Glasgow Math. J. **50** (2008) 75–81. © 2008 Glasgow Mathematical Journal Trust.
doi:10.1017/S0017089507004053. Printed in the United Kingdom

COEFFICIENTS OF THE PROBABILISTIC FUNCTION OF A MONOLITHIC GROUP*

PAZ JIMÉNEZ–SERAL
Departamento de Matemáticas, Universidad de Zaragoza, 50009 Zaragoza, Spain
e-mail: paz@unizar.es

(Received 13 November, 2006; revised 9 October, 2007; accepted 14 October, 2007)

Abstract. We relate the coefficients of the probabilistic zeta function of a finite monolithic group to those of an almost simple group.

2000 *Mathematics Subject Classification.* 20P05, 20D60, 20D30.

Introduction. For a finite group G and a non-negative integer t, let $P_G(t)$ be the probability that a randomly chosen ordered t-tuple from G generates G. In **[7]** P. Hall gave an explicit formula for $P_G(t)$, exhibiting the latter as a finite Dirichlet series $\sum_n a_n n^{-t}$, with $a_n \in \mathbb{Z}$ and $a_n = 0$ unless n divides $|G|$; that is,

$$P_G(t) = \sum_{H \leq G} \frac{\mu(G, H)}{|G : H|^t},$$

where μ is the Möbius function of the subgroup lattice of G.

In view of Hall's formula, we can speak of $P_G(s)$ for any arbitrary complex number s. The function $P_G(s)$ is the multiplicative inverse of a zeta function for G, as described by Mann **[9]** and Boston **[2]**. If N is a normal subgroup of G and t is an integer with $P_{G/N}(t) \neq 0$, we define $P_{G,N}(t) = P_G(t)/P_{G/N}(t)$; this is the probability that a t-tuple generates G, given that it generates the G module N. W. Gaschütz (see **[5]**) gave a formula for $P_{G,N}(t)$, generalizing Hall's formula. As noted in **[3]**, although Gaschütz avoided explicit mention of the Möbius function, his formula can be written as

$$P_{G,N}(t) = \sum_{HN=G} \frac{\mu(G, H)}{|G : H|^t}.$$

The identity $P_G(s) = P_{G,N}(s)P_{G/N}(s)$ holds initially for sufficiently large positive integers t, but it remains valid as an identity in the ring of Dirichlet series. In **[4]**, E. Detomi and A. Lucchini gave the following factorization, which is independent of the choice of the chief series,

$$P_G(t) = \prod_{A} \prod_{1 \leq i \leq \delta_G(A)} \overline{P}_{L_A,i}(t),$$

where A runs over the set of irreducible G-groups G-equivalent to a non-Frattini chief factor of G and $\delta_G(A)$ is the number of non-Frattini chief factors G-equivalent to A,

*Partially supported by DGICYT, MTM2004-08219-C02-01.

https://doi.org/10.1017/S0017089507004053 Published online by Cambridge University Press

given a chief series of G. The monolithic primitive group associated with A is defined as $L_A = (G/C_G(A))[A]$ if A is abelian and $L_A = G/C_G(A)$ otherwise. The factors $\overline{P}_{L_A,i}(t)$ are defined in [**4**] as follows. Let L be a monolithic group, and let N be its socle. Then

$$\overline{P}_{L,1}(t) = P_{L,N}(t)$$

and for $i > 1$ it is

$$\overline{P}_{L,i}(t) = P_{L,N}(t) - \frac{(1 + q_N + \cdots + q_N^{i-2})\gamma_N}{|N|^t},$$

where $\gamma_N = |C_{\mathrm{Aut}(N)}(L/N)|$ and $q_N = |\mathrm{End}_L(N)|$, if N is abelian and $q_N = 1$ otherwise.

Using this factorization the coefficients of $P_G(t)$ can be given in terms of those of $P_{L,\mathrm{soc}(L)}(t)$ for a monolithic group L. The present paper arose from an attempt to describe the coefficients of $P_{L,\mathrm{soc}(L)}(t)$ by means of $P_{X,\mathrm{soc}(X)}(t)$ for certain almost simple groups X. More precisely, let G be a monolithic primitive group whose socle is nonabelian, S a simple component of $\mathrm{soc}(G)$ and $N = N_G(S)$. Consider the almost simple group $X = N/C_G(S)$. In Theorem 5 for those integers m with $|S| \nmid m$ we describe the coefficients a_m of $P_{G,\mathrm{soc}(G)}(t)$ in terms of those of $P_{X,\mathrm{soc}(X)}(t)$.

It was proved in [**3**] that $P_G(-1)$ is precisely the Euler characteristic of the coset poset of G. (This was first noticed by Bouc; see [**3**].) In the same paper, certain divisibility properties of $P_G(-1)$ were studied. Our Theorem 5 provides a relationship between divisibility properties of $P_G(-1)$ and of $P_X(-1)$.

The following theorem, proved by Gross–Kovács [**6**], will be essential to the proof of Proposition 3.

