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VARIATIONS ON THE HAMILTONIAN THEME

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1. **Introduction.** As its name implies, this paper consists of observations on various topics in graph theory that stem from the concept of Hamiltonian cycle. We shall mainly adopt the notation and terminology of Harary [5]. However, we use *vertices* and *edges* for what are called "points" and "lines" in [5]. $V(G)$, $E(G)$ respectively will denote the sets of vertices and edges of graph G, and $|X|$ will denote the cardinal of the set X. $|V(G)|$ is the *order* of G, and $|E(G)|$ the *size* of G. Throughout n is reserved for the order of G.

A *Hamiltonian cycle* in a graph G is a simple cycle which passes through every vertex of G; if G has a Hamiltonian cycle, G is *Hamiltonian*; G is *hypo-Hamiltonian* if G is non-Hamiltonian but every vertex-deleted subgraph $G-v$ of G is Hamiltonian.

Our first variation is concerned with extremal non-Hamiltonian graphs. We show that, except for $n=5$, there is a unique non-Hamiltonian graph of order n and maximum possible size. We then investigate non-Hamiltonian graphs whose supergraphs are all Hamiltonian, and show that the minimum possible size of such graphs is $3n/2$. In §3 we exhibit some new hypo-Hamiltonian graphs.

2. **Extremal non-Hamiltonian Graphs.**

2.1. It was proved by Ore [8] that every graph of order n and size greater than $\frac{1}{2}(n^2-3n+4)$ is Hamiltonian. Ore noted that this result is best possible in that $K_{n-1}\cdot K_2$ (the connected graph with blocks K_{n-1} and K_2) and $K_2+\bar{K}_3$ (for $n=5$) are non-Hamiltonian and have exactly $\frac{1}{2}(n^2-3n+4)$ edges. We show in Theorem 1 that these are the only extremal graphs. Our proof makes use of the following lemma.

LEMMA 1. (Chvátal [3]). *Let G be a graph with partition $\{d_i\}_1^n$, where $d_1\le d_2 \le\cdots\le d_n$ and $n>2$. If*

$$d_k \le k < \frac{n}{2} \Rightarrow d_{n-k} \ge n-k,$$

then G is Hamiltonian.

THEOREM 1. *Let G be a non-Hamiltonian graph of order n and size $\frac{1}{2}(n^2-3n+4)$. Then either $G\simeq K_{n-1}\cdot K_2$ or $G\cong K_2+\bar{K}_3$.*

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Proof. Suppose that G satisfies the hypotheses of Theorem 1. If G has partition $\{d_i\}_1^n$ with $d_1 \le d_2 \le \cdots \le d_n$ then, by Lemma 1, there is some k such that

$$d_k \le k < \frac{n}{2} \quad \text{and} \quad d_{n-k} < n-k.$$

It follows that

$$(1) \qquad n^2-3n+4 = 2\,|E(G)| = \sum_{i=1}^{n} d_i \le k^2+(n-2k)(n-k-1)+k(n-1).$$

Hence

$$(2) \qquad 3k^2+k-4 \ge (2k-2)n \ge (2k-2)(2k+1).$$

Simplifying,

$$(k-1)(k-2) \le 0.$$

Therefore either $k=1$ (with equality holding in (1)) or $k=2$ (with equality holding in (1) and (2)). If $k=1$, $d_1=1$, $d_2=\cdots=d_{n-1}=n-2$, $d_n=n-1$, and $G \cong K_{n-1}\cdot K_2$. If $k=2$, $n=5$ and $d_1=d_2=d_3=2$, $d_4=d_5=4$. So $G \cong K_2+\bar{K}_3$.

