ONE SIDED INVERTIBILITY AND LOCALISATION

by C. R. HAJARNAVIS

(Received 26 April, 1991)

1. Introduction. In general, a prime ideal P of a prime Noetherian ring need not be classically localisable. Since such a localisation, when it does exist, is a striking property; sufficiency criteria which guarantee it are worthy of careful study. One such condition which ensures localisation is when P is an invertible ideal [**5**, Theorem 1.3]. The known proofs of this result utilise both the left as well as the right invertiblity of P. Such a requirement is, in practice, somewhat restrictive. There are many occasions such as when a product of prime ideals is invertible [**6**] or when a non-idempotent maximal ideal is known to be projective only on one side [**2**], when the assumptions lead to invertibilty also on just one side. Our main purpose here is to show that in the context of Noetherian prime polynomial identity rings, this one-sided assumption is enough to ensure classical localisation [Theorem 3.5]. Consequently, if a maximal ideal in such a ring is invertible on one side then it is invertible on both sides [Proposition 4.1]. This result plays a crucial role in [**2**]. As a further application we show that for polynomial identity rings the definition of a unique factorisation ring is left-right symmetric [Theorem 4.4].

2. Notation and preliminaries. All rings are associative and have identity. Let R be a ring with a quotient ring Q. Let I be an ideal of R and M a right or left R-module. We define

$\mathscr{C}(I) = \{c \in R \mid c + I \text{ regular in } R/I\}$

$I^* = \{q \in Q \mid qI \subseteq R\}$

$I^\# = \{q \in Q \mid Iq \subseteq R\}$

$|M_R|$ = Krull dimension of M_R

$|{}_RM|$ = Krull dimension of ${}_RM$

PI ring = a ring satisfying a polynomial identity

$P^{(n)}$ = the n-th symbolic power of P defined by Goldie [**8**]

R_P = the ring of fractions formed when $\mathscr{C}(P)$ is an Ore set

$\rho_r(M_R)$ = the reduced rank of M_R

$\rho_l({}_RM)$ = the reduced rank of ${}_RM$

R is said to be as *local* ring if R/J is a simple Artinian ring where J is the Jacobson radical of R. When R is a prime right Noetherian ring and a prime ideal P satisfies the right Ore condition with respect to $\mathscr{C}(P)$, we may form the right localisation R_P which is a local ring with Jacobson radical PR_P. Further, under two sided assumptions the left localisation coincides with the right localisation. In this case we have $PR_P = R_PP$.

The ideal I is said to be *left invertible* if $I^*I = R$, *right invertible* if $II^\# = R$ and *invertible* if $I^*I = R = II^\#$. When I is invertible it is easily seen that $I^* = I^\#$.

Let R be a ring with a simple Artinian quotient ring. Let I be a non-zero ideal of R. The dual basis lemma [**4**, Proposition 3.1, p132] shows that I_R is projective if and only if $1 \in II^*$. Similarly ${}_RI$ is projective if and only if $1 \in I^{\#}I$.

The ring R is said to be a *maximal order* if there is no larger order in Q equivalent to R. A convenient characterisation is as follows: Let R be a prime Noetherian ring. Then R is a maximal order if and only if for each non-zero ideal I of R and $q \in Q$, $Iq \subseteq I \Rightarrow q \in R$ and $qI \subseteq I \Rightarrow q \in R$.

It is easily seen that the property $I^* = I^{\#}$ also holds in a maximal order.

A prime ideal P is said to have *height* 1 if P does not properly contain a chain of two distinct prime ideals. By [**11,** Proposition 13.8.2] in a Noetherian prime *PI* ring every non-zero prime ideal contains a height 1 prime.

R is said to be a *Krull symmetric* ring if for each R-R-bimodule M which is finitely generated on both sides we have $|{}_RM| = |M_R|$.

Let M be a module over a semi-prime right Noetherian ring R. We say that M is a *torsion module* if given $m \in M$ there exists c regular in R such that $mc = 0$. The term *torsion-free* is defined analogously.

