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INTEGRATION OF NON-MEASURABLE FUNCTIONS (II)

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This is a supplement to a previous paper [4] in which integration was developed for *arbitrary* extended-real functions over *arbitrary* sets in an outer measure space $(S, \mathscr{M}^*, m^*)$ where m^*, also written m, is a regular outer measure on all subsets of an arbitrary set $S \neq \varnothing$, and $\mathscr{M}^*$ is the family of all m^*-*measurable* sets.(1) Our present objective is to strengthen some previous theorems by considering convergence *in measure*, and *nets* of not necessarily measurable functions $f: S \to E^*$ (E^* being the extended real number system).(2) The notation and definitions of [4] are presupposed and continued In particular, the operations in E^* are defined as usual, with two additional conventions: $\infty - \infty = +\infty$ and $0 \cdot \infty = 0$ (∞ stands for $\pm\infty$). As before, we write "$f \geq g$ on A" iff $f(x) \geq g(x)$ for all $x \in A$, and set $A(f > a) = \{x \in A \mid f(x) > a\}$, $A(|f| \leq g) = \{x \in A \mid |f(x)| \leq g(x)\}$, etc. Continuing the numbering of propositions of [4], we now add §6 (the preceding sections being quoted as §§1–5).

6. **Convergence in measure.** In the sequel, $\{f_i \mid i \in I\}$ denotes a net of extended-real functions on S where i ranges over a *directed* set I, i.e. a partially ordered set in which every two elements have an upper bound. Given also a function f such that $|f| < +\infty$ on a set $A \subseteq S$, we say that the net $\{f_i\}$ *converges in measure* to f on A, and write $f_i \to f$ (meas.) on A, if for every real $q > 0$, $\lim_i m^*A(|f_i - f| \geq q) = 0$; that is, given any real q, $\varepsilon > 0$, there is $i_0 \in I$ such that the outer measure of the set $\{x \in A \mid |f_i(x) - f(x)| \geq q\}$ is less than δ whenever $i \geq i_0$ in I. Similarly we write $f_i \to +\infty$ (meas.) on A if $\lim_i m^*A(f_i \leq n) = 0$, $n = 1, 2, \ldots$; and $f_i \to -\infty$ (meas.) if $\lim_i m^*A(f_i \geq -n) = 0$, $n = 1, 2, \ldots$. More generally, we write $f_i \to f$ (meas.), or $\lim_i f_i \approx f$ (meas.), on A if $f_i \to \pm\infty$ (meas.) on $A(f = \pm\infty)$, and $f_i \to f$ (meas.) on $A(|f| < +\infty)$, as defined above. Analogous definitions apply to *uniform* (*unif.*) and *almost uniform* (*a.unif.*) convergence (cf. §5). If, in addition, $\{f_i\}\uparrow$ on A [i.e. if $i \leq j$ in I implies $f_i \leq f_j$ a.e. on A] we write $f_i \nearrow f$ instead of $f_i \to f$. Similarly for "$f_i \searrow f$". As usual, "a.e. on A" means "on $A - Q$, for some $Q \subseteq A$ with $m^*Q = 0$". We denote by $\mathscr{A}$ the family of all sets of the form $A \cap X$, with $X \in \mathscr{M}^*$. As was noted in §3, the restriction of m^* to $\mathscr{A}$ is a countably additive *measure*,

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(1) We are using the terminology of Munroe [3], pp. 50–57.

(2) For "nets", see Kelley [2], p. 65 ff. Sequences are a special case.

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called m_A. We start with some simple propositions, omitting proofs when trivial. The space $(S, \mathscr{M}^*, m^*)$ is fixed henceforth.

