REGULAR ω-SEMIGROUPS

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Let S be a semigroup whose set E of idempotents is non-empty. We define a partial ordering $\geqq$ on E by the rule that $e \geqq f$ if and only if $ef = f = fe$. If $E = \{e_i : i \in N\}$, where N denotes the set of all non-negative integers, and if the elements of E form the chain

$$e_0 > e_1 > e_2 > \dots,$$

then S is called an *ω-semigroup.*

The purpose of this paper is to give a complete classification of regular ω-semigroups in terms of groups and group homomorphisms. The main problem is that of determining the structure of a simple regular ω-semigroup. It should be noted that if S is a simple semigroup containing a primitive idempotent (an idempotent that is minimal under the partial ordering of idempotents described above) then S is regular and its structure known [7; see also 3, Chapters 2,3]; we say that S is *completely simple*. The study of simple regular ω-semigroups can be regarded as a natural next step beyond that of completely simple semigroups.

In §1 some special cases of regular ω-semigroups are discussed; reference is made to them in later sections. Bisimple ω-semigroups constitute one important case; these semigroups, of which the bicyclic semigroup is an example, have been classified by Reilly [8].

A regular ω-semigroup S is necessarily an inverse semigroup. It is convenient to distinguish between the case in which S has a kernel and that in which it has not. In §2 it is shown that S has no kernel if and only if it is the union of a semilattice of groups, the semilattice in this case being an ω-chain. The structure of a regular ω-semigroup with no kernel is therefore determined by an infinite sequence of groups G_i and homomorphisms γ_i,

$$G_0 \overset{\gamma_0}{\to} G_1 \overset{\gamma_1}{\to} \dots \to G_n \overset{\gamma_n}{\to} \dots,$$

in accordance with a theorem of Clifford [2, §3; see also 3, Chapter 4]. On the other hand, if S has a kernel K then K is a simple regular ω-semigroup; further, if $K \neq S$ then the multiplication in S can be expressed in terms of that of K and of finitely many groups by means of certain connecting homomorphisms (Theorem 2.7).

In §3 we construct a simple regular ω-semigroup $S(d;\ G_i;\ \gamma_i)$ from a sequence of groups G_i and homomorphisms γ_i of the form

$$G_0 \overset{\gamma_0}{\to} G_1 \overset{\gamma_1}{\to} \dots \to G_{d-1} \overset{\gamma_{d-1}}{\to} G_0.$$

The integer d is characterised as the number of distinct $\mathscr{D}$-classes in $S(d;\ G_i;\ \gamma_i)$. It is then proved in §4 that this construction provides the most general simple regular ω-semigroup. Putting $d = 1$ we obtain the main theorems of [8]. The results of §§2, 3 and 4 combine to show that a regular ω-semigroup with a proper kernel K is determined by a sequence of groups G_i and homomorphisms γ_i of the form

$$G_0 \xrightarrow{\gamma_0} G_1 \xrightarrow{\gamma_1} \dots \to G_l \xrightarrow{\gamma_l} \dots \to G_{l+d-1} \xrightarrow{\gamma_{l+d-1}} G_l$$

for some $l > 0$ and $d > 0$.

Finally, in §5, necessary and sufficient conditions are given for two simple regular ω-semigroups, $S(d; G_i; \gamma_i)$ and $S(d^*; G_i^*; \gamma_i^*)$, to be isomorphic. This result is extended to the case of regular ω-semigroups with proper kernels.

1. Some examples of regular ω-semigroups. With a few minor exceptions, we shall throughout use the notation and terminology of [3]. The set of all non-negative integers will be denoted by N.

It is convenient to begin by listing various types of regular ω-semigroups to which we shall refer later.

(1.1) *The union of an ω-chain of groups.*

Let $\{G_i: i \in N\}$ be a set of pairwise-disjoint groups and for each $i \in N$ let γ_i be a homomorphism of G_i into G_{i+1}. For each pair $(i,j) \in N \times N$ such that $i < j$ let

$$\alpha_{i,j} = \gamma_i \gamma_{i+1} \cdots \gamma_{j-1}$$

and for each $i \in N$ let $\alpha_{i,i}$ denote the identity automorphism of G_i. Let $S = \bigcup_{i=0}^{\infty} G_i$ and define a multiplication on S by the rule that

$$a_i b_j = (a_i \alpha_{i,t})(b_j \alpha_{j,t}) \quad (a_i \in G_i, b_j \in G_j),$$

where $t = \max\{i,j\}$. Then S is a regular ω-semigroup. In fact, if e_i denotes the identity of G_i for all $i \in N$, then $e_i \geqq e_j$ if and only if $i \leqq j$. Write $T_n = \bigcup_{i=n}^{\infty} G_i$ $(n \in N)$. Then it is clear from the law of multiplication that T_n is an ideal of S for all $n \in N$. Moreover,

$$\bigcap_{n=0}^{\infty} T_n = \emptyset.$$

Hence S has no kernel.

Semigroups of the above type are a special case of those first studied by Clifford in [2, §3].

(1.2) *The bicyclic semigroup B.*

Let $B = N \times N$ and define a multiplication in B by the rule that

$$(m, n)(p, q) = (m-n+t, q-p+t),$$

where $t = \max\{n,p\}$. Then B is a bisimple ω-semigroup [3, p. 43 and Theorem 2.53]. The set of idempotents of B is $\{(n,n): n \in N\}$ and

$$(m, m) \geqq (n, n) \Leftrightarrow m \leqq n.$$

We call B the *bicyclic semigroup*. It occurs as a subsemigroup of every simple semigroup that contains a non-primitive idempotent [3, Theorem 2.54].

(1.3) *The semigroup* $S(G, \alpha)$.

The bicyclic semigroup can be generalised as follows. Let G be any group and let α be an endomorphism of G. Let $S = N \times G \times N$ and define a multiplication in S by

$$(m; g; n)(p; h; q) = (m-n+t;\ g\alpha^{t-n} . h\alpha^{t-p};\ q-p+t),$$

where $t = \max\{n, p\}$ and α^0 denotes the identity automorphism of G. Then S is a bisimple ω-semigroup, which we denote by $S(G, \alpha)$; moreover, every bisimple ω-semigroup is, to within isomorphism, of this type [**8**, Theorems 2.2 and 3.5]. Such a semigroup is necessarily regular [**3**, Theorem 2.11].

(1.4) *The semigroup* B_d.

Let d be any positive integer and let B_d be defined by

$$B_d = \{(m, n) \in B:\ m \equiv n \pmod d\},$$

where B is the bicyclic semigroup (1.2). Then B_d is a subsemigroup of B. Furthermore, it can be shown that B_d is a simple regular ω-semigroup with exactly d $\mathcal{D}$-classes. The $\mathcal{D}$-classes are the subsets

$$D_i = \{(m, n) \in B:\ m \equiv i \pmod d \quad \text{and} \quad n \equiv i \pmod d\} \quad (0 \leqq i < d),$$

and each D_i is a subsemigroup of B_d isomorphic to B itself.

(1.5) *The Bruck extension of the union of a finite chain of groups.*

Let A be any semigroup with an identity and let S denote the set $N \times A \times N$. Define a multiplication on S by the rule that

$$(m; a; n)(p; b; q) = \begin{cases} (m-n+p; b; q) & \text{if} \quad n < p, \\ (m; ab; q) & \text{if} \quad n = p, \\ (m; a; q-p+n) & \text{if} \quad n > p. \end{cases}$$

Then S is a simple semigroup with an identity. This construction was first used by Bruck [**1**, Theorem 8.3] to show that every semigroup can be embedded in a simple semigroup with an identity. We call S *the Bruck extension of* A [see also **10**, p. 569]. It can be verified that $(m; a; n)$ is an idempotent of S if and only if $m = n$ and $a^2 = a$. Further, $(m; a; n)$ is a regular element of S if and only if a is a regular element of A [**3**, Theorem 8.48].

Now let $\{G_i: i = 0, \ldots, d-1\}$ be a set of d pairwise-disjoint groups and, if $d > 1$, let γ_i be a homomorphism of G_i into G_{i+1} ($i = 0, \ldots, d-2$). Let $A = \bigcup_{i=0}^{d-1} G_i$ and let multiplication in A be defined as in (1.1), where $\alpha_{i,j}$ denotes $\gamma_i \ldots \gamma_{j-1}$ ($i < j$) and, for each i, $\alpha_{i,i}$ denotes the identity automorphism of G_i. Let e_i denote the identity of G_i. Then A is a regular semigroup with idempotents e_i ($i = 0, \ldots, d-1$); furthermore, $e_0 > e_1 > \ldots > e_{d-1}$. We call A *the union of a finite chain of groups.* Let S be the Bruck extension of A. Then S is regular since A is regular. Also, the set of idempotents of S is

$$\{(m; e_i; m): m \in N; i = 0, \ldots, d-1\}$$

and it can be verified that

$$(m; e_i; m) > (n; e_j; n) \quad \Leftrightarrow \quad \textit{either } m < n \textit{ or } (m = n \textit{ and } i < j).$$

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It follows that S is an ω-semigroup. Thus S is a simple regular ω-semigroup. If, for each i, we take $G_i = \{e_i\}$ then S reduces to the semigroup B_d of (1.4). (In fact, the mapping $(m; e_i; n) \to (md+i, nd+i)$ is an isomorphism of S onto B_d.)

