sum_k t_{\nu k}, \qquad \sum_j s_{j\mu} = \sum_j t_{j\mu} \qquad (\nu = 1, \ldots, n; \mu = 1, \ldots, m). \tag{8}$$

By means of (8) the set of powers $\{P\}$ is subdivided into disjoint subsets whose members need not be orthogonal to each other. The elements of a subset are made into an orthonormal set by the Gram Schmidt formulas, thus giving a CONS of homogeneous polynomials $\{\phi\}$ on D. We note that Hua has constructed a CONS of functions of class L^2 on D using representation theory **(6)**.

In § 4 applications of the CONS $\{\phi_\nu\}$ are given. First an Abelian theorem is obtained and then a Tauberian theorem for the orthogonal series $\sum a_\nu \phi_\nu$ as z approaches $[I, 0]$ of B along the matrix $[r, 0]$ where r is the diagonal matrix $[r_1, \ldots, r_n]$, $0 \leqslant r_j < 1$. Next a Cauchy's inequality is obtained for the Fourier coefficients a_ν. Finally two mean value theorems, which generalize analogous theorems for the unit circle, are proved.

§2. Poisson summability.

1. *Reduction of integral* (1.3) *to normal form.* Rauch outlined this reduction to me. The transformation

(1) $$w = zu_0^{-1}, \qquad u_0 u_0^* = I,$$

takes $z = u_0$ into $w = I$ and also leaves D and B invariant since under it $I - ww^* = I - zz^*$. Also if $u \to v$ under (1) $P(z, u) \to P(w, v)$ and

$$dV_u = \frac{\dot{u}}{\det^n u} = \frac{\partial(v)}{\partial(u)} \frac{\dot{v}}{\det^n v \det^n u_0},$$

where

$$\dot{u} = (-1)^{-\frac{1}{4}n(n+1)} \prod_{j,k} du_{jk}$$

(14). Now $du = dv\, u_0$, the Jacobian of which is $\partial(u)/\partial(v) = \det^n u_0$ **(3)**. Thus $dV_u \to dV_v$. Also $f(u) \to f(vu_0) = f_0(v)$ so that $I(f, z) \to I(f_0, w)$.

If $w \to I$ along the set of points ρI $(0 \leqslant \rho < 1)$, then

$$\det(I - ww^*) = (1 - \rho^2)^n$$

and

$$Q = (I - wv^*)(I - vw^*) = I - wv^* - vw^* + ww^*$$
$$= I(1 + \rho^2) - \rho(v + v^*).$$

Now v is unitary equivalent to a diagonal matrix v_D which is also unitary **(10**, Theorem 41.41**)**, that is,

(2) $$v = U^* v_D U.$$

Thus if $v_D = [d_1, \ldots, d_n]$, then $d_j \bar{d}_j = 1$ and we can write $d_j = e^{i\theta_j}(0 \leqslant \theta_j < 2\pi)$. Hence

$$Q = U^*[I(1 + \rho^2) - \rho(v_D^* + v_D)]U$$

and

$$\det Q = \det[I(1 + \rho^2) - \rho(v_D^* + v_D)] = \prod_{j=1}^{n} (1 - 2\rho\cos\theta_j + \rho^2).$$

Let

(3) $$p(\rho, \theta_j) = \frac{1 - \rho^2}{1 - 2\rho\cos\theta_j + \rho^2},$$

which is 2π times the Poisson kernel for the unit circle. Then

(4) $$P(\rho I, v) = V^{-1} \prod_{j=1}^{n} p^n(\rho, \theta_j)$$

and

(5) $$I(f_0, \rho I) = V^{-1} \int_B \prod_j p^n(\rho, \theta_j) f_0(v) dV_v.$$

According to Weyl **(19**, p. 197**)** if (2) holds, then

(6) $$dV_v = [dV_U]\Delta\bar{\Delta}d\theta_1 \ldots d\theta_n,$$

where

$$(7) \qquad \Delta\bar{\Delta} = \prod_{j<k} |e^{i\theta_j} - e^{i\theta_k}|^2 = 4 \prod_{j<k} \sin^2\tfrac{1}{2}(\theta_j - \theta_k)$$

and $dV_0 = [dV_U]$ is a constant times the Euclidean volume element on the other $n^2 - n$ parameters defining B. Let B_0 be this part of B. Now $P(\rho I, v)$, considered as a function of v, is independent of the other $n^2 - n$ parameters and hence

$$(8) \qquad I(f_0, \rho I) = \int_0^{2\pi} \dots \int_0^{2\pi} P(\rho I, v) F(\theta) \Delta\bar{\Delta} d\theta_1 \dots d\theta_n,$$

where

$$(9) \qquad F(\theta) = F(\theta_1, \dots, \theta_n) = \int_{B_0} f_0(v) dV_0.$$