Let P be a prime ideal of a Noetherian ring R. The *symbolic powers* $P^{(n)}$ of P that we require are those described by Goldie [**8**]. These have the property that

$$\mathscr{C}(P) = \mathscr{C}(P^{(n)}) \quad \text{for all} \quad n \geq 1.$$

R is said to be a right *unique factorisation ring* (*UFR*) if every height 1 prime ideal of R is principal as a right ideal.

Finally, where relevant, the absence of the adjectives right or left will imply that the given condition is meant to hold on both sides.

3. The main theorem.

LEMMA 3.1. *Let R be a Noetherian prime Krull symmetric ring. Let M be a bimodule finitely generated on both sides. Then* $\rho_r(M) = 0 \Leftrightarrow \rho_l(M) = 0$.

Proof. Assume that $\rho_l(M) = 0$. Then ${}_RM$ is a torsion module so by [**11,** Proposition 6.3.11] we have

$$|{}_RM| < |{}_RR|. \tag{i}$$

Suppose that $\rho_r(M) \neq 0$. Then M_R is not a torsion module. By factoring by the torsion submodule of M_R (and observing that this is a subbimodule) we may assume that M_R is torsion-free. Since R is prime, M_R must be faithful. Now ${}_RM$ is finitely generated. Let $M = Rm_1 + \ldots + Rm_k$ where $m_i \in M$. The map $R \to M \oplus \ldots \oplus M$ (k times) given by $r \to (m_1r, \ldots, m_kr)$ for $r \in R$ shows that R_R is isomorphic to a submodule of $(M \oplus \ldots \oplus M)_R$. It follows that

$$|R_R| \leq |M_R|. \tag{ii}$$

(i) and (ii) conflict with the assumption of Krull symmetry. This contradiction shows that $\rho_r(M) = 0$.

LEMMA 3.2. *Let I be an ideal of a Noetherian Krull symmetric ring R. Suppose that the maximal nilpotent ideal N of R is a prime ideal.*

Then $\rho_r(I) = 0 \Leftrightarrow \rho_l(I) = 0$.

Proof. Suppose that $\rho_1(I)=0$.

Consider the chain $I \supseteq I\cap N \supseteq I\cap N^2 \supseteq \dots \supseteq I\cap N^k=0$. By addivity of the reduced rank we have

$$\rho_1(I)=\sum_{i=0}^{k-1}\rho_i\left(\frac{I\cap N^i}{I\cap N^{i+1}}\right).$$

Since the reduced rank is a non-negative integer and

$$\rho_1(I)=0 \quad \text{we have} \quad \rho_1\left(\frac{I\cap N^i}{I\cap N^{i+1}}\right)=0 \quad \text{for} \quad i=0,\dots,k-1.$$

Now each $\dfrac{I\cap N^i}{I\cap N^{i+1}}$ is an R/N-module (on both sides). So by Lemma 3.1

$$\rho_r\left(\frac{I\cap N^i}{I\cap N^{i+1}}\right)=0 \quad \text{for} \quad i=0,\dots,k-1.$$

Since

$$\rho_r(I)=\sum_{i=0}^{k-1}\rho_r\left(\frac{I\cap N^i}{I\cap N^{i+1}}\right)$$

it follows that $\rho_r(I)=0$.

Proposition 3.3. *Let P be a prime ideal of a Noetherian Krull symmetric ring. Then for each $n\geq 1$ there exists $d_n\in\mathscr{C}(P)$ such that $P^{(n)}d_n\subseteq P^n$.*

Proof. By induction on n. Assume that $P^{(n-1)}d_{n-1}\subseteq P^{n-1}$ where $d_{n-1}\in\mathscr{C}(P)$. By [**8,** §3 and 4] there exist c, $d\in\mathscr{C}(P)$ such that $cP^{(n)}d\subseteq PP^{(n-1)}$. Hence

$$cP^{(n)}dd_{n-1}\subseteq P^n. \tag{*}$$

Let ρ_r denote the reduced rank of right modules over the ring R/P^n and let ρ_1 be the analogous reduced rank on the left. Consider $I=[P^{(n)}dd_{n-1}R+P^n]/P^n$ an ideal of R/P^n. By (*) we have $\rho_1(I)=0$. It follows by Lemma 3.2 that $\rho_r(I)=0$. Since I is finitely generated as a left ideal it follows that $P^{(n)}d_n\subseteq P^n$ for some $d_n\in\mathscr{C}(P)$.