In [1] the author extended Ore's result to $(n-1)$-cycles: if G has order n and size greater than $\frac{1}{2}(n^2-5n+12)$ then G has a cycle of length $n-1$. More recently the author [2] and Woodall [10] have obtained much more general results on the existence of $(n-r)$-cycles. By the method of proof of Theorem 1 (but using Theorem 1 of [2]) one can show that, for small values of r, the only graphs with no $(n-r+1)$-cycles and with maximum possible size are $K_{n-r}\cdot K_{r+1}$, $K_r+\bar{K}_{2r}$ (when $n=3r$) and $K_{r+1}+\bar{K}_{2r+1}$ (when $n=3r+2$). It would seem that such a result should hold for a wide range of values of r.

2.2. A graph G is *ready* for a Hamiltonian cycle if G is not Hamiltonian but has Hamiltonian paths between every pair of nonadjacent edges; that is, if the addition of any new edge to G results in a Hamiltonian graph. The graphs of §2.1 are ready for Hamiltonian cycles, and they have the maximum possible size consistent with this. We now look at the question of the minimum possible size that such graphs can have. Graphs realizing this minimum will be called *lower extremal graphs* (ready for a Hamiltonian cycle).

LEMMA 2.1. *If G is nonseparable and ready for a Hamiltonian cycle, then no vertex of degree* 2 *can be adjacent to a vertex of degree less than* 4 *in G.*

Proof. Let $d(x)=2$ and let y, z be adjacent to x. Then y and z are adjacent. For otherwise there would have to be a Hamiltonian path from y to z, and this is clearly impossible if $n>3$. (If $n=3$ G is separable, and this case is excluded.)

Clearly $d(y)\ge 2$. But $d(y)\ne 2$ since this would imply that either G was the cycle $xyzx$, or that z was a cut-vertex of G. Suppose that $d(y)=3$, and let y be adjacent to w (in addition to x and z). Since x is not adjacent to w, there is a Hamiltonian path from x to w, say $xyzu_1\ldots u_{n-4}w$. But then $yxzu_1\ldots u_{n-4}wy$ is a Hamiltonian cycle of G. Therefore $d(y)\ge 4$ and similarly $d(z)\ge 4$.

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LEMMA 2.2. *Let G be ready for a Hamiltonian cycle and let v and w be vertices of degree* 2 *in G with the same neighbours, x and y. Then* $G-\{v, w\}$ *is complete and hence the size of G is exactly* $\frac{1}{2}(n^2-5n+14)$.

Proof. Let u_1, u_2 be vertices of $G-\{v, w\}$. No path in G connecting u_1 and u_2 can contain both v and w, and hence no such path can be Hamiltonian.

LEMMA 2.3. *Let G be ready for a Hamiltonian cycle. If G contains vertices* $u_1, \ldots, u_k, v_1, \ldots, v_{k+1}$ *such that* $d(u_i)=2$ *and* u_i *is adjacent to* v_i *and* v_{i+1}, $1 \le i \le k$, *then* $k \le 2$. *Moreover, if* $k=2$ *then* $d(v_2)=n-1$.

Proof. If k were greater than 2 there would be no Hamiltonian path connecting v_2 and u_3. If $k=2$, there is no Hamiltonian path connecting v_2 with any other vertex.

THEOREM 2. *Let G be ready for a Hamiltonian cycle. If G has order* $n>6$ *and m vertices of degree* 2, *then the size of G is at least* $\frac{1}{2}(3n+m)$.

Proof. If G is separable, every block of G must be complete. The theorem is then easily verified.

By Lemma 2.2 we can assume that no two vertices of degree 2 have the same neighbours since, when $n>6$,

$$\tfrac{1}{2}(n^2-5n+14) \ge 2n \ge \tfrac{1}{2}(3n+m).$$

If G is nonseparable, let $u_1, \ldots, u_m$ be the vertices of degree 2 in G. Then, by Lemmas 2.1 and 2.3, $G'=G-\{u_1, \ldots, u_m\}$ has minimum degree at least 3. Hence

$$|E(G)| = |E(G')|+2m \ge \tfrac{3}{2}(n-m)+2m = \tfrac{1}{2}(3n+m).$$

COROLLARY 2.1. *If G is ready for a Hamiltonian cycle, and if G has order* $n>6$ *and size* $3n/2$, *then G is regular of degree* 3.