2. *Convergence theorem for* (8). It is sufficient to consider (8) for $n = 2$, in which case replacing θ_1 by s and θ_2 by t

$$(10) \qquad I(f_0, \rho I) = 4V^{-1} \int_0^{2\pi} \int_0^{2\pi} p^2(\rho, s) p^2(\rho, t) \sin^2\tfrac{1}{2}(s - t) F(s, t) ds dt$$

and we consider $\lim_{\rho \to 1} I(f_0, \rho I)$. Subtracting S from each side we reduce (10) by well-known methods in Fourier series **(20)** to a consideration of

$$(11) \qquad I = 4V^{-1} \int_0^{\pi} \int_0^{\pi} p^2(\rho, s) p^2(\rho, t) \sin^2\tfrac{1}{2}(s - t) G(s, t) ds dt$$

and

$$(11a) \qquad I_1 = 4V^{-1} \int_0^{\pi} \int_0^{\pi} p^2(\rho, s) p^2(\rho, t) \sin^2\tfrac{1}{2}(s + t) G(2\pi - s, t) ds dt,$$

where

$$(12) \qquad G(s, t) = F(s, t) + F(2\pi - s, 2\pi - t) - 2V_0 S,$$

V_0 being the volume of B_0. We show that under certain hypotheses on $G(s, t)$, $I \to 0$ as $\rho \to 1$ and the proof is similar for I_1. In (11) integrate by parts with respect to s and t. Since

$$H(s, t) = \iint G(s, t) ds dt$$

is zero for $s = 0$ or $t = 0$, $\sin^2\frac{1}{2}(s - t) = 0$ for $s = t = \pi$, $\sin^2\frac{1}{2}(\pi - \theta) = \cos^2\frac{1}{2}\theta$ and $p(\rho, \pi) = (1 - \rho)/(1 + \rho)$, we get

$$(13) \qquad \begin{aligned} I = &-\left(\frac{1-\rho}{1+\rho}\right)^2 \int_0^{\pi} \frac{\partial}{\partial t} [p^2(\rho, t) \cos^2\tfrac{1}{2}t] H(\pi, t) dt \\ &-\left(\frac{1-\rho}{1+\rho}\right)^2 \int_0^{\pi} \frac{\partial}{\partial s} [p^2(\rho, s) \cos^2\tfrac{1}{2}s] H(s, \pi) ds \\ &+ \int_0^{\pi} \int_0^{\pi} \frac{\partial^2}{\partial s \partial t} [p^2(\rho, s) p^2(\rho, t) \sin^2\tfrac{1}{2}(s - t)] H(s, t) ds dt \\ \equiv\ &I_1 + I_2 + I_3. \end{aligned}$$

https://doi.org/10.4153/CJM-1961-006-9 Published online by Cambridge University Press

The following inequalities are used in considering I_1, I_2, and I_3 **(20)**:

(14) (i) $$0 \leqslant p(\rho, \theta) \leqslant \frac{1+\rho}{1-\rho}$$

(ii) $$0 \leqslant p(\rho, \theta) \leqslant \frac{1-\rho^2}{4 \sin^2 \frac{1}{2}\theta}$$

(iii) $$\lim_{\theta \to 0} \frac{\theta}{\sin \theta} = 1$$

(iv) $$\int_0^\pi p(\rho, \theta) d\theta = \pi$$

(v) $$\int_\delta^\pi p(\rho, \theta) d\theta = o(1) \quad \text{as} \quad \rho \to 1 \quad \text{for} \quad 0 < \delta \leqslant \pi.$$

Also

$$\frac{\partial}{\partial \theta} p(\rho, \theta) = 2\rho(1-\rho^2)\sin \theta \, d^{-2}(\rho, \theta)$$

where

$$d(\rho, \theta) = 1 - 2\rho\cos\theta + \rho^2 = \frac{1-\rho^2}{p(\rho, \theta)}$$

and

$$\frac{\partial^2}{\partial s \partial t}[p^2(\rho, s)p^2(\rho, t)\sin^2 \tfrac{1}{2}(s-t)] = p(\rho, s)p(\rho, t) \,.$$
$$[16\rho^2(1-\rho^2)^2 \sin s \sin t \sin^2 \tfrac{1}{2}(s-t) d^{-2}(\rho, s) d^{-2}(\rho, t) - \tfrac{1}{2} p(\rho, s) p(\rho, t)$$
$$. \cos(s-t) - 2\rho(1-\rho^2) \sin s \sin(s-t) p(\rho, t) d^{-2}(\rho, s)$$
$$+ 2\rho(1-\rho^2)\sin t \sin(s-t) p(\rho, s) d^{-2}(\rho, t)].$$

Concerning the function $G(s, t)$ we assume that

(15) $$\lim_{h,k \to 0} \frac{1}{hk} \int_0^h \int_0^k |G(s, t)| ds dt = 0$$

$$\int_0^h \int_0^k |G(s, t)| |ds dt| \leqslant C|hk|, \qquad (0 < |h|, |k| \leqslant \pi),$$

C an absolute constant. (See **(11)** where these conditions were used in a similar connection.) Then using (15) we find

$$|I_1| \leqslant \pi(1-\rho^2) C \int_0^\pi |2p(\rho, t)\cos^2 \tfrac{1}{2}t \partial p(\rho, t)/\partial t - p^2(\rho, t) \sin \tfrac{1}{2}t \cos \tfrac{1}{2}t| t dt$$

$$= 0((1-\rho) \int_0^\pi p(\rho, t) dt) = 0(1-\rho)$$

and similarly for I_2. Now

$$I_3 = \int_0^\delta \int_0^\delta + \left\{ \int_0^\delta \int_\delta^\pi + \int_\delta^\pi \int_0^\delta + \int_\delta^\pi \int_\delta^\pi \right\} \equiv I_{31} + I_{32}.$$

By (15) it is found that

$$|I_{31}| \leqslant \epsilon \int_0^{\pi} \int_0^{\pi} p(\rho, s) p(\rho, t) ds dt,$$

where ϵ is an arbitrary positive number, and

$$I_{32} = 0\left(\frac{1-\rho}{\sin^2 \frac{1}{2}\delta}\right).$$

Consequently given $\epsilon > 0$ choose $\delta > 0$ so that

$$\left| \int_0^{s} \int_0^{t} G(s, t) ds dt \right| < \epsilon |st|,$$

if $|s| < \delta$ and $|t| < \delta$. With fixed δ choose $1 - \rho$ sufficiently small. Then $I = 0(\epsilon)$ for ρ sufficiently close to 1. Consequently we have proved