(6.1) *If $f_i \to f$ (meas.) on A and on B, then $f_i \to f$ (meas.) on $A \cup B$.*

(6.2) *If $f_i \to f$ (meas.) on A, then:*

(i) $|f_i| \to |f|$ *(meas.) on A;*
(ii) $f_i^+ \to f^+$ *and* $f_i^- \to f^-$ *(meas.) on A* $(f^+ = \sup\{f, 0\}, f^- = \sup\{-f, 0\}$;
(iii) $af_i \to af$ *(meas.) on A, for every real (finite) constant a.*

(6.3)(Riesz) *If $f_i \to f$ (meas.) on A, there is a sequence $i_1 \le i_2 \le \cdots i_n \le i_{n+1} \cdots$ in I such that $\lim_{n\to\infty} f_{i_n} \approx f$ (a.unif.) [hence also (a.e.)] on A.*

Proof. Let $A_0 = A(|f| < +\infty)$, $A_1 = A(f = +\infty)$ and $A_2 = A(f = -\infty)$. Our assumption implies that for each integer $n > 0$ there is $i_n \in I$ such that

6.3.1. $$m^*A_0(|f_i - f| \ge 1/n) + m^*A_1(f_1 \le n) + m^*A_2(f_i \ge -n) < 2^{-n}, \quad \text{for} \quad i \ge i_n$$

As I is *directed*, we can choose the i_n inductively in such a manner that $i_n \le i_{n+1}$, $n = 1, 2, \ldots$, with 6.3.1 preserved for all n. Now, given $\varepsilon > 0$, fix k such that $\sum_{n=k}^{\infty} 2^{-n} < \varepsilon$, and set $D = \bigcup_{n=k}^{\infty} [A_0(|f_{j_n} - f| \ge 1/n) \cup A_1(f_{i_n} \le n) \cup A_2(f_{i_n} \ge -n)]$, so that $m^*D < \sum_{n=k}^{\infty} 2^{-n} < \varepsilon$, by 6.3.1. Also, let $E = A \cap \bar{D}$ where $\bar{D} \in \mathscr{M}^*$, $\bar{D} \supseteq D$ and $m^*D = m\bar{D}$ (i.e. $\bar{D}$ is a "*measurable cover*" of D; it exists by the assumed regularity of m^*; cf. [3], pp. 50–57). Then $E \in \mathscr{A}$ and $mE \le m^*D < \varepsilon$. Moreover, $A_0 - E \subseteq A_0 - D = \bigcap_{n=k}^{\infty} A_0(|f_{i_n} - f| < 1/n)$, $A_1 - E \subseteq \bigcap_{n=k}^{\infty} A_1(f_{i_n} > n)$ and $A_2 - E \subseteq \bigcap_{n=k}^{\infty} A_2(f_{i_n} < -n)$.[3] Thus, for all $n \ge k$, we have $|f_{i_n} - f| < 1/n$ on $A_0 - E$, $f_{i_n} < n$ on $A_1 - E$, and $f_{i_n} < -n$ on $A_2 - E$. It easily follows that $f_{i_n} \to f$ (unif.) on $A_0 - E$, $f_{i_n} \to +\infty$ (unif.) on $A_1 - E$ and $f_{i_n} \to -\infty$ (unif.) on $A_2 - E$. As such an E exists for *any* $\varepsilon > 0$, we have, by definition, $f_{i_n} \to f$ (a.unif.) on $A_0 \cup A_1 \cup A_2$, hence on A. This, in turn, implies $f_{i_n} \to f$ (a.e.) on A, as is well known. QED.

This extension of the classical Riesz theorem enables us to extend also other classical results to nets of *arbitrary* functions $f_i: S \to E^*$. In particular, we immediately obtain (by choosing a *sequence* $f_{i_n} \to f$ (a.e.)):

(6.4) *If $f_i \to f$ (meas.) on A and if there is a function g such that $f_i \le g$ a.e. on A, for all $i \in I$, then $f \le g$ a.e. on A. Similarly in case $f_i \ge g$ a.e. on A, for all i.*

(6.5) *Let $f_i \to f$ (meas.) on A. Then $f = g$ a.e. on A iff $f_i \to g$ (meas.) on A.*

Proof. The "only if" is obvious. The "if" follows by choosing the sequence $\{f_{i_n}\}$ of 6.2, in such a manner that *both* $f_{i_n} \to f$ and $f_{i_n} \to g$ a.e. on A. This is possible because I is *directed*.