The regular ω-semigroups in (1.2), (1.3), (1.4) and (1.5) are simple. We conclude this section with an example of a simple ω-semigroup that fails to be regular. Take A to be the three-element semigroup $\{0, a, 1\}$, where 0 and 1 are, respectively, the zero and identity elements of A and $a^2 = 0$. Let S be the Bruck extension of A. Then a is not a regular element of A and so S is not regular. The set of idempotents of S is $\{(m;\ e;\ m) \colon m \in N,\ e = 0 \text{ or } 1\}$ and it is easily seen that

$$(0;1;0) > (0;0;0) > (1;1;1) > (1;0;1) > (2;1;2) > (2;0;2) > \dots .$$

Thus S is an ω-semigroup.

2. Preliminary results. In this section we shall reduce the problem of determining the structure of regular ω-semigroups to that of determining the structure of *simple* regular ω-semigroups.

First, [8, Lemma 2.1] and [5, Theorem 3.2] combine to give

THEOREM 2.1. *Let S be a regular ω-semigroup. Then S is an inverse semigroup with an identity and $\mathcal{H}$ is a congruence on S.*

We now establish some notation that will be used throughout the remainder of the paper. To save repetition, the full hypotheses will not be restated for successive lemmas.

Let S be a regular ω-semigroup and let $\{e_n : n \in N\}$ be the set of idempotents of S, where $e_m \geqq e_n$ if and only if $m \leqq n$. Let the $\mathcal{R}$-[$\mathcal{L}$-]class of S containing e_n be denoted by R_n [L_n] for all $n \in N$. With the usual partial ordering of the $\mathcal{R}$- and $\mathcal{L}$-classes [3, §6.6] we then have

$$R_0 > R_1 > R_2 > \dots \quad \text{and} \quad L_0 > L_1 > L_2 > \dots .$$

Write $H_{i,j} = R_i \cap L_j$. The following statements are easily seen to be equivalent:

$$\text{(i) } H_{i,j} \neq \emptyset, \qquad \text{(ii) } (e_i, e_j) \in \mathcal{D}, \qquad \text{(iii) } H_{j,i} \neq \emptyset.$$

The non-empty sets $H_{i,j}$ are just the $\mathcal{H}$-classes of S. We note that if $x \in H_{i,j}$ then $xx^{-1} = e_i$ and $x^{-1}x = e_j$. Evidently $e_i \in H_{i,i}$ and so each $H_{i,i}$ is a group [3, Theorem 2.16].

LEMMA 2.2. *Let $i, j \in N$ and let $t = \max\{i,j\}$. Then*

(i) *$H_{i,i}H_{j,j} \subseteq H_{t,t}$ and $H_{j,j}H_{i,i} \subseteq H_{t,t}$;*

(ii) *$e_i b_j = b_j e_i$ for all $b_j \in H_{j,j}$.*

Proof. We have that $e_i e_j = e_t = e_j e_i$. But, by Theorem 2.1, $\mathcal{H}$ is a congruence on S. Hence (i) holds. Let $b_j \in H_{j,j}$. Then $e_i b_j \in H_{t,t}$ and so $e_i b_j = e_t e_i b_j e_t = e_t b_j e_t$. Similarly, $b_j e_i = e_t b_j e_t$. Thus we obtain (ii).

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By Lemma 2.2(i), $\bigcup_{n=0}^{\infty} H_{n,n}$ is a subsemigroup of S. Since it is both a union of groups and an ω-semigroup, it has the structure described in (1.1) [3, Theorem 4.11].

Write $S_i = e_i S e_i$ $(i \in N)$. The main properties of S_i are described in the next lemma.

Lemma 2.3.

(i) *S_i is a regular ω-semigroup with identity e_i and group of units $H_{i,i}$.*

(ii) $S_i = \bigcup\{H_{r,s}\colon\ r \geqq i \ \text{ and } \ s \geqq i\}$.

(iii) *Let $(e_i, e_j) \in \mathscr{D}$. Then there exists an isomorphism θ of S_i onto S_j such that $(x, x\theta) \in \mathscr{D}$ for all $x \in S_i$.*

Proof. (i) It is clear that S_i is a subsemigroup of S with identity e_i and group of units $H_{i,i}$ (the maximal subgroup of S containing e_i). Let $x \in S_i$. Then $x = e_i x e_i$ and so $x^{-1} = e_i x^{-1} e_i \in S_i$. Thus S_i is regular. Also $e_j \in S_i$ for all $j \geqq i$ and so S_i is an ω-semigroup.

(ii) Let $x \in S_i$. Then $x \in e_i S$ and therefore $x \in R_r$ for some $r \geqq i$. Similarly, $x \in L_s$ for some $s \geqq i$. Hence $x \in H_{r,s}$ for some $r \geqq i$ and $s \geqq i$. Conversely, let $y \in H_{r,s}$ for some $r \geqq i$, $s \geqq i$. Then $y = e_r y e_s = e_i(e_r y e_s)e_i \in S_i$.

(iii) Since $(e_i, e_j) \in \mathscr{D}$ it follows that $H_{i,j} \neq \emptyset$. Let $a \in H_{i,j}$. It can readily be shown that $a^{-1}xa \in S_j$ for all $x \in S_i$ and that the mapping $\theta\colon S_i \to S_j$ defined by $x\theta = a^{-1}xa$ $(x \in S_i)$ is an isomorphism of S_i onto S_j [6, Lemma 1]. Let $x \in S_i$. Then $xaa^{-1} = xe_i = x$ and so $(xa, x) \in \mathscr{R}$. Also $a(a^{-1}xa) = e_i xa = xa$; therefore $(xa, a^{-1}xa) \in \mathscr{L}$. Thus $(x, a^{-1}xa) \in \mathscr{D}$ and this completes the proof.

The maximal subgroups of S are the sets $H_{n,n}$ and we have already noted that $\bigcup_{n=0}^{\infty} H_{n,n}$ is a regular ω-subsemigroup of S with the structure described in (1.1). It will now be shown that if $S \neq \bigcup_{n=0}^{\infty} H_{n,n}$ then S has a kernel.

Lemma 2.4. *Let S be such that $R_i \neq H_{i,i}$ for some $i \in N$ and let l be the least such integer i. Then S_l is the kernel of S and is a simple regular ω-semigroup. If $l > 0$ then*

$$S = A \cup S_l, \quad A \cap S_l = \emptyset,$$

where A is the subsemigroup $\bigcup_{i=0}^{l-1} H_{i,i}$ of S.

Proof. By Lemma 2.3(ii), $S_l = \bigcup\{H_{r,s}\colon r \geqq l \text{ and } s \geqq l\}$. If $H_{i,j} \neq \emptyset$ for $i \neq j$ and $j < l$, then $H_{j,i} \neq \emptyset$ and so $R_j \neq H_{j,j}$, which contradicts the definition of l. Thus, for $i \geqq l$, $R_i = \bigcup\{H_{i,j}\colon j \geqq l\} \subseteq S_l$. Similarly, $L_i \subseteq S_l$ $(i \geqq l)$.

We show first that S_l is an ideal of S. Let $a \in H_{r,s}$ for some $r \geqq l$, $s \geqq l$ and let $x \in S$. Then $ax \in R_i$ for some $i \geqq r$. Since $i \geqq l$ it follows that $ax \in S_l$. Similarly $xa \in S_l$.

Next we show that S_l is simple. Let $h \in R_l \backslash H_{l,l}$ and let $k = h^{-1}$. Then h and k lie in S_l. Also $hk = e_l$, $kh \neq e_l$ and so S_l contains the infinite descending chain of idempotents

$$e_l = hk > kh > k^2h^2 > \ldots > k^nh^n > \ldots \tag{2.4a}$$

[3, Lemma 1.31]. Let T be any ideal of S_l and let $x \in T$. Then $xx^{-1} = e_n$ for some $n \in N$; also $e_n \in T$. From (2.4a), $e_n \geqq k^nh^n$. Hence

$$e_l = (h^nk^n)^2 = h^n(k^nh^ne_n)k^n \in T$$

and so $S_l \subseteq T$. Thus S_l is simple. Being an ideal of S, S_l is the kernel of S. Moreover, by 2.3(i), S_l is a regular ω-semigroup.