THEOREM 2.1. *Let $F(s, t)$ be an integrable function. If $G(s, t) = F(s, t) + F(2\pi - s, 2\pi - t) - 2V_0 S$ satisfies conditions* (15), *then*

$$\lim_{\rho \to 1^-} 4V^{-1} \int_0^{2\pi} \int_0^{2\pi} p^2(\rho, s) p^2(\rho, t) \sin^2 \tfrac{1}{2}(s - t) F(s, t) ds dt = S.$$

For $n > 2$ it would be sufficient to assume that

$$\text{(16)} \quad \lim_{\theta_j \to 0} \frac{1}{\theta_1 \ldots \theta_n} \int_0^{\theta_n} \ldots \int_0^{\rho_n} |G(\theta)| d\theta_1 \ldots d\theta_n = 0,$$

$$\int_0^{\theta_1} \ldots \int_0^{\theta_n} |G(\theta)| |d\theta_1 \ldots d\theta_n| \leqslant C|\theta_1 \ldots \theta_n|, \ (0 < |\theta_j| \leqslant \pi, j = 1, \ldots, n),$$

where $G(\theta)$ is defined similarly to $G(s, t)$. We obtain

THEOREM 2.2. *Let f be an integrable function on the unitary group B such that the function $F(\theta)$ defined by* (9), *where f_0 is the transform of f under* (1), *satisfies* (16). *Then the Poisson integral* (1.3) *has a limit if z approaches the point u_0 on B radially.*

§3. Complete orthonormal systems on D. Orthogonal developments.

1. *Integration over D.* Let z be an n by m matrix ($n \leqslant m$), z_p its pth row and Z_p the submatrix consisting of the first p rows ($p = 1, \ldots, n$). The inner product (f, g) defined by (1.7) may be transformed into an iterated integral over the product of n hyperspheres by a procedure due to Hua **(12)**, giving

$$\text{(1)} \qquad (f, g) = \left(\frac{1}{2i}\right)^{mn} \int_{w_1 w_1^* < 1} (1 - w_1 w_1^*)^{n-1} \dot{w}_1 \ldots \int_{w_n w_n^* < 1} f \bar{g} \, \dot{w}_n,$$

where

$$\text{(2)} \qquad \dot{w}_p = \prod_{k=1}^{m} dw_{pk} d\bar{w}_{pk}$$

$$w_p = z_p \Gamma_{p-1}$$

$$z_p = w_p \Gamma_{p-1}^{-1} \qquad (p = 1, \ldots, n, \ \Gamma_0 = 1),$$

https://doi.org/10.4153/CJM-1961-006-9 Published online by Cambridge University Press

and Γ_{p-1} is a unique positive definite matrix such that

$$\Gamma_{p-1}\Gamma^*_{p-1} = (I - Z^*_{p-1}Z_{p-1})^{-1}. \tag{3}$$

2. *Construction of the matrix* $\Gamma_{p-1}{}^{-1}(2 \leqslant p \leqslant n)$. Let $U_p(q) = U(q)$ be the minor formed from the first q rows and columns of

$$U_p = U = (u_{jk}) = I - Z^*_{p-1}Z_{p-1}, \tag{4}$$

$\mathscr{U}_p(q) = \mathscr{U}(q)$ the corresponding submatrix and $u_{m-j} = (u_{m-j,1}, \ldots, u_{m-j,m-j-1})$. Since $I - Z_{p-1}Z^*_{p-1}$ is the leading $(p-1)$th principal submatrix of $I - zz^*$, it is positive definite, hence U is positive definite and $U(q)$ are positive. Thus the hermitian matrix U may be reduced to diagonal form by the well-known Kronecker reduction **(1)** whose $(j+1)$th step is

$$V_{j+1}\ldots V_1UV^*_1\ldots V^*_{j+1} = \begin{pmatrix} \mathscr{U}(m-j-1) & & 0 \\ & X_{m,m-j} & \\ 0 & & \ddots \\ & & & X_{mm} \end{pmatrix}.$$

$(0 \leqslant j \leqslant m-2)$, where

$$V_{j+1} = \begin{pmatrix} I^{(m-j-1)} & 0 \\ -u_{m-j}\mathscr{U}^{-1}(m-j-1) & I^{(j+1)} \\ 0 & \end{pmatrix},$$

$$X_{m,m-j} = U(m-j)U^{-1}(m-j-1),$$

$$X_{m1} = U(1).$$

Also

$$u_{m-j}\mathscr{U}^{-1}(m-j-1) = U^{-1}(m-j-1)\cdot$$
$$\left((-1)^{k+m-j+1}\,U\begin{pmatrix}1,\ldots,[k],\ldots,m-j\\1,\ldots,m-j-1\end{pmatrix}\right), \quad (1 \leqslant k \leqslant m-j-1),$$

where

$$U\begin{pmatrix}1,\ldots,[k],\ldots,m-j\\1,\ldots,m-j-1\end{pmatrix}$$

is the minor of U formed from rows $1, \ldots, m-j$ with row k omitted and columns $1, \ldots, m-j-1$.