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[3] Because the sets A_0, A_1 and A_2 are *disjoint*.

Henceforth we write "$f \approx g$" ("f is *equivalent* to g") for "$f=g$ a.e.", and "$f \preccurlyeq g$" for "$f \leq g$ a.e." (all on A), omitting "A" if $A=S$. This partially orders the set E^{*S} of all functions $f: S \to E^*$. A function g is an *essential upper* (*lower*) *bound* of $\{f_i\}$ iff $f_i \preccurlyeq g$ ($f_i \succcurlyeq g$), for all i. We write $f \approx \text{essup} f_i$ ($f \approx \text{essinf} f_i$) on A if $f_i \preccurlyeq f \preccurlyeq g$ ($f_i \succcurlyeq f \succcurlyeq g$) on A, for all such g and all i. Clearly, $\text{essup} f_i$ and $\text{essinf} f_i$ are unique to within equivalence.[4]

(6.6) (i) *If* $f_i \nearrow f$ *(meas.) on* A, *then* $f \approx \text{essup} f_i$ *on* A.
(ii) *If* $f_i \searrow f$ *(meas.) on* A, *then* $f \approx \text{essinf} f_i$ *on* A.

Proof of (i). Fix any $i_0 \in I$. As I is directed, the sequence $\{f_{i_n}\}$ of 6.2 can be so chosen that $i_0 \leq i_1$; so we *include* f_{i_0} in $f_i \nearrow f$ a.e. on A. It easily follows that; $f_{i_0} \preccurlyeq f$ on A. As i_0 is arbitrary, f is an ess. upper bound of $\{f_i\}$, on A. Now, if g is another such bound, 6.4 yields $f \preccurlyeq g$ on A. Thus $f \approx \text{essup} f_i$, proving (i). Similarly for (ii).

As in [4], we identify $\int f$ with the *upper* integral, $\overline{\int} f$, of f (with respect to m). We consider lower integrals, $\underline{\int} f$, also. Both were defined for *any* function $f: S \to E^*$, over *any* set $A \subseteq S$ (cf. §2). We now obtain:

6.7. Lemma. *For any functions* $f, g: S \to E^*$ *such that* $\int_A |f| < +\infty$ *and* $\int_A |g| < +\infty$,[5] *we have* $|\int_A f - \int_A g| \leq \int_A |f-g|$ *and* $|\underline{\int}_A f - \underline{\int}_A g| \leq \overline{\int}_A |f-g|$.

Proof. By 2.9(b) and (2.3(f) in §2,

$$\int_A (f-g) = \int_A [f+(-g)] \geq \int_A f + \int_A (-g) = \int_A f - \int_A g.$$

Thus, if $\int_A f \geq \int_A g$, we obtain $|\int_A f - \int_A g| \leq \int_A |f-g|$. Similarly in case $\int_A g \geq \int_A f$. Thus all is proved for *upper* integrals. Now replacing f and g by $-f$ and $-g$, respectively, we obtain

$$\int_A |f-g| \geq \left| \int_A (-f) - \int_A (-g) \right| = \left| -\underline{\int}_A f + \underline{\int}_A g \right|,$$

proving the result for lower integrals, also.