COROLLARY 2.2. *The Petersen graph is a lower extremal graph ready for a Hamiltonian cycle.*

3. **Hypo-Hamiltonian Graphs.** It is a simple observation that the minimum degree of any hypo-Hamiltonian graph must be at least 3. Cubic hypo-Hamiltonian graphs are therefore of special interest, since they have minimum possible size for their order. Only two cubic hypo-Hamiltonian graphs seem to be known. One is the Petersen graph, which is the smallest hypo-Hamiltonian graph [4], and the first member of an infinite family of hypo-Hamiltonian graphs constructed independently by Sousselier [6] and Lindgren [7]. The other is a graph of order 18, also found by Sousselier [6].

We first show that the Coxeter graph C_{28}, which Tutte proved non-Hamiltonian

[9], is in fact hypo-Hamiltonian. C_{28} is cubic, of order 28, and can be described as follows:

$$V(C_{28}) = \{a_i, b_i, c_i, d_i\}_{i=1}^{7}$$
$$E(C_{28}) = \begin{Bmatrix} (a_i, a_{i+1}), (b_i, b_{i+3}), (c_i, c_{i+2}) \\ (a_i, d_i), (b_i, d_i), (c_i, d_i) \end{Bmatrix}_{i=1}^{7}$$

(where indices are taken modulo 7).

C_{28} has the following 27-cycles:

$$a_2a_3a_4a_5a_6a_7d_7b_7b_3d_3c_3c_1d_1b_1b_4d_4c_4c_6d_6b_6b_2b_5d_5c_5c_7c_2d_2a_2$$

(using every vertex but a_1) and

$$a_1a_2d_2b_2b_5b_1b_4b_7d_7c_7c_2c_4d_4a_4a_3d_3b_3b_6d_6c_6c_1c_3c_5d_5a_5a_6a_7a_1$$

(using every vertex but d_1).

Using this, and the existence of the three automorphisms

(i) $i \to i+k$ (mod 7)
(ii) $i \to 2i$ (mod 7); $a \to c \to b \to a$
(iii) $i \to 3i$ (mod 7); $a \to b \to c \to a$,

it follows that C_{28} is hypo-Hamiltonian.

It has already been remarked that the Petersen graph is hypo-Hamiltonian. On a suggestion of Dr. U. S. R. Murty I investigated some generalized Petersen graphs $G_{k,l}$, defined by

$$V(G_{k,l}) = \{a_i, b_i\}_{i=1}^{k},$$
$$E(G_{k,l}) = \{(a_i, a_{i+1}), (b_i, b_{i+l}), (a_i, b_i)\}_{i=1}^{k}$$

(where indices are taken modulo k).

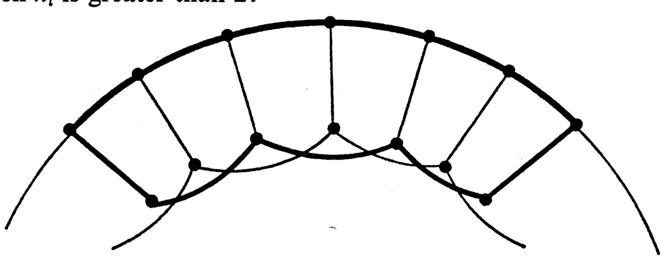
We shall call the cycle $a_1a_2\ldots a_ka_1$ the *rim* of $G_{k,l}$.

THEOREM 3. *$G_{k,2}$ is non-Hamiltonian if and only if $k=3m+2$ with m odd.*

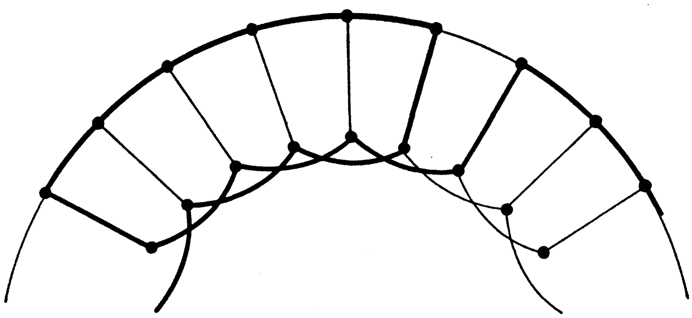
Proof. We shall indicate the proof mainly with the aid of diagrams as a detailed proof would be unnecessarily complicated.