Now $V_{j+1}{}^{-1}$ equals V_{j+1} with the sign of the matrix $-u_{m-j}\mathscr{U}^{-1}(m-j-1)$ changed. Also X_{mk} is real. Hence we can take

$$\Gamma^{*-1}_{p-1} = V_1^{-1}\ldots V^{-1}_{m-1}[X^{\frac{1}{2}}_{m1}, \ldots, X^{\frac{1}{2}}_{mm}]. \tag{5}$$

By an inductive proof we may show that

$$(6)\quad V_1^{-1}\dots V_{m-1}^{-1} = \left(\frac{U\binom{j}{1}}{U(1)}\,\frac{U\binom{1,j}{1,2}}{U(2)}\cdots\frac{U\binom{1,2,\dots,[j-1],j}{1,2,\dots,j-1}}{U(j-1)}\,1\,0\dots 0\right),$$

$(1 \leqslant j \leqslant m)$. (For details of the proof cf. **(15)**.) From (4) it follows that

$$(7)\quad U\binom{1,\dots,[i],r}{1,\dots,i} = U_p\binom{1,\dots,[i],r}{1,\dots,i} = \det\left(\delta_{jk} - \sum_{l=1}^{p-1} \bar{z}_{lj}z_{lk}\right),$$

$$(j = 1,\dots,i-1,r;\ k = 1,\dots,i;\ i+1 \leqslant r \leqslant m,\ 1 \leqslant i \leqslant m-1).$$

3. *Formula for z_{pr}.* From (2), (5), and (6) follows

$$(8)\quad z_{pr} = \sum_{i=1}^{r-1} c_{pi}\bar{U}_p\binom{1,\dots,[i],r}{1,\dots,i}w_{pi} + c_{prr}w_{pr}\ (p \geqslant 2)$$

$$z_{1r} = w_{1r}\left(1 \leqslant r \leqslant m;\ \sum_{i=1}^{0} = 0\right),$$

where

$$(9)\quad \begin{aligned} c_{pi} &= [U_p(i)\,U_p(i-1)]^{-\frac{1}{2}}\ (1 \leqslant i \leqslant r-1)\\ c_{prr} &= [U_p(r)/U_p(r-1)]^{\frac{1}{2}}\\ c_{1i} &= c_{1rr} = 1.\end{aligned}$$

The formula

$$(10)\quad U_p(q) = \prod_{k=1}^{p-1}\left(1 - \sum_{j=1}^{q} w_{kj}\bar{w}_{kj}\right)\qquad (p \geqslant 2)$$

holds.

We prove (10) by induction on p. Since by (8)

$$U_2(q) = \det(I - z_1^*z_1) = \det(I - z_1z_1^*) = 1 - \sum_{j=1}^{q} w_{1j}\bar{w}_{1j},$$

where $z_1 = (z_{11},\dots, z_{1q})$, (10) is true for $p = 2$. Assume (10) holds for $U_p(q)$ and prove for $U_{p+1}(q)$. Since $U_p(q) \neq 0$, there exists a unimodular matrix A such that the matrix $\mathscr{V}_{p+1}(q) = (\delta_{jk} - \sum_{l=1}^{q} z_{jl}\bar{z}_{kl})$ $(1 \leqslant j, k \leqslant p)$ is equal to

$$A[\mathscr{V}_p(q),\ 1 - z_p\mathscr{U}_p^{-1}(q)z_p^*]A^*,$$

(12), and thus

$$U_{p+1}(q) = \det \mathscr{V}_{p+1}(q) = \det \mathscr{V}_p(q)(1 - z_p\mathscr{U}_p^1(q)z_p^*).$$

Since $\det \mathscr{V}_p(q) = U_p(q)$, using (10) we need only consider the last factor on the right, which equals

$$E = 1 - \sum_{j,k=1}^{q} z_{pj}U_{kj}\bar{z}_{pk}/U_p(q),$$

where U_{kj} is the cofactor of the element u_{kj} in the matrix $\mathscr{U}_p(q)$. Substitute (8) for z_{pj} and $\bar{z}_{pk}$. The term w_{pr} occurs when $j = r$ or when $i = r$ and $j = r + 1, \ldots, q$ and $\bar{w}_{p\lambda}$ occurs when $k = \lambda$ or $i = \lambda$ and $k = \lambda + 1, \ldots, q$. Thus the coefficient, $C_{r\lambda}$, of $w_{pr}\bar{w}_{p\lambda}(1 \leqslant r, \lambda \leqslant q)$ is

$$C_{r\lambda} = U_p^{-1}(q)\left[c_{pr}\sum_{j=r+1}^{q}\bar{U}_p\begin{pmatrix}1, \ldots, [r], j\\ 1, \ldots, r\end{pmatrix}D_{j\lambda} + c_{prr}D_{r\lambda}\right],$$

where

$$D_{j\lambda} = c_{p\lambda}\sum_{k=\lambda+1}^{q}U_{kj}U_p\begin{pmatrix}1, \ldots, [\lambda], k\\ 1, \ldots, \lambda\end{pmatrix} + U_{\lambda j}c_{p\lambda\lambda}.$$

By means of elementary properties of determinants it is not difficult to prove in case $\lambda \leqslant r$ that $D_{j\lambda} = 0$ for $j = r, \lambda < r$ and $j = r + 1, \ldots, q, \lambda \leqslant r$. Hence $C_{r\lambda} = 0$ for $\lambda \neq r$. Also $C_{rr} = 1$. A similar proof holds for $r < \lambda$. Thus

$$E = 1 - \sum_{j=1}^{q} w_{pj}\bar{w}_{pj}$$

and $U_{p+1}(q)$ has the desired form, which proves (10).

4. *Structure of the* CONS *on D.*

THEOREM 3.1. $(P, Q) \neq 0$ *implies equations* (1.8).