We note that every non-zero ideal of a prime *PI* ring contains a non-zero central element [**11,** Theorem 13.6.4] and when such a ring is Noetherian it satisfies the symmetry condition on Krull dimension required in Proposition 3.3, [**10**] or [**11**, Corollary 13.6.6 and Corollary 6.4.13].

Lemma 3.4. *Let P be a prime ideal of a prime Noetherian PI ring. Suppose that P is right invertible. Then* (i) $\bigcap_{n=1}^{\infty}P^{(n)}=0$ *and* (ii) $\mathscr{C}(P)\subseteq\mathscr{C}(0)$.

Proof. (i) Suppose not. Then $\bigcap_{n=1}^{\infty}P^{(n)}$ contains a non-zero central element—p say. By Proposition 3.3 we have $pc_n\in P^n$ for some $c_n\in\mathscr{C}(P)$. We shall show that $p\in P^n$ for all $n\geq 1$. Assume by induction that $p\in P^{n-1}$. Then $p(P^{\#})^{n-1}\subseteq R$. Now since $pc_n\in P^n$ we have $pc_n(P^{\#})^{n-1}\subseteq P$ and so $c_np(P^{\#})^{n-1}\subseteq P$. As $c_n\in\mathscr{C}(P)$ and $p(P^{\#})^{n-1}\subseteq R$, we obtain

$p(P^{\#})^{n-1} \subseteq P$. Thus $(P^{\#})^{n-1}p \subseteq P$. Hence $Rp \subseteq P^n$ which gives $p \in P^n$. So we have $0 \neq p \in \bigcap_{n=1}^{\infty} P^n$ which is a contradiction since $\bigcap_{n=1}^{\infty} P^n = 0$ by [**6**, Lemma 3.1].

(ii) Follows from the above noting the property of symbolic powers that $\mathscr{C}(P) = \mathscr{C}(P^{(n)})$ for all n.

A special case of our next theorem was proved in [**6**] for maximal ideals under an extra hypothesis.

Recall that a *pri* (*pli*) ring R is a ring in which every right (left) ideal of R is principal.

THEOREM 3.5. *Let R be a prime Noetherian PI ring. Let P be a right invertible prime ideal of R. Then P is localisable and the localised ring R_p is a pri and pli ring. In particular, P has height* 1.

Proof. Let $a, c \in R$ with $c \in \mathscr{C}(P)$. By Lemma 3.4 we have $c \in \mathscr{C}(0)$. Hence cR is an essential right ideal and so by [**1**, Theorem 7] or [**11**, Corollary 13.2.9], cR contains a non-zero ideal. Thus cR contains a non-zero central element. Let A be a maximal left invertible ideal contained in cR. Suppose that $A \subseteq P$. Since $c \in \mathscr{C}(P)$ we have $A \subseteq cP$. So we have $AP^{\#} \subseteq cPP^{\#} = cR$ since P is right invertible. Now clearly $AP^{\#} \subseteq A$ and $PA^*AP^{\#} = PRP^{\#} = PP^{\#} = R$. So $AP^{\#}$ is left invertible. By the maximality of A we have $A = AP^{\#}$. Therefore $A^*A = A^*AP^{\#}$ and so $R = P^{\#}$. Hence $P = PR = PP^{\#} = R$. This is a contradiction and so $A \nsubseteq P$. So we may select $c_1 \in A \cap \mathscr{C}(P)$. Now $aA \subseteq A \subseteq cR$. Therefore we have $ac_1 = ca_1$ for some $a_1 \in R$. Thus the right Ore condition is satisfied with respect to $\mathscr{C}(P)$ and so R is right localisable at P. By [**3**, Theorem A] R is also left localisable at P.