Consider a Hamiltonian cycle (if such exists) of $G_{k,2}$. The intersection of this cycle with the rim of $G_{k,2}$ will be a sequence of paths of lengths $n_1, \ldots, n_r$ taken in order round the rim. The sequence $n_1, \ldots, n_r, n_{r+1}(=n_1)$ is restricted by the following observations:

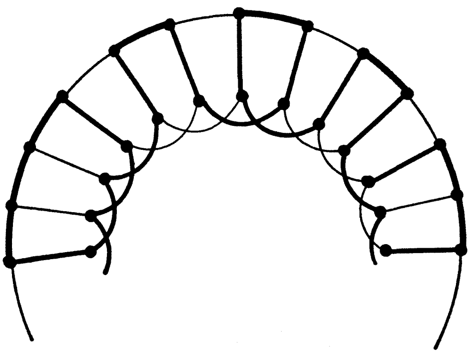
(i) No even n_i is greater than 2:



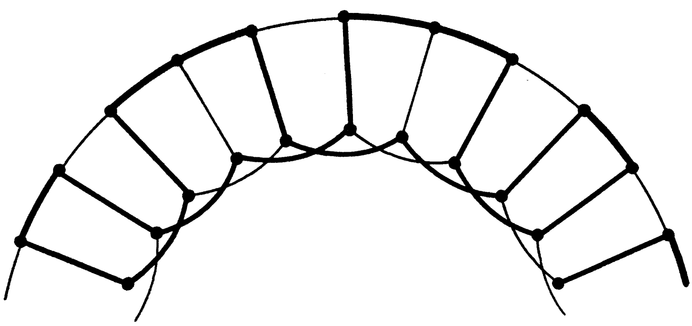
(ii) If $n_i > 2$, then $n_{i-1} = n_{i+1} = 1$:



(iii) If $n_i > 2$ then no $n_j = 2$:



(iv) If no $n_i > 2$, then either $n_j = n_{j+1} = 1$ for some j and $n_i = 2$ otherwise, or $n_i = 1$ for all i, or $n_i = 2$ for all i.



Now, by (i)–(iii), if $n_i > 2$ for some i then k must be even. If no $n_i > 2$ then, by (iv), either $k = 3r - 2$ or $k = 3r$ or $k = 2r$. Hence, if $k = 3m + 2$ with m odd, then $G_{k,2}$ is non-Hamiltonian. It is an easy matter (using details of this proof) to construct a Hamiltonian cycle for $G_{k,2}$ if k is not of this form.

COROLLARY 3.1. *If $k = 3m + 2$ with m odd, then $G_{k,2}$ is hypo-Hamiltonian.*

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Proof. We must show that $G_{k,2}-a_i$ and $G_{k,2}-b_i$ are Hamiltonian for $1 \leq i \leq k$. But clearly all the vertices a_i are similar, and all the vertices b_i are similar, and so we need only exhibit Hamiltonian cycles of $G-a_1$, and of $G-b_1$. $G-a_1$ has the Hamiltonian cycle

$$a_2a_3 \ldots a_kb_kb_{k-2} \ldots b_1b_{k-1} \ldots b_2a_2$$

and $G-b_1$ has the Hamiltonian cycle

$$a_1a_2b_2b_4 \ldots b_{k-1}a_{k-1}a_{k-2} \ldots a_3b_3b_5 \ldots b_ka_ka_1.$$

It is interesting, in this context, to note the following result obtained also by the above methods.

THEOREM 4. *$G_{k,3}$ is always Hamiltonian, with the sole exception of the Petersen graph ($k=5$).*

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