Proof. We first show that

$$(11) \qquad z_{pr} = w_{pr}\sum B_i w_{pi}\bar{w}_{pr}\prod (w_{\lambda_1\alpha_1}\bar{w}_{\lambda_1\beta_1}\ldots w_{\lambda_l\alpha_l}\bar{w}_{\lambda_l\beta_l}w_{\lambda_i r}\bar{w}_{\lambda_i\beta_i}),$$

where B_i is a function of $w_{jk}\bar{w}_{jk}$ $(1 \leqslant j \leqslant p - 1)$; $\lambda_1, \ldots, \lambda_l, \lambda_i$ take on values in the set $1, 2, \ldots, p - 1$; $\alpha_1, \ldots, \alpha_l$ is a subset of $1, \ldots, i - 1$ and $(\beta_1, \ldots, \beta_l, \beta_i)$ is a permutation of $(\alpha_1, \ldots, \alpha_l, i)$ $(i = 1, \ldots, r - 1)$. (Notice that each term of (11) can be grouped into pairs $w_{\alpha\beta}\bar{w}_{\gamma\delta}$ in two ways: (i) each pair belongs to the same row, (ii) each pair belongs to the same column.) Since $z_{1r} = w_{1r}$, (11) holds for $p = 1$. Now assume (11) for $p - 1$, $1 \leqslant r \leqslant m$, and prove for p. Upon expanding

$$\bar{U}_p\begin{pmatrix}1, \ldots, [i], r\\ 1, \ldots, i\end{pmatrix}$$

(given by (7)) and multiplying out the resulting factors, we find that its general term is

$$z_{\lambda_1\alpha_1}\bar{z}_{\lambda_1\beta_1}\ldots z_{\lambda_s\alpha_s}\bar{z}_{\lambda_s\beta_{\alpha_s}}z_{\lambda_{s+1}r}\bar{z}_{\lambda_{s+1}\beta_i},$$

where $\lambda_\alpha(\alpha = 1, \ldots, s + 1)$ takes on values from $1, 2, \ldots, p - 1$; $\alpha_1, \ldots, \alpha_s$ is a subset of $1, \ldots, i - 1$ and

$$\beta_{\alpha_1}, \ldots, \beta_{\alpha_s}, \beta_i'$$

is a permutation of $\alpha_1, \ldots, \alpha_s, i$. Thus the general term of z_{pr} would be w_{pr} times

https://doi.org/10.4153/CJM-1961-006-9 Published online by Cambridge University Press

$$c'_{pi} z_{\lambda_1 \alpha_1} \bar{z}_{\lambda_1 \beta \alpha_1} \ldots z_{\lambda_s \alpha_s} \bar{z}_{\lambda_s \beta \alpha_s} z_{\lambda_s+1\, r} \bar{z}_{\lambda_s+1\, \beta_i} w_{p\,i} \bar{w}_{p\,r},$$

$c_{pi}' = c_{pi}/w_{pr}\bar{w}_{pr}$. Replace

$$z_{\lambda_1 \alpha_1 \lambda} \bar{z}_{\lambda_1 \beta \alpha_1} \ldots \text{ by } w_{\lambda_1 \alpha_1} \bar{w}_{\lambda_1 \beta \alpha_1} \ldots$$

times a factor which by induction already has the required form and (11) follows.

Now consider $z_{pr}^{s_{pr}}$. From (11) we see that except for the first factor $w_{pr}^{s_{pr}}$ if $w_{\nu j}^{\sigma}$ occurs, then a factor $\bar{w}_{\nu k}^{\sigma}$ also occurs and if $w_{i\mu}^{\tau}$ appears, then $\bar{w}_{l\mu}^{\tau}$ also appears. Consequently in the expression for $z_{pr}^{s_{pr}}$ for each $\nu(\nu = 1, \ldots, p-1)$ the sum of the exponents of the factors $w_{\nu j} (j = 1, \ldots, m)$ equals the sum of the exponents of the $\bar{w}_{\nu k}$ $(k = 1, \ldots, m)$ and the sum of the exponents of w_{pj} equals the sum of the exponents of $\bar{w}_{pk}$ increased by s_{pr}. Similarly for the columns. Thus P can be expressed in the form

$$P(z) = P_0(w\bar{w}) \prod w_{jk}^{s_{jk}} = P_0(w\bar{w})P(w), \tag{12}$$

where $P_0(w\bar{w})$ contributes the same exponents to the sum of the elements in the νth row of w and of $\bar{w}$ and similarly for the columns of w and $\bar{w}$. An analogous expression holds for Q.

In (P, Q) replace P and Q by (12). Since $\{w_{\nu k}^{s_{\nu k}}\}$ $(k = 1, \ldots, m)$ forms an orthogonal set on $w_\nu w^*_\nu < 1$ **(2)**, $(P, Q) \neq 0$ if and only if for each k the exponent of $w_{\nu k}$ equals the exponent of $\bar{w}_{\nu k}$. Consequently if we sum the exponents of the νth row, owing to the form of $P_0(w\bar{w})$, $Q_0(w\bar{w})$ we obtain the first of equations (1.8) and summing the exponents of the μth column the second of equations (1.8). Thus the theorem is proved.