Finally, let $l > 0$. Since $A = \bigcup_{i=0}^{l-1} R_i$ and $S_l = \bigcup_{i=l}^{\infty} R_i$ we see that $S = A \cup S_l$ and $A \cap S_l = \emptyset$. Furthermore, by Lemma 2.2(i), A is a subsemigroup of S.

Corollary 2.5. *S is simple if and only if $R_0 \neq H_{0,0}$.*

Proof. Let $R_0 \neq H_{0,0}$. Then, by Lemma 2.4, S_0 is simple and $S = S_0$. Conversely, suppose that S is simple. If $R_n = H_{n,n}$ for all $n \in N$ then S would be the union of an ω-chain of groups (1.1) and so would possess proper ideals. Hence $R_i \neq H_{i,i}$ for some $i \in N$. Let l be the least such i. Then, by Lemma 2.4, S_l is an ideal of S and is proper if $l > 0$; hence $l = 0$.

We now give a characterisation of a regular ω-semigroup without a kernel.

Theorem 2.6. *Let S be a regular ω-semigroup. The following conditions on S are equivalent.*

(i) *S has no kernel.*

(ii) *The idempotents of S are central.*

(iii) *S is the union of an ω-chain of groups.*

Proof. We first show the equivalence of (i) and (iii). Let S have no kernel. Then, by Lemma 2.4, $R_n = H_{n,n}$ for all $n \in N$ and so $S = \bigcup_{n=0}^{\infty} H_{n,n}$. This establishes (iii). Conversely, as was shown in (1.1), the union of an ω-chain of groups has no kernel.

Liber [4] has shown that an inverse semigroup is a union of groups if and only if its idempotents are central. The equivalence of (ii) and (iii) is a special case of this result.

The final result of this section concerns the structure of a regular ω-semigroup with a proper kernel.

Theorem 2.7. *Let $G_0, \ldots, G_{l-1}$ be a set of pairwise-disjoint groups for some $l > 0$ and let K be a simple regular ω-semigroup, disjoint from each G_i, with group of units G. Write $G_l = G$. For each i such that $0 \leqq i \leqq l-1$ let γ_i be a homomorphism of G_i into G_{i+1}. For $0 \leqq i < j \leqq l$ define $\alpha_{i,j}$ to be $\gamma_i\gamma_{i+1}\cdots\gamma_{j-1}$ and let $\alpha_{i,i}$ be the identity automorphism of G_i $(0 \leqq i \leqq l-1)$. Let $S = G_0 \cup G_1 \cup \ldots \cup G_{l-1} \cup K$. Define a multiplication $(\circ)$ in S, extending that of K and of each G_i, as follows:*

(i) $a_i \circ b_j = (a_i \alpha_{i,t})(b_j \alpha_{j,t})$,

(ii) $a_i \circ x = (a_i \alpha_{i,l})x, \quad x \circ a_i = x(a_i \alpha_{i,l})$,

(iii) $x \circ y = xy$,

where $a_i \in G_i$ $(0 \leqq i \leqq l-1)$, $b_j \in G_j$ $(0 \leqq j \leqq l-1)$, $t = \max\{i,j\}$ and $x, y \in K$. Then S is a regular ω-semigroup with kernel K.

Conversely, if S is a regular ω-semigroup with kernel $K \neq S$ then S is isomorphic to a semigroup constructed as above.

Proof. Let $A = \bigcup_{i=0}^{l-1} G_i$. Then by [3, Theorem 4.11], A is a semigroup under $(\circ)$. Define a mapping $\theta: A \to S$ by $a_i\theta = a_i\alpha_{i,l}$ $(a_i \in G_i;\ 0 \leqq i \leqq l-1)$. It is easily verified from (i) that θ is a homomorphism. Applying [3, Theorem 4.19], we see that $S(= A \cup K)$ is a semigroup. Since K is a simple ideal of S, it is the kernel of S. Let e_i be the identity of G_i $(0 \leqq i \leqq l-1)$ and let $\{f_i : i \in N\}$ be the set of idempotents of K, where $f_0 > f_1 > f_2 > \dots$. From (i) we see that

$$e_0 > e_1 > \dots > e_{l-1}$$

and, from (ii), that $e_{l-1} \circ f_0 = f_0 = f_0 \circ e_{l-1}$; that is, $e_{l-1} > f_0$. Thus S is an ω-semigroup. That S is regular follows from the fact that its subsemigroups K, G_i $(i = 0, \dots, l-1)$ are regular.

Conversely, let S be a regular ω-semigroup with kernel $K \neq S$. We use the notation established earlier. By Theorem 2.6, $R_i \neq H_{i,i}$ for some $i \in N$. Let l be the least such integer i. By Lemma 2.4, $K = S_l$. Since $K \neq S$, it follows that $l > 0$ and so $S = A \cup K$, where $A = \bigcup_{i=0}^{l-1} H_{i,i}$. Write $G_i = H_{i,i}(0 \leqq i \leqq l)$. Then, for $0 \leqq i \leqq l$, $0 \leqq j \leqq l$ and $t = \max\{i,j\}$, we have that $G_i G_j \subseteq G_t$, by Lemma 2.2(i). Also $a_i e_{i+1} = e_{i+1} a_i$ for all $a_i \in G_i$ $(0 \leqq i \leqq l-1)$ by Lemma 2.2(ii) and so the mapping $\gamma_i : G_i \to G_{i+1}$ defined by $a_i \gamma_i = a_i e_{i+1}$ is a homomorphism (see [3, Theorem 4.11]). It then follows easily that the structure of A is as described in (i). Now let $a_i \in G_i$ $(0 \leqq i \leqq l-1)$ and let $x \in K$. Then

$$a_i x = a_i(e_l x) = (a_i e_{i+1} e_{i+2} \dots e_l)x = (a_i \alpha_{i,l})x,$$

where $\alpha_{i,l} = \gamma_i \gamma_{i+1} \cdots \gamma_{l-1}$. Similarly, $xa_i = x(a_i \alpha_{i,l})$. This completes the proof.

(2.8) It is easily verified that the $\mathscr{J}$-classes of S in the above theorem are the sets $G_0, \dots, G_{l-1}$, K and that, under the natural ordering of these classes [3, § 6.6],

$$G_0 > G_1 > \dots > G_{l-1} > K.$$

Now suppose that a second regular ω-semigroup S^* is defined similarly in terms of a simple regular ω-semigroup K^*, groups G_i^* $(i = 0, \dots, l^*)$, where $G_{l^*}^*$ is the unit of group K^*, and homomorphisms γ_i^* $(i = 0, \dots, l^*-1)$. Then $S \cong S^*$ if and only if the following conditions are satisfied:

(i) $l = l^*$;

(ii) there exists an isomorphism ϕ of K onto K^*;

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(iii) for $i = 0, \ldots, l$ there exists an isomorphism θ_i of G_i onto G_i^* and, in particular, $\theta_l = \phi \mid G_l$;

(iv) $\theta_i \gamma_i^* = \gamma_i \theta_{i+1} \quad (i = 0, \ldots, l-1)$.

We omit the proof.

3. **The semigroup** $S(d; G_i; \gamma_i)$. In this section we give a process for constructing a simple regular ω-semigroup from a finite set of groups and homomorphisms. It will then be shown (§4) that this construction yields the most general type of simple regular ω-semigroup.

Let d be a positive integer and let $\{G_i : i = 0, \ldots, d-1\}$ be a set of d pairwise-disjoint groups. Let γ_{d-1} be a homomorphism of G_{d-1} into G_0 and, if $d > 1$, let γ_i be a homomorphism of G_i into G_{i+1} $(i = 0, \ldots, d-2)$. Thus we have the sequence

$$G_0 \xrightarrow{\gamma_0} G_1 \xrightarrow{\gamma_1} \ldots \to G_{d-1} \xrightarrow{\gamma_{d-1}} G_0 .$$

For all $n \in N$ let $\gamma_n = \gamma_{n (\mathrm{mod}\ d)}$. For $m, n \in N$ and $m < n$ write

$$\alpha_{m,n} = \gamma_m \gamma_{m+1} \cdots \gamma_{n-1}$$

and for all $n \in N$ let $\alpha_{n,n}$ denote the identity automorphism of $G_{n (\mathrm{mod}\ d)}$. Let S be the set of all ordered triples

$$(m; a_i; n),$$

where $m, n \in N$, $0 \leqq i \leqq d-1$ and $a_i \in G_i$. Define a multiplication in S by the rule that

$$(m; a_i; n)(p; b_j; q) = (m-n+t; (a_i \alpha_{u,w})(b_j \alpha_{v,w}); q-p+t), \tag{3.1}$$

where $t = \max\{n, p\}$, $u = nd+i$, $v = pd+j$ and $w = \max\{u, v\}$. Denote the groupoid so formed by $S(d; G_0, \ldots, G_{d-1}; \gamma_0, \ldots, \gamma_{d-1})$ or, more compactly, by $S(d; G_i; \gamma_i)$.