Let S and J denote respectively the localised ring R_P and its Jacobson radical $PR_P = R_PP$. Since P is right invertible it is easy to see that $JJ^{\#} = S$ where $J^{\#}$ is taken with respect to the ring S. Let a be a non-zero central element of S. By [**6**, Lemma 3.1] we have $\bigcap_{n=1}^{\infty} J^n = 0$. So there exists an integer k such that $a \in J^k$ but $a \notin J^{k+1}$. Since $aS \subseteq J^k$ we have $aS(J^{\#})^k \subseteq S$. Clearly $aS(J^{\#})^k$ is an ideal of S. Suppose that $aS(J^{\#})^k \subseteq J$. Then since a is central we obtain $aS \subseteq J^{k+1}$ which is a contradiction. Thus $aS(J^{\#})^k \nsubseteq J$. Since S is a local ring we must have $aS(J^{\#})^k = S$. Hence $aS = J^k$. It is clear now that J must be an invertible ideal of S. It follows by [**9**, Proposition 1.3] that S is a *pri* and *pli* ring. It is standard to show that J has height 1 in S and thus P has height 1 in R.

REMARK. If only the conclusion that P has height 1 is required then a proof independent of localisation can be given.

4. Applications.

PROPOSITION 4.1. *Let R be a Noetherian prime PI ring and let M be a maximal ideal of R. Then*

$$M \text{ is right invertible} \Leftrightarrow M \text{ is left invertible} \tag{i}$$

$$M \text{ is a principal right ideal} \Leftrightarrow M \text{ is a principal left ideal.} \tag{ii}$$

Proof. (i) Suppose that M is right invertible. By Theorem 3.5 the ring R_M exists and is a *pri* and *pli* ring. The rest of the proof can proceed as in [**6**, Lemma 4.1].

(ii) Follows from the above noting the equality of the left and the right inverses of M (see the proof of Theorem 4.4).

REMARK. We have no information on the status of Proposition 4.1 when M is a non-maximal prime ideal.

Our next application is a part of the joint work with A. Braun [2].

PROPOSITION 4.2. *Let R be a Noetherian prime PI ring. Let M be a maximal ideal such that M_R is projective. Then M is either idempotent or invertible. In the latter case M has height* 1.

Proof. Since M is a maximal ideal and $M \subseteq M^*M \subseteq R$, we have either $M^*M = M$ or $M^*M = R$. Suppose that $M^*M = M$. Since M_R is projective we have by the dual basis lemma $1 \in MM^*$. Therefore $M = 1M \subseteq MM^*M = M^2$ and M is idempotent. Otherwise we have $M^*M = R$ and then by Proposition 4.1 M is invertible.

In the next lemma the intersection is taken in Q the quotient ring of R.

LEMMA 4.3. *Let R be a prime Noetherian PI ring. Suppose that R is a left UFR. Then $R = \cap R_p$ where P runs over the height* 1 *prime ideals of R. Moreover each R_P is a pri and pli ring. In particular, R is a maximal order.*