5. CONS. *Orthogonal development.* A CONS is constructed from the set of powers $\{P(z)\}$ as follows. Let $\alpha = (\alpha_1, \ldots, \alpha_{m+n})$ be a set of non-negative integers with $\sum_{j=1}^{n} \alpha_j = \sum_{k=1}^{m} \alpha_{k+n} = p$. The powers of the set $S(\alpha) = S(\alpha_1, \ldots, \alpha_{m+n})$ such that

$$\sum_k s_{\nu k} = \alpha_\nu, \qquad \sum_j s_{j\mu} = \alpha_{\mu+n}$$

need not be orthogonal to each other. (There exist sets $S(\alpha)$ whose members are not all orthogonal to each other—for example, in the 2 by 2 case the elements $z_{11}z_{22}$, $z_{12}z_{21}$ are not orthogonal **(12)**.) However if $P \in S(\alpha)$, $Q \in S(\beta)$, where $\alpha_j \neq \beta_j$ for some j, then by Theorem 3.1 $(P, Q) = 0$. We order the elements of the set $S(\alpha)$ in some convenient manner into a sequence $P_0, P_1, \ldots, P_{p(\alpha)}$. An ONS is constructed from these elements by the Gram-Schmidt formulas

$$\phi_\nu^{(p)}(z) = \det\begin{pmatrix} P_0 \ldots P_\nu \\ a_{00} \ldots a_{\nu 0} \\ \ldots \\ a_{0,\nu-1} \ldots a_{\nu,\nu-1} \end{pmatrix} / (D_{\nu-1}D_\nu)^{\frac{1}{2}}, \tag{13}$$

where

https://doi.org/10.4153/CJM-1961-006-9 Published online by Cambridge University Press

$$D_\mu = \det(a_{\alpha\beta}) \ (0 \leqslant \alpha, \beta \leqslant \mu; \mu = \nu - 1 \text{ or } \nu, \nu \neq 0), \ D_{-1} = 1.$$
$$a_{ij} = (P_i, P_j), \ (0 \leqslant i, j \leqslant \nu).$$

Now order the system $\{\phi_\nu^{(p)}\}$ into a sequence $\phi_1, \phi_2, \ldots$. The orthogonal development of any $f \in L^2$ with respect to the ONS is

(14) $$\sum a_q \phi_q,$$

where a_q are the Fourier coefficients, (f, ϕ_q), of f. From Bergman's theory **(2)** it is known that (14) converges absolutely and continuously to f on D (continuous convergence means that the series converges uniformly on any compact set contained in D).

§4. Applications of the CONS.

1. *Abelian and Tauberian theorems.* Let $\{a_q\}$ be an arbitrary sequence of numbers and consider the behaviour of (3.14) as $z \in D$ approaches a point $u_0 \in B$. In particular let $u_0 = [I, 0]$, $I = I^{(n)}$, and $z \to u_0$ along the set of points $[r, 0]$ where r is the diagonal matrix $[r_1, \ldots, r_n]$, $0 \leqslant r_j < 1$. When $z = [r, 0]$ it is seen that P_ν of the set $S(\alpha)$ is either equal to 0 or to

$$\prod r_j^{p_j}$$

in case $\alpha_j = \alpha_{j+n} = p_j$ for $j = 1, \ldots, n$ and $\alpha_{j+n} = 0$ for $j > n$. Consequently for a fixed set α either all P_ν are zero or there is one P_ν different from zero in case $\alpha_j = \alpha_{j+n}$ for $j = 1, \ldots, n$ and $\alpha_{j+n} = 0$ for $j > n$. Order the elements of the set $S(\alpha)$ so that the non-zero term is the last term of the set. Then in (3.13) only $\phi_t^{(p)}(z)$, $t = p(\alpha)$, is different from zero when $z = [r, 0]$ and

$$\phi_t^{(p)}(z) = [D_{t-1}/D_t]^{\frac{1}{2}} P_t(z) = [D_{t-1}/D_t]^{\frac{1}{2}} r_1^{p_1} \ldots r_n^{p_n}.$$

Thus (3.14) reduces to a multiple power series. Let this series be summed by the usual method for power series. Then

(1) $$S(r) = \sum_{p_1=0}^{\infty} \ldots \sum_{p_n=0}^{\infty} c_{p_1 \ldots p_n} r_1^{p_1} \ldots r_n^{p_n},$$
$$c_{p_1 \ldots p_n} = a_q/(D_{t-1}/D_t)^{\frac{1}{2}},$$

where $\phi_t^{(p)} = \phi_q$ is the ordering of the ONS $\{\phi_t^{(p)}\}$ into a simple sequence. Let

$$S_{q_1 \ldots q_n}$$

be the partial sum of $S(r)$:

$$S_{q_1 \ldots q_n} = \sum_{p_1=0}^{q_1} \ldots \sum_{p_n=0}^{q_n} c_{p_1 \ldots p_n} r_1^{p_1} \ldots r_n^{p_n}.$$

The following Abelian theorem is valid:

THEOREM 4.1. *If $S(I)$ exists and*

$$|S_{q_1 \ldots q_n}| < C,$$

where C is an absolute constant, then $S(r)$ *is uniformly convergent for* $0 \leqslant r_j < 1$ *and* $\lim_{r\to I} S(r) = S(I)$. ($r \to I$ means $[r, 0] \to [I, 0]$.) See **(4)** for a proof.

Also a Tauberian theorem proved by Knopp **(7)** for double series may be extended to multiple series.

THEOREM 4.2 (Tauberian theorem). *Let the series* $S(r)$ *converge for each* $[r, 0] \in D$, $r = [r_1, \ldots, r_n]$, *and for these* r *let* $|S(r)| \leqslant K$, *where* K *is an absolute constant. If*

$$(2) \qquad |c_{p_1 \ldots p_n}| (p_1^2 + \ldots + p_n^2)^{\frac{1}{2}n} < M < \infty,$$

then $\lim_{r\to I} S(r) = S$ *implies* $S(I) = S$, *that is,*

$$(3) \qquad \sum_{p_1=0}^{\infty} \ldots \sum_{p_n=0}^{\infty} c_{p_1 \ldots p_n} = S.$$

In order to prove Theorem 4.2, Theorems 3 of § 3 and the proofs in § 4 of Knopp's paper must be proved for n-fold series ($n \geqslant 3$). Using condition (2) Theorem 3 has been extended to multiple series in **(18)**. Also by means of (2) the proofs in § 4 follow for multiple series. In addition see **(13)** where a similar condition is utilized for multiple series summed spherically.