The main result of this section (Theorem 3.3) is that $S(d; G_i; \gamma_i)$ is a simple regular ω-semigroup with exactly d $\mathscr{D}$-classes.

Remarks. Let $m, n \in N$ and let $m \leqq n$. Then $\alpha_{m,n}$ is a homomorphism of $G_{m(\mathrm{mod}\ d)}$ into $G_{n(\mathrm{mod}\ d)}$ and, for all $r \in N$,

$$\alpha_{m,n} = \alpha_{m+rd, n+rd} .$$

Moreover,

$$\alpha_{m,n} \alpha_{n,p} = \alpha_{m,p} \qquad (m \leqq n \leqq p).$$

The following special case of (3.1) should be noted:

$$(m; a_i; n)(n; b_j; q) = (m; (a_i \alpha_{i,t})(b_j \alpha_{j,t}); q),$$

where $t = \max\{i, j\}$. In particular, taking $i = j$ we have that

$$(m; a_i; n)(n; b_i; q) = (m; a_i b_i; q).$$

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LEMMA 3.2. *$S(d; G_i; \gamma_i)$ is a semigroup.*

Proof. Since the multiplication in (3.1) is such that the outer components of the triples reflect the multiplication in the bicyclic semigroup B (1.2), it is enough to consider the behaviour of the central components.

Let $a = (m; a_i; n)$, $b = (p; b_j; q)$, $c = (r; c_k; s)$. We shall show that the central components of $(ab)c$ and $a(bc)$ are the same. This will establish the lemma. To simplify the proof we make use of the subsemigroup B_d of B discussed in (1.4). Define elements a', b', c' of B_d by

$$a' = (md+i, nd+i), \quad b' = (pd+j, qd+j), \quad c' = (rd+k, sd+k).$$

Then

$$a'b' = ((m-n+t_1)d+u_1, \ (q-p+t_1)d+u_1),$$

where

$$t_1 d+u_1 = \max\{nd+i, pd+j\}, \qquad 0 \leqq u_1 < d,$$

and so

$$(a'b')c' = ((m-n+p-q+t_2)d+u_2, (s-r+t_2)d+u_2),$$

where

$$t_2 d+u_2 = \max\{(q-p+t_1)d+u_1, rd+k\}, \quad 0 \leqq u_2 < d.$$

A similar argument shows that

$$a'(b'c') = ((m-n+t_4)d+u_4, (s-r+q-p+t_4)d+u_4),$$

where

$$t_3 d+u_3 = \max\{qd+j, rd+k\}, \quad 0 \leqq u_3 < d,$$

and

$$t_4 d+u_4 = \max\{nd+i, (p-q+t_3)d+u_3\}, \quad 0 \leqq u_4 < d.$$

Comparing $(a'b')c'$ and $a'(b'c')$, we see from the associativity of B_d that

$$p-q = t_4-t_2 \quad \text{and} \quad u_2 = u_4. \tag{3.2a}$$

We use the same notation below. Consider the product $(ab)c$ in $S(d; G_i; \gamma_i)$. The central component of ab is

$$(a_i \alpha_{nd+i,\ t_1d+u_1})(b_j \alpha_{pd+j,\ t_1d+u_1}).$$

This lies in the group G_{u_1}; denote it by x_{u_1}. It then follows that the central component of $(ab)c$ is

$$(x_{u_1} \alpha_{(q-p+t_1)d+u_1,\ t_2d+u_2})(c_k \alpha_{rd+k,\ t_2d+u_2}).$$

Since $p-q+t_2 = t_4$, by (3.2a), we have that

$$x_{u_1} \alpha_{(q-p+t_1)d+u_1,\ t_2d+u_2} = x_{u_1} \alpha_{t_1d+u_1,\ t_4d+u_2} = (a_i \alpha_{nd+i,\ t_4d+u_2})(b_j \alpha_{pd+j,\ t_4d+u_2}).$$

Hence the central component of $(ab)c$ is

$$(a_i \alpha_{nd+i,\ t_4d+u_2})(b_j \alpha_{pd+j,\ t_4d+u_2})(c_k \alpha_{rd+k,\ t_2d+u_2}).$$

In the same way, it can be shown that the central component of $a(bc)$ is

$$(a_i \alpha_{nd+i,\ t_4d+u_4})(b_j \alpha_{qd+j,\ t_2d+u_4})(c_k \alpha_{rd+k,\ t_2d+u_4}).$$

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But since $t_4-p=t_2-q$ and $u_2=u_4$, by (3.2a), we have that

$$\alpha_{pd+j,\,t_4d+u_2}=\alpha_{qd+j,\,t_2d+u_4}$$

and so the central components of $(ab)c$ and $a(bc)$ are equal.

This completes the proof.

THEOREM 3.3. *$S(d;\,G_i;\,\gamma_i)$ is a simple regular ω-semigroup with exactly d $\mathscr{D}$-classes.*

Proof. Write $S=S(d;\,G_i;\,\gamma_i)$. Let f_i denote the identity of the group G_i and, for each $a_i\in G_i$, let a_i^{-1} denote the inverse of a_i in G_i $(i=0,\dots,d-1)$.

By Lemma 3.2, S is a semigroup. Let $(m;\,a_i;\,n)\in S$. Since $(m;\,a_i;\,n)(n;\,a_i^{-1};\,m)(m;\,a_i;\,n) = (m;\,a_i;\,n)$, S is regular.

We prove next that $\{(m;f_i;m): m\in N;\, i=0,\dots,d-1\}$ is the set of idempotents of S. It is clear that $(m;f_i;m)$ is an idempotent. Conversely, let $x=(m;\,a_i;\,n)$, where $x^2=x$. Then $m-n+t=n-m+t$, where $t=\max\{m,n\}$; hence $m=n$. Thus $x^2=(m;\,a_i^2;\,m)$ and so $a_i=f_i$. Now

$$(m;f_i;m)(n;f_j;n)=(t;f_k;t)=(n;f_j;n)(m;f_i;m),$$

where

$$td+k=\max\{md+i,nd+j\},\quad 0\leqq k<d.$$

It follows that, under the natural ordering, the idempotents form a chain

$$\begin{aligned}&(0;f_0;0)>(0;f_1;0)>\dots>(0;f_{d-1};0)\\ &>(1;f_0;1)>(1;f_1;1)>\dots>(1;f_{d-1};1)\\ &>(2;f_0;2)>(2;f_1;2)>\dots>(2;f_{d-1};2)\\ &>\dots.\end{aligned}$$

Thus S is a regular ω-semigroup. The identity of S is $(0;f_0;0)$ and it is readily verified that

$$(m;a_i;n)^{-1}=(n;a_i^{-1};m).$$

To show that S is simple it is enough to prove that $(0;f_0;0)$ lies in the ideal generated by an arbitrarily-chosen element $(m;\,a_i;\,n)$ of S. We have that

$$\begin{aligned}&(0;a_i^{-1}\alpha_{i,\,d};m+1)(m;a_i;n)(n+1;f_0;0)\\ &=(0;(a_i^{-1}\alpha_{i,\,d})(a_i\alpha_{md+i,\,(m+1)d});n+1)(n+1;f_0;0)\\ &=(0;(a_i^{-1}\alpha_{i,\,d})(a_i\alpha_{i,\,d});0)\\ &=(0;f_0;0)\end{aligned}$$

and this establishes the result.

Finally, we have to show that S has exactly d $\mathscr{D}$-classes. Since S is regular each $\mathscr{D}$-class contains an idempotent; hence it is sufficient to show that

$$((m;f_i;m),\,(n;f_j;n))\in\mathscr{D}\;\Leftrightarrow\; i=j.$$

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Write $e=(m;f_i;m)$, $e'=(n;f_j;n)$. We use the fact that $(e,e')\in\mathscr{D}$ if and only if there exists $x\in S$ such that $xx^{-1}=e$, $x^{-1}x=e'$.

First suppose that $(e,e')\in\mathscr{D}$. Let $x=(r;a_k;s)$, where $xx^{-1}=e$, $x^{-1}x=e'$. Since $xx^{-1}=(r;f_k;r)$ and $x^{-1}x=(s;f_k;s)$ we see, in particular, that $i=k=j$. Conversely, suppose that $i=j$. Take $x=(m;f_i;n)$. Then $xx^{-1}=e$ and $x^{-1}x=e'$, which shows that $(e,e')\in\mathscr{D}$.

This completes the proof of the theorem.