Proof. Let P be a height 1 prime. Since P is a principal left ideal with a regular generator, P is a right invertible ideal. So by Theorem 3.5 the localisation R_P exists and is a *pri* and *pli* ring. Let $q \in \cap R_P$. Define $X = \{r \in R \mid qr \in R\}$. Then X is a right ideal of R. Since R is a prime *PI* ring, by Posner's theorem [**11**, Theorem 13.6.5] $q = \alpha\lambda^{-1}$ where $\alpha \in R$ and λ lies in the centre of R. Thus X contains a non-zero ideal of R. Since R is a prime Noetherian ring, every non-zero ideal of R contains a product of non-zero prime ideals. Since every non-zero prime ideal of R contains a height 1 prime ideal, there exist height 1 prime ideals $P_1, \ldots, P_k$ such that $P_1 \ldots P_k \subseteq X$. Thus $qP_1 \ldots P_k \subseteq R$. Since $q \in R_{P_k}$ we have $q = c^{-1}a$ for some $a \in R$ and $c \in \mathscr{C}(P)$. Thus $aP_1 \ldots P_{k-1}P_k \subseteq cR$. Since $c \in \mathscr{C}(P_k)$ it follows that $aP_1 \ldots P_{k-1}P_k \subseteq cP_k$. Now $P_k = Rp_k$ for some $p_k \in P_k$ since R is a left *UFR*. Thus $aP_1 \ldots P_{k-1}Rp_k \subseteq cRp_k$. Since R is a prime ring p_k must be a regular element. Therefore $aP_1 \ldots P_{k-1} \subseteq cR$ and so $qP_1 \ldots P_{k-1} \subseteq R$. Proceeding in this way we obtain $q \in R$. Hence $R = \cap R_P$. Now R_P being a *pri* and *pli* ring is a maximal order by [**12**, Corollary 4.6] (or by the criterion mentioned in §2). As an arbitrary ideal of R_P is of the form IR_P it is easily seen that R is also a maximal order.

THEOREM 4.4. *Let R be a prime Noetherian PI ring. Then*

$$R \text{ is a right UFR} \Leftrightarrow R \text{ is a left UFR}.$$

Proof. Suppose that R is a left *UFR*. Let P be a height 1 prime ideal of R. By assumption $P = Rp$ for some $p \in P$. Hence $P^{\#} = p^{-1}R$. By Lemma 4.3 R is a maximal order and so we have $P^* = p^{-1}R$. Thus $p^{-1}RP \subseteq R$ and hence $P \subseteq pR$. It follows that $P = pR$. Therefore R is a right *UFR*.

In the context of Proposition 4.1 it is interesting to note that in a ring, a maximal ideal which is projective on one side need not be projective on the other, even when the ring is prime and a finitely generated module over its Noetherian centre.

EXAMPLE 4.5. Consider

$$R=\begin{bmatrix} k[x,y] & (x,y) \\ k[x,y] & k[x,y] \end{bmatrix}$$

where k is a field and (x, y) is the ideal generated by x and y. Then R is a prime ring and a finite module over its centre which is isomorphic to $k[x, y]$. The two maximal ideals

$$M=\begin{bmatrix} (x,y) & (x,y) \\ k[x,y] & k[x,y] \end{bmatrix}$$

and

$$M'=\begin{bmatrix} k[x,y] & (x,y) \\ k[x,y] & (x,y) \end{bmatrix}$$

are projective on one side but not the other. Noting that (x, y) is not an invertible ideal of the domain $k[x, y]$ we have

$$M^*=\begin{bmatrix} k[x,y] & (x,y) \\ k[x,y] & k[x,y] \end{bmatrix} \quad \text{and} \quad M^{\#}=\begin{bmatrix} k[x,y] & k[x,y] \\ k[x,y] & k[x,y] \end{bmatrix}.$$

It is easily seen that $1 \in M^{\#}M$ but $1 \notin MM^*$. Thus M is left projective but not right projective. Now R is obtained as an idealizer at a semi-maximal right ideal of the full 2×2 matrix ring. So by [**13**, Theorem 2.8] R is a ring of global dimension 2. It follows that M_R has projective dimension 1.

It is easy to see that the ring considered in the above example is not a maximal order. Indeed, in this case, we can prove the following.