On the other hand if we let $z \to [I, 0]$ along the set $[\rho I, 0]$, $0 \leqslant \rho < 1$, and sum series (3.14) by diagonals:

$$(4) \qquad S_0(\rho) = \sum_{p=0}^{\infty} b_p \rho^p,$$

where

$$(4a) \qquad b_p = \sum_{p_1 + \ldots + p_n = p} c_{p_1 \ldots p_n},$$

then (4) is a simple series and the boundedness conditions on $S_{q_1 \ldots q_n}$ and $S(r)$ can be omitted in Theorems 4.1 and 4.2 **(17)**. Abel's theorem reads *if* $S_0(I)$ *exists, then* $S_0(\rho I)$ *is uniformly convergent for* $0 \leqslant \rho < 1$ *and* $\lim_{\rho\to 1} S_0(\rho I) = S_0(I)$ and Theorem 4.2 becomes

Let the series $S_0(\rho I)$ *converge for each* $[\rho I, 0] \in D$, $(0 \leqslant \rho < 1)$. *If* $b_p = 0(1/p)$, *then* $\lim_{\rho\to 1} S_0(\rho I) = S_0$ *implies* $S_0(I) = S_0$.

2. *Cauchy's inequality and mean value theorem.* In the next paragraphs it is convenient to group the elements of the CONS $\{\phi_\nu\}$ of same degree, hence, let

$$\phi_1^{(p)}, \ldots, \phi_{M_p}^{(p)}$$

be the terms of degree p. Then for any $f \in L^2$ on D

$$(5) \qquad f(z) = \sum_{p=0}^{\infty} \sum_{j=1}^{M_p} a_j^{(p)} \phi_j^{(p)}(z),$$

where the convergence is continuous and absolute for $z \in D$. Multiply (5) by $\bar{f}$ and integrate over the domain

$$(6) \qquad D_\rho = [z | \rho^2 I - zz^* > 0, 0 < \rho < 1].$$

Clearly $D_\rho \subsetneqq D$ for $0 < \rho < 1$. Since the convergence of (5) is uniform and absolute on $\bar{D}_\rho$

$$(7)\quad I(\rho) = V_\rho^{-1} \int_{D\rho} |f|^2 dW_z = V_\rho^{-1} \sum_{p=0}^{\infty} \sum_{j=1}^{M_p} \sum_{q=0}^{\infty} \sum_{k=1}^{M_q} a_j^{(p)} \bar{a}_k^{(q)} \int_{D\rho} \phi_j^{(p)} \bar{\phi}_k^{(q)} dW_z$$

(V_ρ being the volume of D). In the integral

$$I = \int_{D\rho} \phi_j^{(p)} \bar{\phi}_k^{(q)} dW_z$$

set $z = \rho w$. Then $D_\rho \to D$, $dW_z \to \rho^{2mn} dW_w$ (cf. § 3.1) and

$$\phi_j^{(p)}(z) = \sum {}_{\Sigma \alpha_{jk} = p} \text{constant} \prod z_{jk}^{\alpha_{jk}} \to \rho^p \phi_j^{(p)}(w).$$

Hence

$$I = \rho^{2mn+p+q} (\phi_j^{(p)}, \phi_k^{(q)}) = \rho^{2mn+p+q} \delta_{pq} \delta_{jk}.$$

Also $V_\rho = \rho^{2mn} V_0$ where V_0 is the volume of D. Thus (7) becomes

$$(8)\qquad I(\rho) = \frac{1}{V_\rho} \int_{D\rho} |f|^2 dW = \frac{1}{V_0} \sum_{p=0}^{\infty} \rho^{2p} \sum_{j=0}^{M_p} |a_j^{(p)}|^2.$$

From (8)

$$\frac{1}{V_0} \sum_{j=0}^{M_p} |a_j^{(p)}|^2 \leqslant (1/\rho^{2p}) \max_{z \in D\rho} |f(z)|^2.$$

Now according to a theorem proved by Hua **(6)** if f is analytic on the closed circular domain $\bar{D}_\rho$, then f attains its maximum modulus on the circular manifold

$$(9)\qquad B_\rho = [z | zz^* = \rho^2 I].$$

Hence we get the Cauchy inequality:

$$(10)\qquad \frac{1}{V} \sum_{j=0}^{M_p} |a_j^{(p)}|^2 \leqslant (1/\rho^{2p}) \max_{z \in B\rho} |f(z)|^2.$$

THEOREM 4.3 (mean value theorem). *$I(\rho)$ defined by (7) is a monotone increasing function of $\rho (0 < \rho < 1)$ and $\log I(\rho)$ is a convex function of $\log \rho$.*

Proof. The monotonicity of $I(\rho)$ is obvious from (8). The proof of convexity is the same as the proof in **(17**, p. 174**)** for the one variable case.

3. *A mean value theorem over B_ρ.*

THEOREM 4.4. *In the case $n = m$ the integral*

$$(11)\qquad I_1(\rho) = \int_{B\rho} |f|^2 dV$$

is a monotone increasing function of ρ and $\log I_1(\rho)$ is a convex function of $\log \rho$.