(3.4) With the notation of §2 it can be verified that

$$R_{md+i}=\{(m;a_i;n)\in S: a_i\in G_i, n\in N\}\quad (m\in N; i=0,\dots,d-1),$$

$$L_{nd+i}=\{(m;a_i;n)\in S: a_i\in G_i, m\in N\}\quad (n\in N; i=0,\dots,d-1)$$

and so

$$H_{md+i,\,nd+i}=\{(m;a_i;n)\in S: a_i\in G_i\}\quad (m,n\in N; i=0,\dots,d-1).$$

We conclude this section by discussing two special cases.

First take $d=1$. By Theorem 3.3, $S(1;G_0;\gamma_0)$ is a bisimple ω-semigroup. Write $G=G_0$ and $\alpha=\gamma_0$. Then α is an endomorphism of G. Also if $m,n\in N$ and $m\leqq n$ then the mapping $\alpha_{m,n}: G\to G$ is just α^{n-m}, where α^0 is interpreted as the identity automorphism of G. It follows that the multiplication in (3.1) reduces to that of the semigroup $S(G,\alpha)$ described in (1.3).

Next consider the case in which $\gamma_{d-1}: G_{d-1}\to G_0$ is the "zero homomorphism" defined by

$$x\gamma_{d-1}=f_0\quad (x\in G_{d-1}),$$

where f_0 is the identity of G_0. Let $u,v\in N$ and suppose that $u\leqq v$. Write $u=md+i$, $v=nd+j$ $(0\leqq i<d, 0\leqq j<d)$. Then

$$a_i\alpha_{u,v}=\begin{cases} f_j & \text{if } m<n,\\ a_i\alpha_{i,j} & \text{if } m=n,\end{cases}$$

where f_j is the identity of G_j. Let $A=\bigcup_{i=0}^{d-1} G_i$ and define a multiplication $(\circ)$ in A by

$$a_i\circ b_j=(a_i\alpha_{i,t})(b_j\alpha_{j,t}),$$

where $a_i\in G_i$, $b_j\in G_j$ and $t=\max\{i,j\}$. Then A is a semigroup and, from (3.1), we have that

$$(m;a_i;n)(p;b_j;q)=\begin{cases}(m-n+p;b_j;q) & \text{if } n<p,\\ (m;a_i\circ b_j;q) & \text{if } n=p,\\ (m;a_i;q-p+n) & \text{if } n>p.\end{cases}$$

Thus in this case $S(d;G_i;\gamma_i)$ reduces to the Bruck extension of A (see (1.5)).

4. The structure of a simple regular ω-semigroup. This section is devoted to showing that any simple regular ω-semigroup S is isomorphic to a semigroup of the type $S(d;G_i;\gamma_i)$. In particular, the number of $\mathscr{D}$-classes of S is finite.

We shall follow the notation of §2. Let S be a simple regular ω-semigroup. Then, by

Corollary 2.5, $R_0 \neq H_{0,0}$; that is, R_0 is the union of $H_{0,0}$ and certain other non-empty sets $H_{0,n}$. Let d be the smallest positive integer n such that $H_{0,n} \neq \emptyset$. Thus $(e_0, e_d) \in \mathscr{D}$ and $(e_0, e_i) \notin \mathscr{D}$ for any i such that $0 < i < d$.

LEMMA 4.1. *$(e_r, e_{nd+r}) \in \mathscr{D}$ for all $r, n \in N$.*

Proof. By Lemma 2.3 (iii) there exists an isomorphism θ of $S(=S_0)$ onto S_d such that $(x, x\theta) \in \mathscr{D}$ for all $x \in S$. Hence $\{e_r\theta : r \in N\}$ is the set of idempotents of S_d and

$$e_d = e_0\theta > e_1\theta > e_2\theta > \dots .$$

But the set of idempotents of S_d is $\{e_{d+r} : r \in N\}$. Thus $e_r\theta = e_{d+r}$ for all $r \in N$ and so

$$(e_r, e_{d+r}) \in \mathscr{D} \qquad (r \in N).$$

The result now follows by induction.

Since every idempotent of S is of the form e_{nd+i} for some $n \in N$ and some i such that $0 \leqq i < d$, the result shows that S has at most d distinct $\mathscr{D}$-classes. The next lemma establishes that there are exactly d $\mathscr{D}$-classes.

LEMMA 4.2. *Let $0 < i < j < d$. Then $(e_i, e_j) \notin \mathscr{D}$.*

Proof. Suppose that $(e_i, e_j) \in \mathscr{D}$. Then $H_{i,j} \neq \emptyset$. Let $h \in H_{i,j}$ and let $k = h^{-1}$. Then $hk = e_i$, $kh = e_j$ and $h, k \in S_i$. Hence by [3, Lemma 1.31] we have the strictly descending chain of idempotents

$$e_i > kh > k^2h^2 > \dots > k^nh^n > \dots .$$

By hypothesis, $e_0 > e_i > e_j > e_d$. Choose n such that

$$k^nh^n \geqq e_d \geqq k^{n+1}h^{n+1}.$$

Then it is easily seen that $h^ne_dk^n$ is an idempotent and that

$$h^n(k^nh^n)k^n \geqq h^ne_dk^n \geqq h^n(k^{n+1}h^{n+1})k^n .$$

But $h^n(k^nh^n)k^n = e_i$ and $h^n(k^{n+1}h^{n+1})k^n = e_i(kh)e_i = e_j$; hence

$$e_i \geqq h^ne_dk^n \geqq e_j . \tag{4.2a}$$

Now $(e_dk^n)h^n = e_d$ and so $(e_dk^n, e_d) \in \mathscr{R}$; further, $k^n(h^ne_dk^n) = e_dk^n$ and so $(h^ne_dk^n, e_dk^n) \in \mathscr{L}$. It follows that $(h^ne_dk^n, e_d) \in \mathscr{D}$. Hence $(h^ne_dk^n, e_0) \in \mathscr{D}$. This, together with (4.2a), contradicts the definition of d. Thus $(e_i, e_j) \notin \mathscr{D}$.

From the previous two lemmas we can now state which of the sets $H_{m,n}$ are non-empty.

LEMMA 4.3. *For any $m, n \in N$ the following conditions are equivalent.*

(i) $H_{m,n} \neq \emptyset$, (ii) $(e_m, e_n) \in \mathscr{D}$, (iii) $m \equiv n \pmod d$.

Proof. The equivalence of (i) and (ii) has already been noted. By Lemma 4.1, (iii) implies

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(ii). Now let $(e_m, e_n) \in \mathscr{D}$ and write $m = rd+i$, $n = sd+j$, where $0 \leqq i < d$, $0 \leqq j < d$. Then $(e_i, e_j) \in \mathscr{D}$ by Lemma 4.1. Hence $i = j$, by Lemma 4.2, and so $m \equiv n \pmod{d}$. Thus we have shown that (ii) implies (iii).

Let the $\mathscr{D}$-class of S containing e_i be denoted by D_i and let the group $H_{i,i}$ be denoted by G_i $(i = 0, \ldots, d-1)$. Evidently $S = \bigcup_{i=0}^{d-1} D_i$ and $D_i \cap D_j = \emptyset$ if $i \neq j$.

LEMMA 4.4.

(i) *D_i is a bisimple ω-semigroup with identity e_i and group of units G_i.*

(ii) *The $\mathscr{R}$- [$\mathscr{L}$-] classes of D_i are the sets $R_{nd+i}[L_{nd+i}]$ $(n \in N)$.*

Proof. (i) Since S is regular and its idempotents form a chain, each D_i is a bisimple inverse subsemigroup of S [9]. Moreover, by Lemma 4.3, the set of idempotents of D_i is $\{e_{nd+i} : n \in N\}$. Thus D_i is a bisimple ω-semigroup with identity e_i. The group of units of D_i is the maximal subgroup of D_i containing e_i. But $G_i \subseteq D_i$ and G_i is the maximal subgroup of S containing e_i. Hence the group of units of D_i is G_i.

(ii) From Lemma 4.3 we have that

$$D_i = \bigcup_{n=0}^{\infty} R_{nd+i} = \bigcup_{n=0}^{\infty} L_{nd+i}.$$

It is clear that each $\mathscr{R}$-class of D_i is contained in an $\mathscr{R}$-class of S. Let $n \in N$ and let $a, b \in R_{nd+i}$. To show that R_{nd+i} is an $\mathscr{R}$-class of D_i it is enough to prove that a and b are $\mathscr{R}$-equivalent in D_i. Since $(a, b) \in \mathscr{R}$ there exist elements $x, y \in S$ such that $a = bx$, $b = ay$. Write $x' = b^{-1}bxa^{-1}a$, $y' = a^{-1}ayb^{-1}b$. Then $a = bx'$, $b = ay'$. Further, $(a, x') \in \mathscr{L}$ and $(b, y') \in \mathscr{L}$. Hence $x', y' \in D_i$. This gives the required result.