THEOREM 4.6. *Let R be a Noetherian prime PI ring which is a maximal order. Let I be a ideal of R. Then I_R projective $\Leftrightarrow {}_RI$ projective. Consequently, if either condition holds then I is an invertible ideal.*

Proof. Assume that I_R is projective. Then $1 \in II^*$. Since R is a maximal order $I^* = I^{\#}$. This implies that II^* is an ideal of R and so $II^* = R$. Thus I is right invertible. Note that for each $m \geq 1$ we have $(I^*)^m I^m \subseteq R$. Moreover $[(I^*)^m I^m]^2 = [(I^*)^m I^m][(I^*)^m I^m] = (I^*)^m R I^m = (I^*)^m I^m$. Thus each $(I^*)^m I^m$ is an idempotent ideal of R. By [14, Theorem 3] R has only a finite number of idempotent ideals. Thus there exist two integers n and k with $k > 0$ such that $(I^*)^n I^n = (I^*)^{n+k} I^{n+k}$. Therefore $I^n(I^*)^n I^n (I^*)^n = I^n (I^*)^{n+k} I^{n+k} (I^*)^n$. Hence $R = R(I^*)^k I^k R$. Thus $(I^*)^k I^k = R$. It follows easily from this that I is left invertible and left projective.

REMARKS. It is possible to give a 'first principles' proof of Theorem 3.5 without reference to Goldie's symbolic powers. The key step is to observe that under the hypothesis of Lemma 3.2, R has the Ore condition with respect to $\mathscr{C}(N)$. This is proved by induction on the index of nilpotency of N. The induction hypothesis shows that $T = \{x \in R \mid xc = 0 \text{ for some } c \in \mathscr{C}(N)\}$ is an ideal of R. Now $\rho_r(T) = 0$ and so $\rho_1(T) = 0$. Thus for any $d \in \mathscr{C}(N)$ we have $\rho_1[l(d)] = 0$ giving $\rho_1(R/Rd) = 0$. The left Ore condition with respect to $\mathscr{C}(N)$ now follows.

Finally we note that the symmetry hypothesis on the Krull dimension can be replaced by a function with similar formal properties.

NOTE ADDED IN PROOF. We have been able to extend Theorem 3.5 to the case in which R is a semi-prime ring.

REFERENCES

1. S. A. Amitsur, Prime rings having polynomial identities with arbitrary coefficients, *Proc. London Math. Soc.*, (3) **17** (1967), 470–486.

2. A. Braun and C. R. Hajarnavis, Affine PI rings of global dimension 2, *J. Algebra*, to appear.

3. A. Braun and R. B. Warfield Jr., Symmetry and localization in Noetherian prime *PI* rings, *J. Algebra*, **118** (1988), 322–334.

4. H. Cartan and S. Eilenberg, *Homological Algebra*, (Princeton University Press, 1956).

5. A. W. Chatters and S. M. Ginn, Localisation in hereditary rings, *J. of Algebra*, **22** (1972), 82–88.

6. A. W. Chatters and C. R. Hajarnavis, Ideal arithmetic in Noetherian *PI* rings, *J. Algebra*, **122** (1989), 475–480.

7. E. Formanek, Central polynomials for matrix rings, *J. Algebra*, **23** (1972), 129–132.

8. A. W. Goldie, Localisation in non-commutative Noetherian rings, *J. Algebra*, **5** (1967), 89–105.

9. C. R. Hajarnavis and T. H. Lenagan, Localisation in Asano orders, *J. Algebra*, **21** (1972), 441–449.

10. A. V. Jategaonkar, Jacobson's conjecture and modules over fully bounded Noetherian rings, *J. Algebra*, **30** (1947), 103–121.

11. J. C. McConnell and J. C. Robson, Noncommutative Noetherian rings. (Wiley. 1988).

12. J. C. Robson, Artinian quotient rings, *Proc. London Math. Soc.*, (3) **17** (1967), 600–616.

13. J. C. Robson, *Idealizer rings*, *ring theory* (*Ed. R. Gordon*), *Proceedings of a conference held in Park City*, *Utah.* (Academic Press, 1972).

14. J. C. Robson and L. W. Small, Idempotent ideals in *PI* rings, *J. London Math. Soc.*, (2) **14** (1976), 120–122.

MATHEMATICS INSTITUTE
UNIVERSITY OF WARWICK
COVENTRY CV4 7AL
ENGLAND