Proof. Hua **(6)** has shown how to construct a set $\{\psi_\nu\}$ orthonormalized with respect to the inner product

$$(\psi_\nu, \psi_\mu)_B = \int_B \psi_\nu \bar{\psi}_\mu dV$$

from the CONS $\{\phi_\nu\}$ as follows. Since B is a circular space $(\phi_j^{(p)}, \phi_k^{(q)})_B = 0$ if $p \neq q$. Define the vector

$$z_p = (\phi_1^{(p)}, \ldots, \phi_{M_p}^{(p)}).$$

Then

$$(z_p', z_p)_B = ((\phi_j^{(p)}, \phi_k^{(p)})_B) = K_p$$

is a matrix of constants. Since $z_p'\bar{z}_p > 0$ if

$$z_p z_p^* = |\phi_1^{(p)}|^2 + \ldots + |\phi_{M_p}^{(p)}|^2$$

is positive, $K_p > 0$ and there exists a unitary matrix U such that $U^* K_p U = \Lambda$, where Λ is a diagonal matrix with positive elements on the diagonal. Now $\{y_p\}$, defined by

$$y_p = z_p \bar{U} = (\theta_1^{(p)}, \ldots, \theta_{M_p}^{(p)}),$$

is a CONS on D if $\{z_p\}$ is, since

$$((z_p\bar{U})', z_p U) = U^*(z_p', z_p)U = U^*((\phi_j^{(p)}, \phi_k^{(p)}))U = U^* I U = I.$$

Let

$$\psi_j^{(p)} = \theta_j^{(p)} / ||\theta_j^{(p)}||_B.$$

Then $\{\psi_j^{(p)}\}$ is an ONS with respect to integration over B and the orthogonal development (5) of $f \in L^2$ can be written as

$$f(z) = \sum_{p=0}^{\infty} \sum_{j=1}^{M_p} b_j^{(p)} \psi_j^{(p)}, \tag{12}$$

where

$$b_j^{(p)} = (f, \psi_j^{(p)}) / ||\psi_j^{(p)}||^2.$$

Multiply (12) by $\bar{f}$ and integrate over B_ρ. By a procedure similar to that in paragraph 3 we obtain the formula

$$I_1(\rho) = \int_{B_\rho} |f|^2 dV = \sum_{p=0}^{\infty} \rho^{2p} \sum_{j=1}^{M_p} |b_j^{(p)}|^2, \tag{13}$$

from which the conclusions of the theorem follow.

Note added in Proof. Recently it has come to my attention that Hua and Look **(21)** have proved that $F(z) \to f(u_0)$ as $z \to u_0$ in any manner. Further for continuous f on B, the solution F given by (3) is unique. They also consider Abel summability for continuous functions of the unitary group.

References

1. A. A. Albert, *Modern higher algebra* (Chicago, 1937).
2. S. Bergman, *The kernel function and conformal mapping*, Math. Surveys, No. V (1950).
3. S. Bochner, *Group invariance of Cauchy's formula in several variables*, Ann. Math., *45* (1944), 686–707.

4. T. J. I'A Bromwich and G. H. Hardy, *Some extensions to multiple series of Abel's theorem on continuity of power series*, Proc. London Math. Soc., *2* (1905), 161–189.

5. E. Cartan, *Sur les domains bornés homogènes de l'espace de n variables complexes*, Abh. Math. Sem. Univ. Hamburg, *11* (1936), 116–162.

6. L. K. Hua, *On the theory of functions of several complex variables*, I-III, Acta Math. Sinica, *2* (1953), 288–323; *5* (1955), 1–25, 205–242; (in Chinese), M.R., *17* (1956), p. 191. Also *Harmonic analysis of the classical domain in the study of analytic functions of several complex variables*, mimeographed notes (about 1956).

7. K. Knopp, *Limitierungs-Umkehrsätze für Doppelfolgen*, Math. Z., *45* (1939), 573–589.

8. D. B. Lowdenslager, *Potential theory and a generalized Jensen-Nevalinna formula for functions of several complex variables*, Jn. Math. Mech., *7* (1958), 207–218.

9. ——— *Potential theory in bounded symmetric homogeneous complex domains*, Ann. Math., *67* (1958), 467–484.

10. C. C. Macduffee, *The theory of matrices* (Ergebnisse der Mat. und ihrer Grenzgebiete [Chelsea, 1946]).

11. J. Mitchell, *On double Sturm-Liouville series*, Amer. J. Math., *65* (1943), 616–636.

12. ——— *An example of a complete orthonormal system and the kernel function in the geometry of matrices*, Proceedings of the Second Canadian Mathematical Congress (Vancouver, 1949), 155–163.

13. ——— *On the spherical summability of multiple orthogonal series*, Trans. Amer. Math. Soc., *71* (1951), 136–151.

14. ——— *Potential theory in the geometry of matrices*, Trans. Amer. Math. Soc., *79* (1955), 401–422.

15. ——— *Orthogonal systems on matric spaces*, West. Research Lab. Pub., Scientific Paper 60-94801-1-P2 (1956).

16. K. Morita, *On the kernel functions of symmetric domains*, Sci. Reports of Tokyo Kyoiku Daigaku Sec. A, *5* (1956), 190–212.

17. E. C. Titchmarsh, *The theory of functions* (2nd ed., Oxford University Press, 1939).

18. S. H. Tung, *Tauberian theorems for multiple series*, unpublished Master's thesis (The Pennsylvania State University).

19. H. Weyl, *The classical groups* (Princeton University Press, 1946).

20. A. Zygmund, *Trigonometrical series*, Monog. Mat., Tom V (Warsaw, 1935).

21. L. K. Hua and K. H. Look, *Theory of harmonic functions in classical domains*, Sci. Sinicd, 8, No. 10 (1959), 1032-1094.

The Pennsylvania State University