6.8. *Let* $f_i \to f$ *(meas.) on* A. *Suppose there is* $g: S \to E^*$, *with* $\int_A g < +\infty$ *and* $|f_i| \preccurlyeq g$ *on* A, *for all* i. *Then* $\lim_i \int_A |f_i - f| = 0$ *and* $\lim_i \int_A f_i = \int_A f$. *We also have:* $\lim_i \underline{\int}_A |f_i - f| = 0$ *and* $\lim_i \underline{\int}_A f_i = \underline{\int}_A f$.[6]

Proof. By 3.4, there is an *A-measurable* (i.e. measurable on A, with respect to the measure m_A mentioned above) function $h \geq g \succcurlyeq |f_i|$, such that $\int_A h = \int_A g < +\infty$. Hence, by 5.1, h vanishes on $A - C$ where $C = \bigcup_{n=1}^{\infty} C_n$ for some disjoint sets $C_n \in \mathscr{A}$, with $m_A C_n < +\infty$, $n = 1, 2, \ldots$. By the countable additivity of the

(4) Provided *they exist* (we leave this question open at this time).

(5) Note that these assumptions ensure that $\int_A f$ and $\int_A g$ are "orthodox" (§2).

(6) As before (and thereafter), f and f_i need not be measurable.

integral, $\int_A h = \int_{A-C} h + \int_C h = 0 + \sum_{n=1}^{\infty} \int_{C_n} h$. As $\int_A h < +\infty$, the series $\sum \int_{C_n} h$ converges. Thus, given $\varepsilon > 0$, there is an integer p such that $\int_A h - \varepsilon < \sum_{n=1}^{p} \int_{C_n} h = \int_{H_p} h$ where $H_p = \bigcup_{n=1}^{p} C_n \in \mathscr{A}$, and $m^* H_p = m_A H_p < +\infty$.[7] Hence $\int_{A-H_p} h = \int_A h - \int_{H_p} h < \varepsilon$.

As $|f_i| \leq h$ on A, 6.4 and 6.2(i) yield $|f| \leqslant h$. Thus $|\int_A f_i| \leq \int_A h < +\infty$ and $|\int_A f| \leq \int_A h < +\infty$. Also, $|f_i - f| \leqslant 2h$ on A. Hence by 6.7, we get, for all i,

$$(6.8.1.)\quad \left| \int_A f_i - \int_A f \right| \leq \int_A |f_i - f| \leq \int_{H_p} |f_i - f| + \int_{A-H_p} 2h \leq \int_{H_p} |f_i - f| + 2\varepsilon.$$

Next, recalling that $m^* H_p < +\infty$, with ε fixed, we choose a real $q > 0$ such that $q \cdot m^* H_p < \varepsilon$. We also set, for all i, $A_i = H_p(|f_i - f| \geq q)$ and $B_i = H_p - A_i = H_p(|f_i - f| < q)$. As $H_p = A_i \cup B_i$, we have, by 2.11 (noting that $|f_i - f| < q$ on B_i):

$$(6.8.2)\quad \int_{H_p} |f_i - f| \leq \int_{A_i} |f_i - f| + \int_{B_i} |f_i - f| \leq \int_{A_i} |f_i - f| + q \cdot m * B_i.$$

Here $q \cdot m^* B_i \leq q \cdot m^* E \leq \varepsilon$. Thus, combining 6.8.2 and 6.8.1, we obtain

$$(6.8.3)\quad \left| \int_A f_i - \int_A f \right| \leq \int_A |f_i - f| \leq \int_{H_p} |f_i - f| + 2\varepsilon \leq \int_{A_i} |f_i - f| + 3\varepsilon,$$

for all i.

Now, as h is A-measurable, with $\int_A h < +\infty$, it is m_A-integrable in the sense of the traditional theory of integration. Thus, as is well known, $\int h\, dm_A$ is absolutely continuous (cf. [1], 12.34); so, with ε as above, there is $\delta > 0$ such that $2\int_X h < \varepsilon$, whenever $X \in \mathscr{A}$ and $m^* X = m_A X < \delta$. Fixing this δ, and recalling that $f_i \to f$ (meas.) on A, we find an $i_0 \in I$ such that $m^* A_i = m^* H_p(|f_i - f| \geq q) < \delta$, for all $i \geq i_0$.[8] Let $\bar{A}_i \in \mathscr{M}^*$ be a measurable cover of A_i and set $X_i = A \cap \bar{A}_i$, $i \in I$. Then $X_i \in \mathscr{A}$, $X_i \supseteq A_i$ and $m^* X_i = m^* A_i < \delta$ for $i \geq i_0$, implying: $\int_{A_i} |f_i - f| \leq 2\int_{A_i} h \leq 2\int_{X_i} h < \varepsilon$, for such i. Hence, by 6.8.3,

$$\left| \int_A f_i - \int_A f \right| \leq \int_A |f_i - f| < 4\varepsilon, \quad \text{for all } i > i_0.$$