From Lemma 4.4 we obtain

LEMMA 4.5. *Let $h_i \in H_{i,d+i}$ $(0 \leqq i < d)$ and let $k_i = h_i^{-1}$. Take $h_i^0 = k_i^0 = e_i$. Then*

$$k_i^m a_i h_i^n \in H_{md+i,\, nd+i} \qquad (m, n \in N; a_i \in G_i)$$

and the mapping $\psi : G_i \to H_{md+i,\, nd+i}$ defined by

$$x\psi = k_i^m x h_i^n \qquad (x \in G_i)$$

is a bijection.

This is essentially [8, Lemma 3.4] applied to the bisimple ω-semigroup D_i. We omit the proof.

Now choose an element $h \in H_{0,d}$ and let $k = h^{-1}$. Thus $hk = e_0$ and $kh = e_d$. For the remainder of this section h and k will be kept fixed. We make the convention that $h^0 = k^0 = e_0$.

LEMMA 4.6. $e_i h \in H_{i,d+i}$ $(i = 0, \ldots, d-1)$.

Proof. First, $(e_i h)(e_i h)^{-1} = e_i h k e_i = e_i e_0 e_i = e_i$ and so $e_i h \in R_i$. Let $e_i h \in L_n$. From

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Lemma 4.3, to show that $n = d+i$ it suffices to show that $d \leqq n < 2d$. We note that $ke_i h = (e_i h)^{-1}(e_i h) \in L_n$; hence $ke_i h = e_n$. Since $e_0 = hk \geqq e_i > kh = e_d$ it follows easily that

$$k(hk)h \geqq ke_i h \geqq k^2 h^2;$$

that is,

$$e_d \geqq e_n \geqq k^2 h^2.$$

Now from Lemma 4.5 we have that $k^2h^2 = k^2 e_0 h^2 \in H_{2d,\, 2d}$ and so $k^2h^2 = e_{2d}$. Furthermore, if $e_n = k^2h^2$ then $e_i = h(ke_i h)k = h(k^2h^2)k = kh = e_d$, which is a contradiction. Hence $e_d \geqq e_n > e_{2d}$ and this gives the required result.

LEMMA 4.7. *Every element of S is uniquely expressible in the form* $k^m a_i h^n$ $(m, n \in N;$ $0 \leqq i < d;\ a_i \in G_i)$.

Proof. Let $0 \leqq i < d$. We first show that $(e_i h)^n = e_i h^n$ $(n = 1, 2, 3, \ldots)$. This holds trivially for $n = 1$. Assume that $(e_i h)^r = e_i h^r$ for some positive integer r. By Lemma 4.6, $e_i h \in R_i \subseteq S_i$ and so $(e_i h)^r e_i = (e_i h)^r$. Hence

$$(e_i h)^{r+1} = (e_i h)^r h = e_i h^{r+1}$$

and the result follows.

Next, $ke_i = (e_i h)^{-1}$ and $(ke_i)^n = [(e_i h)^n]^{-1} = (e_i h^n)^{-1} = k^n e_i$ $(n = 1, 2, 3, \ldots)$. Now take $h_i = e_i h$ in Lemma 4.5. Then $k_i = h_i^{-1} = ke_i$. Also $h_i^n = e_i h^n$ for all positive integers n and this holds also for $n = 0$ since h_i^0 and h^0 are defined to be e_i and e_0 respectively. Similarly $k_i^m = k^m e_i$ $(m \in N)$.

Let $x \in S$. Then $x \in H_{md+i,\, nd+i}$ for some m, n, i $(0 \leqq i < d)$ by Lemma 4.3, and so, by Lemma 4.5,

$$\begin{aligned} x &= k_i^m a_i h_i^n \\ &= k^m e_i a_i e_i h^n \\ &= k^m a_i h^n. \end{aligned}$$

Moreover, if $k^m a_i h^n = k^r b_i h^s$ then $k_i^m a_i h_i^n = k_i^r b_i h_i^s$ and so $m = r$, $n = s$ and $a_i = b_i$ by Lemma 4.5. Thus the expression for x is unique.

LEMMA 4.8. *Let* $0 \leqq j < d$ *and let* $b_j \in G_j$. *Then* $hb_j \in H_{0,\, d}$.

Proof. We have that

$$(hb_j)(hb_j)^{-1} = he_j k = he_j khe_j k = he_j e_d e_j k = he_d k = hkhk = e_0.$$

Hence $hb_j \in R_0$. Also $b_j e_d = e_d b_j$, by Lemma 2.2(ii), and so

$$(hb_j)^{-1}(hb_j) = b_j^{-1} khb_j = b_j^{-1} e_d b_j = b_j^{-1} b_j e_d = e_j e_d = e_d.$$

Thus $hb_j \in L_d$. Therefore $hb_j \in H_{0,\, d}$.

In particular, $hx \in H_{0,d}$ for all $x \in G_{d-1}$. Now, by Lemma 4.5, every element of $H_{0,d}$ is expressible in the form gh for some unique $g \in G_0$. Hence we can define a mapping $\gamma_{d-1}: G_{d-1} \to G_0$ by the equation

$$hx = (x\gamma_{d-1})h \qquad (x \in G_{d-1}).$$

Now suppose that $d > 1$. Let i be such that $0 \leqq i \leqq d-2$. Then for each $x \in G_i$ we have that $xe_{i+1} = e_{i+1}x \in G_{i+1}$ (Lemma 2.2). Define a mapping $\gamma_i: G_i \to G_{i+1}$ by the rule that

$$x\gamma_i = xe_{i+1} \ (x \in G_i).$$

LEMMA 4.9. *γ_i is a homomorphism* ($i = 0, \ldots, d-1$).

Proof. Consider the case $i = d-1$. For all $x, y \in G_{d-1}$

$$(xy)\gamma_{d-1}h = h(xy) = (hx)y = (x\gamma_{d-1})hy = (x\gamma_{d-1})(y\gamma_{d-1})h$$

and so, since every element of $H_{0,d}$ is uniquely expressible in the form gh ($g \in G_0$),

$$(xy)\gamma_{d-1} = (x\gamma_{d-1})(y\gamma_{d-1}).$$

Suppose that $d > 1$. Let $0 \leqq i \leqq d-2$ and let $x, y \in G_i$. Then

$$(xy)\gamma_i = xye_{i+1} = xye_{i+1}^2 = xe_{i+1}\,ye_{i+1} = (x\gamma_i)(y\gamma_i).$$

We now extend the above definitions by writing

$$\gamma_n = \gamma_{n(\mathrm{mod}\ d)} \qquad (n \in N).$$

Thus γ_n is a homomorphism of $G_{n(\mathrm{mod}\ d)}$ into $G_{(n+1)(\mathrm{mod}\ d)}$. For $m, n \in N$ and $m<n$ write

$$\alpha_{m,n} = \gamma_m\gamma_{m+1}\cdots\gamma_{n-1}$$

and for each $n \in N$ let $\alpha_{n,n}$ be the identity automorphism of $G_{n(\mathrm{mod}\ d)}$. Then $\alpha_{m,n} = \alpha_{rd+m,rd+n}$ for all $r \in N$. Also, if $m \leqq n \leqq p$, then

$$\alpha_{m,n}\alpha_{n,p} = \alpha_{m,p}.$$

Furthermore, if $0 \leqq i \leqq j < d$ and $a_i \in G_i$ then

$$a_ie_j = a_i\alpha_{i,j} = e_ja_i.$$

LEMMA 4.10. *Let $a_i \in G_i$, $b_j \in G_j$ ($0 \leqq i < d$, $0 \leqq j < d$) and let r be a positive integer. Then*

(i) $a_ih^rb_j = a_i(b_j\alpha_{j,rd+i})h^r$,

(ii) $a_ik^rb_j = k^r(a_i\alpha_{i,rd+j})b_j$.

Proof. (i) We first note that $he_{d-1} = (e_{d-1}\gamma_{d-1})h = e_0h = h$. Now, by Lemma 4.8, $hb_j \in H_{0,d}$ and so, by Lemma 4.5, $hb_j = gh$ for some $g \in G_0$. Hence

$$(hb_j)e_{d-1} = (gh)e_{d-1} = gh = hb_j.$$

Thus $hb_j = h(b_je_{d-1}) = h(b_j\alpha_{j,d-1}) = (b_j\alpha_{j,d-1}\gamma_{d-1})h = (b_j\alpha_{j,d})h$.

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Consequently,

$$a_i h b_j = a_i(b_j \alpha_{j,d})h = (a_i e_i)(b_j \alpha_{j,d})h = a_i(b_j \alpha_{j,d}\alpha_{d,d+i})h = a_i(b_j \alpha_{j,d+i})h.$$

Thus the result holds for $r = 1$. Assume that it holds for $r = n-1$ $(n > 1)$ and for all i, j such that $0 \leqq i < d$, $0 \leqq j < d$. Then

$$a_i h^n b_j = a_i h^{n-1}(b_j \alpha_{j,d})h = a_i[(b_j \alpha_{j,d})\alpha_{0,(n-1)d+i}]h^{n-1}.h = a_i(b_j \alpha_{j,d}\alpha_{d,nd+i})h^n$$
$$= a_i(b_j \alpha_{j,nd+i})h^n.$$

Hence the result holds for $r = n$ and so, by induction, it holds for all positive integers r.

(ii) From (i) we have that

$$b_j^{-1}h^r a_i^{-1} = b_j^{-1}(a_i^{-1}\alpha_{i,rd+j})h^r$$

and so

$$a_i k^r b_j = (b_j^{-1}h^r a_i^{-1})^{-1} = (h^r)^{-1}(a_i^{-1}\alpha_{i,rd+j})^{-1}b_j = k^r(a_i \alpha_{i,rd+j})b_j.$$

We now come to the main result.

THEOREM 4.11. *Let S be a simple regular ω-semigroup. Then $S \cong S(d; G_i; \gamma_i)$ for some d, G_i, γ_i $(i = 0, \ldots, d-1)$.*

Proof. Let d, h, k, G_i, γ_i be as above. By Lemma 4.7, every element of S is uniquely expressible in the form $k^m a_i h^n$ $(m, n \in N; 0 \leqq i < d; a_i \in G_i)$.

Let $x = (k^m a_i h^n)(k^p b_j h^q)$, where $m, n, p, q \in N$, $0 \leqq i < d$, $0 \leqq j < d$, $a_i \in G_i$ and $b_j \in G_j$. To simplify this product we distinguish three cases.

(i) If $n > p$ then

$$\begin{aligned} x &= k^m a_i h^{n-p} b_j h^q \\ &= k^m a_i(b_j \alpha_{j,(n-p)d+i})h^{q-p+n}, \qquad \text{from Lemma 4.10(i)}, \\ &= k^m a_i(b_j \alpha_{pd+j,nd+i})h^{q-p+n}. \end{aligned}$$

(ii) If $n < p$ then

$$\begin{aligned} x &= k^m a_i k^{p-n} b_j h^q \\ &= k^{m-n+p}(a_i \alpha_{i,(p-n)d+j})b_j h^q, \qquad \text{from Lemma 4.10(ii)}, \\ &= k^{m-n+p}(a_i \alpha_{nd+i,pd+j})b_j h^q. \end{aligned}$$

(iii) If $n = p$ then

$$x = k^m a_i e_0 b_j h^q = k^m a_i b_j h^q = k^m(a_i \alpha_{i,s})(b_j \alpha_{j,s})h^q, \qquad \text{where } s = \max\{i, j\}.$$

All three cases can be combined as follows. Write $t = \max\{n, p\}$, $u = nd+i$, $v = pd+j$, $w = \max\{u, v\}$. Then

$$x = k^{m-n+t}(a_i \alpha_{u,w})(b_j \alpha_{v,w})h^{q-p+t}.$$

Thus the mapping $\theta: S \to S(d; G_i; \gamma_i)$ defined by

$$(k^m a_i h^n)\theta = (m; a_i; n)$$

is an isomorphism. This completes the proof.

(4.12) We conclude this section by combining the results of Theorems 2.7, 3.3 and 4.11. Let l and d be positive integers and let $\{G_i : i = 0, \ldots, l+d-1\}$ be a set of pairwise-disjoint groups. Let γ_{l+d-1} be a homomorphism of G_{l+d-1} into G_l and, for $i = 0, \ldots, l+d-2$, let γ_i be a homomorphism of G_i into G_{i+1}. Thus we have the sequence

$$G_0 \xrightarrow{\gamma_0} G_1 \xrightarrow{\gamma_1} \ldots \to G_l \xrightarrow{\gamma_l} \ldots \to G_{l+d-1} \xrightarrow{\gamma_{l+d-1}} G_l.$$

Write $G_i' = G_{l+i}$, $\gamma_i' = \gamma_{l+i}$ $(i = 0, \ldots, d-1)$ and let $K = S(d; G_i'; \gamma_i')$. Then K is a simple regular ω-semigroup (Theorem 3.3). The unit group of K is isomorphic to G_l. Now let S be constructed from $G_0, \ldots, G_{l-1}$, K and the homomorphisms $\gamma_0, \ldots, \gamma_{l-1}$ as in Theorem 2.7, where we identify γ_{l-1} with the homomorphism $x_{l-1} \to (0; x_{l-1}\gamma_{l-1}; 0)$ of G_{l-1} into the unit group of K. Then S is a regular ω-semigroup with kernel K. Denote it by

$$T(l; d; G_i; \gamma_i).$$

Conversely, let S be a regular ω-semigroup with a proper kernel K. Then from Theorems 2.7 and 4.11 we see that S is isomorphic to a semigroup of the above type. Note that the $\mathscr{R}$- and $\mathscr{L}$-classes of K are just the $\mathscr{R}$- and $\mathscr{L}$-classes of S that are contained in K.

5. The isomorphism theorem.

In the preceding sections we have established a construction for the most general simple regular ω-semigroup in terms of a finite collection of groups and homomorphisms. We now find necessary and sufficient conditions for two semigroups constructed by this process to be isomorphic.

THEOREM 5.1. *Let $S = S(d; G_i; \gamma_i)$ and let $S^* = S(d^*; G_i^*; \gamma_i^*)$. Then $S \cong S^*$ if and only if* (i) *$d = d^*$ and* (ii) *there exist isomorphisms θ_i of G_i onto G_i^* $(i = 0, \ldots, d-1)$ and an inner automorphism ζ^* of G_0^* such that the following diagram is commutative*:

$$\begin{array}{ccccccccc}
G_0 & \xrightarrow{\gamma_0} & G_1 & \xrightarrow{\gamma_1} & \ldots & \longrightarrow & G_{d-1} & \xrightarrow{\gamma_{d-1}} & & & G_0 \\
\downarrow{\scriptstyle \theta_0} & & \downarrow{\scriptstyle \theta_1} & & & & \downarrow{\scriptstyle \theta_{d-1}} & & & & \downarrow{\scriptstyle \theta_0} \\
G_0^* & \xrightarrow[\gamma_0^*]{} & G_1^* & \xrightarrow[\gamma_1^*]{} & \ldots & \longrightarrow & G_{d-1}^* & \xrightarrow[\gamma_{d-1}^*]{} & G_0^* & \xrightarrow[\zeta^*]{} & G_0^*
\end{array} \qquad (5.1a)$$

Proof. We use the notation of §3. Starred quantities refer to S^* throughout. We recall from (3.4) that

$$H_{md+i,\, nd+i} = \{(m; a_i; n) \in S : a_i \in G_i\}$$

for all $m, n \in N$ and for $i = 0, \ldots, d-1$.

First suppose that there exists an isomorphism ϕ of S onto S^*. By Theorem 3.3, d and d^* are the numbers of distinct $\mathscr{D}$-classes in S and S^* respectively. Hence $d = d^*$.

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Consideration of the chain of idempotents in S and in S^* shows that

$$(m;f_i;m)\phi=(m;f_i^*;m) \quad (m\in N; i=0,\dots,d-1), \tag{5.1b}$$

where f_i is the identity of G_i and f_i^* the identity of G_i^*. Now $R_r\phi$ is an $\mathscr{R}$-class of S^* and $L_s\phi$ is an $\mathscr{L}$-class of S^* for all $r,s\in N$. From (5.1b) it follows that

$$R_r\phi = R_r^*, \quad L_s\phi = L_s^*.$$

In particular, $H_{i,i}\phi = H_{i,i}^*$ $(i=0,\dots,d-1)$ and so we can define an isomorphism θ_i of G_i onto G_i^* by the rule that, for all $a_i\in G_i$,

$$(0;a_i;0)\phi = (0;a_i\theta_i;0) \qquad (i=0,\dots,d-1).$$

Also $H_{0,d}\phi = H_{0,d}^*$; hence

$$(0;f_0;1)\phi = (0;z_0^*;1)$$

for some $z_0^*\in G_0^*$. Then for $x_{d-1}\in G_{d-1}$ we have that

$$\begin{aligned}(0;x_{d-1}\gamma_{d-1};1)\phi &= [(0;x_{d-1}\gamma_{d-1};0)(0;f_0;1)]\phi\\ &= (0;x_{d-1}\gamma_{d-1}\theta_0;0)(0;z_0^*;1)\\ &= (0;(x_{d-1}\gamma_{d-1}\theta_0)z_0^*;1).\end{aligned}$$

But

$$\begin{aligned}(0;x_{d-1}\gamma_{d-1};1)\phi &= [(0;f_0;1)(0;x_{d-1};0)]\phi\\ &= (0;z_0^*;1)(0;x_{d-1}\theta_{d-1};0)\\ &= (0;z_0^*(x_{d-1}\theta_{d-1}\gamma_{d-1}^*);1).\end{aligned}$$