As ε is arbitrary, we have $\int_A |f_i - f| \to 0$ and $\int_A f_i \to \int_A f$, as claimed. Since $\underline{\int}_A |f_i - f| \leq \int_A |f_i - f|$ (cf. §2) and $|\underline{\int}_A f_i - \underline{\int}_A f| \leq \underline{\int}_A |f_i - f|$ (by 6.7), the theorem also holds for lower integrals. Thus all is proved.

Simultaneously, we have proved the absolute continuity of the integral, even for *non-measurable* functions $f: S \to E^*$; that is, we have:

6.9. *If* $\int_A |f| < +\infty$, *then for every* $\varepsilon > 0$ *there is* $\delta > 0$ (*dependent on* f *and* ε *only*) *such that* $\int_X |f| < \varepsilon$ *whenever* $X \subseteq A$ *and* $m^* X < \delta$.

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(7) Recall that m_A is a restriction of m^*.

(8) This applies since $\int_A |f| < +\infty$ (implying that $|f| < +\infty$ a.e. on A).

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Indeed we only have to choose an *A-measurable* function $h \geq |f|$, with $\int_A h = \int_A |f|$, and use the absolute continuity of $\int h$, as in the proof of 6.8.

Theorem 6.8 *strengthens* an analogous proposition (5.4) previously proved for uniform and almost uniform convergence only. Indeed, convergence in measure is *more general* since $f_i \to f$ (a.unif) *implies* $f_i \to f$ (meas.), as easily follows from our definitions; moreover, 5.4 was proved for *sequences* $\{f_n\}$ only, not *nets*. It is now obvious that 6.8 holds also with "$f_i \to f$ (meas.)" replaced by the stronger assumption "$f_i \to f$ (a.unif.)" or "(unif.)." A similar generalization for *monotone* convergence (5.3) is as follows.

6.10. (a) *If $f_i \nearrow f$ (meas.) on A and if $\int_A f_i > -\infty$ for some i, then $\int_A f_i \nearrow \int_A f$;* (b) *If $f_i \searrow f$ (meas.) on A and if $\int_A f_i < +\infty$ for some i then $\int_A f_i \searrow \int_A f$. Similarly for lower integrals, provided however that, in case* (a) *at least, $\underline{\int}_A f$ is orthodox (i.e. $\underline{\int}_A f^+ < +\infty$ or $\overline{\int}_A f^- < +\infty$; cf.* §2).

Here, instead of giving an independent proof, it is simpler to use 5.3 (an analogous theorem, for *almost uniform* convergence), combined with 6.3. We prove case (b). If $f_i \searrow f$ (meas.) on A, Theorem 6.3 yields a sequence $i_1 \leq i_2 \cdots \leq i_n \leq i_{n+1} \leq \cdots$ in I such that $f_{i_n} \searrow f$ (*a.unif.*) on A. By assumption, $\int_A f_i < +\infty$ for some $i = i_0$, and we may safely assume $i_0 \leq i_1$. Then, since $\{f_i\}\downarrow$, we also have $\int_A f_{i_n} \leq \int_A f_{i_0} < +\infty$, $n = 1, 2, \ldots$. Thus, by 5.3,

$$\int_A f = \lim_{n\to\infty} \int_A f_{i_n} = \inf_n \int_A f_{i_n} \geq \inf_i \int_A f_i = \lim_i \int_A f_i.$$

But, by 6.6, we also have $f_i \geqslant f$ on A, hence $\int_A f_i \geq \int_A f$, for all i. It follows that

$$\int_A f \leq \inf_i \int_A f_i = \lim_i \int_A f_i \leq \lim_{n\to\infty} \int_A f_{i_n} = \int_A f;$$

so $\int_A f = \lim_i \int_A f_i$, proving assertion (b). Similarly for (a) and for lower integrals [but, for the latter, case (a) requires "*orthodoxy*", by 5.3.]