Hence, for all $x_{d-1}\in G_{d-1}$,

$$(x_{d-1}\gamma_{d-1}\theta_0)z_0^* = z_0^*(x_{d-1}\theta_{d-1}\gamma_{d-1}^*).$$

Let ζ^* denote the inner automorphism $x\to z_0^*xz_0^{*-1}$ of G_0^*. Then

$$\gamma_{d-1}\theta_0 = \theta_{d-1}\gamma_{d-1}^*\zeta^*. \tag{5.1c}$$

Now suppose that $d>1$. For $0\leqq i\leqq d-2$ and any $x_i\in G_i$ we have that

$$\begin{aligned}(0;x_i\gamma_i\theta_{i+1};0) &= (0;x_i\gamma_i;0)\phi\\ &= [(0;x_i;0)(0;f_{i+1};0)]\phi\\ &= (0;x_i\theta_i;0)(0;f_{i+1}^*;0)\\ &= (0;x_i\theta_i\gamma_i^*;0)\end{aligned}$$

and so

$$\gamma_i\theta_{i+1} = \theta_i\gamma_i^* \qquad (i=0,\dots,d-2). \tag{5.1d}$$

From (5.1c) and (5.1d) we see that the diagram (5.1a) is commutative.

Conversely, let $d=d^*$. Suppose also that there exist isomorphisms $\theta_i: G_i\to G_i^*$ $(i=0,\dots,d-1)$ and an inner automorphism ζ^* of G_0^* such that (5.1a) is commutative. Thus (5.1c) holds and, if $d>1$, then (5.1d) holds. We note that, for $0\leqq j<d$,

$$\theta_j\alpha_{j,d}^*\zeta^* = \alpha_{j,d}\theta_0. \tag{5.1e}$$

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Let z_0^* be an element of G_0^* such that

$$x\zeta^* = z_0^* x z_0^{*-1} \qquad (x \in G_0^*).$$

Write $h_* = (0; z_0^*; 1)$ and $k_* = h_*^{-1}$. Then $h_* \in H_{0,d}^*$ and $h_* k_*$ is the identity of S^*. Define $\phi: S \to S^*$ by

$$(m; a_i; n)\phi = k_*^m(0; a_i\theta_i; 0)h_*^n.$$

We take h_*^0 and k_*^0 to be the identity of S^*. Since $\theta_i: G_i \to G_i^*$ is a bijection it follows that

$$H_{i,\,i}^* = \{(0; a_i\theta_i; 0) \in S^*: a_i \in G_i\} \qquad (i = 0, \dots, d-1).$$

Thus, applying Lemma 4.7 to S^*, we see that ϕ is a bijection.

Let $(m; a_i; n), (p; b_j; q) \in S$. We shall show that

$$(m; a_i; n)\phi(p; b_j; q)\phi = [(m; a_i; n)(p; b_j; q)]\phi. \tag{5.1f}$$

It is convenient to consider separately the three cases

$$\text{(i) } n > p, \qquad \text{(ii) } n < p, \qquad \text{(iii) } n = p.$$

Case (i). The left-hand side of (5.1f) is

$$k_*^m(0; a_i\theta_i; 0)h_*^{n-p}(0; b_j\theta_j; 0)h_*^q.$$

Now

$$\begin{aligned} h_*(0; b_j\theta_j; 0) &= (0; z_0^*(b_j\theta_j\alpha_{j,d}^*); 1) \\ &= (0; (b_j\theta_j\alpha_{j,d}^*\zeta^*)z_0^*; 1) \\ &= (0; b_j\theta_j\alpha_{j,d}^*\zeta^*; 0)\,h_* \\ &= (0; b_j\alpha_{j,d}\theta_0;\ 0)h_*, \text{ by (5.1e)}. \end{aligned}$$

Hence

$$\begin{aligned} h_*^2(0; b_j\theta_j; 0) &= h_*(0; b_j\alpha_{j,d}\theta_0;\ 0)h_* \\ &= (0; b_j\alpha_{j,d}\theta_0\alpha_{0,d}^*\zeta^*; 0)h_*^2 \\ &= (0; b_j\alpha_{j,d}\alpha_{0,d}\theta_0; 0)h_*^2 \\ &= (0; b_j\alpha_{j,2d}\theta_0; 0)h_*^2 \end{aligned}$$

and, by induction,

$$h_*^r(0; b_j\theta_j; 0) = (0; b_j\alpha_{j,rd}\theta_0; 0)h_*^r$$

for all positive integers r. Thus

$$\begin{aligned} &(0; a_i\theta_i; 0)h_*^{n-p}(0; b_j\theta_j; 0) \\ &= (0; (a_i\theta_i)(b_j\alpha_{j,(n-p)d}\theta_0\alpha_{0,i}^*); 0)h_*^{n-p} \\ &= (0; (a_i\theta_i)(b_j\alpha_{j,(n-p)d}\alpha_{0,i}\theta_i); 0)h_*^{n-p} \\ &= (0; [a_i(b_j\alpha_{j,(n-p)d+i})]\theta_i; 0)h_*^{n-p}. \end{aligned}$$

It follows that

$$\begin{aligned} (m; a_i; n)\phi(p; b_j; q)\phi &= k_*^m(0; [a_i(b_j\alpha_{j,(n-p)d+i})]\theta_i; 0)h_*^{q-p+n} \\ &= [(m; a_i; n)(p; b_j\ ; q)]\phi. \end{aligned}$$

Case (ii). This is similar to case (i) and we omit the details.

Case (iii). $(m;a_i;n)\phi(n;b_j;q)\phi$

$$\begin{aligned} &= k_*^m(0;a_i\theta_i;0)(0;b_j\theta_j;0)h_*^q \\ &= k_*^m(0;(a_i\theta_i\alpha_{i,s}^*)(b_j\theta_j\alpha_{j,s}^*);0)h_*^q, \quad \text{where } s=\max\{i,j\}, \\ &= k_*^m(0;(a_i\alpha_{i,s}\theta_s)(b_j\alpha_{j,s}\theta_s);0)h_*^q \\ &= k_*^m(0;[(a_i\alpha_{i,s})(b_j\alpha_{j,s})]\theta_s;0)h_*^q \\ &= [(m;a_i;n)(n;b_j;q)]\phi. \end{aligned}$$

This completes the proof.

In the case $d=1$ the theorem reduces to [**8**, Theorem 4.1].

We now extend the result of Theorem 5.1 to the case of a regular ω-semigroup with a proper kernel. Such a semigroup is of the form $T(l;d;G_i;\gamma_i)$ discussed in (4.12).

Let $S=T(l;d;G_i;\gamma_i)$ and let $S^*=T(l^*;d^*;G_i^*;\gamma_i^*)$. Then it follows from (2.8) and Theorem 5.1 that $S\cong S^*$ if and only if the three conditions below are satisfied.

(i) $l=l^*$.

(ii) $d=d^*$.

(iii) There exist isomorphisms θ_i of G_i onto G_i^* $(i=0,\dots,l+d-1)$ and an inner automorphism ζ^* of G_l^* such that the following diagram is commutative:

$$\begin{array}{ccccccccccccccc} G_0 & \xrightarrow{\gamma_0} & G_1 & \xrightarrow{\gamma_1} & \cdots & \to & G_l & \xrightarrow{\gamma_l} & \cdots & \to & G_{l+d-1} & \xrightarrow{\gamma_{l+d-1}} & & & G_l \\ \downarrow{\scriptstyle\theta_0} & & \downarrow{\scriptstyle\theta_1} & & & & \downarrow{\scriptstyle\theta_l} & & & & \downarrow{\scriptstyle\theta_{l+d-1}} & & & & \downarrow{\scriptstyle\theta_l} \\ G_0^* & \xrightarrow[\gamma^*_0]{} & G_1^* & \xrightarrow[\gamma^*_1]{} & \cdots & \to & G_l^* & \xrightarrow[\gamma^*_l]{} & \cdots & \to & G_{l+d-1}^* & \xrightarrow[\gamma^*_{l+d-1}]{} & G_l^* & \xrightarrow[\zeta^*]{} & G_l^* \end{array}$$

Finally, we mention the case of a regular ω-semigroup with no kernel. By Theorem 2.6, such a semigroup is the union of an ω-chain of groups. Let S, S^* be constructed respectively from groups G_i, G_i^* and homomorphisms γ_i, γ_i^* as in (1.1). Then $S\cong S^*$ if and only if there exist isomorphisms θ_i of G_i onto G_i^* such that $\gamma_i\theta_{i+1}=\theta_i\gamma_i^*$ for all $i\in N$ [**2**, p. 1044].

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