As an application, consider families $\{f_t\}$ of functions depending on a real parameter t [we may write $f(t, x)$ for $f_t(x)$, where t is real, and x varies over an arbitrary set A in the outer measure space $(S, \mathcal{M}^*, m^*)$]. Since the reals in any finite or infinite line interval form a directed (even totally ordered) set, the family $\{f_t\}$ is a *net*. Using the classical notation $\int_A f(t, x)\, dm(x)$ for $\int_A f_t$, we easily obtain:

6.11. *Let $f(t, x)$ be a function of two variables, where t varies over a line interval $(a, b) \subset E^*$, and x varies over a set A in a regular outer measure space $(S, \mathcal{M}^*, m^*)$. Suppose $|f(t, x)| \leq g(x)$, $[a < t < b, x \in A]$, for some function $g(x)$, with*

$$\int_A g(x)\, dm(x) < +\infty.$$

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Under these assumptions we have:

(i) *If* $\lim_{t\to a} f(t, x)=f(a, x)$, *in measure, on* A *then* $\lim_{t\to a} \int_A f(t, x)\, dm(x)= \int_A f(a, x)\, dm(x)$, *and* $\lim_{t\to a} \underline{\int}_A f(t, x)\, dm(x)=\underline{\int}_A f(a, x)\, dm(x)$.

(ii) *Analogous statements hold in case* $t\to b$ *or* $t\to t_0$ $(a<t_0<b)$.

Proof. In case $t\to b$, we consider the interval (a, b) in its usual order. Then, as noted above, it is a directed set. Thus, taking $\{f_t\}$ for $\{f_i\}$ in 6.8, we obtain the result. In case $t\to a$, we only have to *reverse* the order in (a, b). Finally, in case $t\to t_0$ $(a<t_0<b)$, we let the set I consist of *pairs* of reals (r, s) such that $a<r<t_0<s<b$, and order I as follows: $(r, s)\leq(r', s')$ iff $r\leq r'\leq t_0\leq s'\leq s$. This again makes I a directed set, and yields the result for the *bilateral* limit as $t\to t_0$ (instead of the one-sided limit at a or b).

Note 1. The endpoints a and b in 6.11 may be infinite. This takes care of the case where $t\to+\infty$ or $t\to-\infty$, over *real* values.

Note 2. The restriction $|f(t, x)|\leq g(x)$ in 6.11 may be relaxed as in Theorem 6.10, if f *is monotone in* t, i.e. $t<t'$ implies $f(t, x)\leq f(t', x)$, [or $t<t'$ implies $f(t, x)\geq f(t', x)$] for almost all $x\in A$.

In conclusion, we would like to repeat a remark made at the end of [4]: It seems to us that the above exposition is so simple that it can be adapted to any course in measure theory and that there is no necessity to limit the theory of integration to measurable functions. Though this paper (along with [4]) is largely expository, Theorems 6.8, 6.10, 6.11 seem to be new in the proposed generality.

References

1. E. Hewitt and K. Stromberg, *Real and Abstract Analysis*, Springer Verlag, New York, Inc. 1965.

2 J. L. Kelley, *General Topology*, D. Van Nostrand Co. Inc., 1960.

3. M. E. Munroe, *Measure and Integration*, 2nd Edition, Addison-Wesley Publishing Co. Reading Mass., 1971.

4. E. Zakon, *Integration of non-measurable functions*, Canad. Math. Bulletin, vol. 9, no. 3, 1966, pp. 307–330.

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