THEOREM 1. (See [**6**] and 1.1.35 of [**1**].) *Let G be a group in which there exists a normal subgroup M of G such that $M = S_1 \times \cdots \times S_n$, where $\{S_1, \ldots, S_n\}$ is the set of all conjugate subgroups of a normal subgroup S_1 of M. Write $N = N_G(S_1)$ and $K = S_2 \times \cdots \times S_n$.*

(1) *Let L/K be a supplement of M/K in N/K. Then, there exists a supplement H of M in G satisfying the following properties.*

(a) *$L = (H \cap N)K$ and $H \cap M = (H \cap S_1) \times \cdots \times (H \cap S_n)$. Further, $H \cap S_1 = L \cap S_1$.*

(b) *Suppose that H_0 is a supplement of M in G such that*

$$(H_0 \cap N)K/K \leq L/K.$$

Then there exists $k \in K$ such that $H_0^k \leq H$.

(c) *H is unique up to conjugacy under K.*

(2) *There is a bijection between, on the one hand, the conjugacy classes in G of supplements H of M in G such that $H \cap M = (H \cap S_1) \times \cdots \times (H \cap S_n)$ and, on the other hand, the conjugacy classes in N/K of supplements L/K of M/K in N/K.*

(3) *Under the bijection above the maximal subgroups of G are in correspondence with maximal subgroups of N/M.*

Let G be a monolithic group with $\mathrm{soc}(G)$ nonabelian and S a simple component of $M = \mathrm{soc}(G)$. Consider the projection $p : M \to S$. The maximal subgroups not containing $\mathrm{soc}(G)$ can be classified in terms of their intersection with $\mathrm{soc}(G)$ as follows.

https://doi.org/10.1017/S0017089507004053 Published online by Cambridge University Press

Type **a**: maximal subgroups H with $p(H \cap M) = S$.
Type **b**: maximal subgroups H with $1 < p(H \cap M) < S$.
Type **c**: maximal subgroups H with $H \cap M = 1$.

The next result can be found in some of the proofs of the O'Nan Scott Theorem.

PROPOSITION 2. (See 1.1.52 and 1.1.53 of [**1**]). *Let G be a monolithic group with $M = \mathrm{soc}(G) = S_1 \times \cdots \times S_n$ nonabelian.*

(a) *If H is a maximal subgroup of type b, then*

$$H \cap M = (H \cap S_1) \times \cdots \times (H \cap S_n) \neq 1.$$

(b) *There exists a bijection between the set of all conjugacy classes of maximal subgroups of $N/(S_2 \times \cdots \times S_n)$ that supplement but do not complement $M/(S_2 \times \cdots \times S_n)$ and the set of all conjugacy classes of core–free maximal subgroups of $N/C_G(S_1)$.*

(c) *Let $L/(S_2 \times \cdots \times S_n)$ be a maximal subgroup of $N/(S_2 \times \cdots \times S_n)$ supplementing $M/(S_2 \times \cdots \times S_n)$. Then $S_2 \times \cdots \times S_n < L \cap M$ if and only if $C_G(S_1) \leq L$.* (See the proof of 1.1.53 of [**1**].)

In 4.3 of [**8**], Kovács determines the number of conjugacy classes of maximal subgroups with a trivial core. Any monolithic group with nonabelian $\mathrm{soc}(G)$ has maximal subgroups of type **b** and it may happen that these are the only maximal subgroups not containing $\mathrm{soc}(G)$. For intersections of such maximal subgroups we have the following result.

PROPOSITION 3. *Let G be a monolithic primitive group with* $\mathrm{soc}(G)$ *nonabelian. Let S be a simple component of* $\mathrm{soc}(G)$*, $N = N_G(S)$, $X = N/C_G(S)$ and $n = |G : N|$. Suppose that H is a supplement of* $\mathrm{soc}(G)$ *in G such that all maximal subgroups containing H are of type* **b**. *Finally denote by $\varphi(H) = (H \cap N)C_G(S)/C_G(S) \leq X$. If $\mu(G, H) \neq 0$, then we have*

$$\mu(G, H) = \mu(X, \varphi(H)).$$

Moreover $|G : H| = |X : \varphi(H)|^n$ and $|H \cap S| = |\varphi(H) \cap \mathrm{Inn}(S)|$.

Proof. Write $M = \mathrm{soc}(G) = S_1 \times \cdots \times S_n$, $S = S_1$ and $K = S_2 \times \cdots \times S_n$. Set

$$\overline{\Omega} = \{H < G \mid HM = G,\ H \cap M = (H \cap S_1) \times \cdots \times (H \cap S_n)\}$$

and

$$\overline{\Sigma} = \{K \leq Y < N \mid (Y/K)(M/K) = N/K\}.$$

We can define a map $\overline{\varphi} : \overline{\Omega} \to \overline{\Sigma}$ by $\overline{\varphi}(H) = (H \cap N)K$. Let $H \in \overline{\Omega}$. As $HM = G$, H acts transitively on the components of M and thus the subgroups $H \cap S_i$ are conjugated. Hence $|H \cap M| = |H \cap S|^n$. Let $R = H \cap S$. We have

$$\overline{\varphi}(H) \cap M = (H \cap N)K \cap M = (H \cap N \cap M)K = (H \cap M)K = (H \cap S) \times K = R \times K.$$

It follows that $\overline{\varphi}(H) \cap S = H \cap S$ and $|R| = |(\overline{\varphi}(H) \cap M)/K|$. Hence

$$|G : H| = |M : M \cap H| = (|S| : |R|)^n = (|M/K| : |(\overline{\varphi}(H)/K \cap (M/K)|)^n = |N : \overline{\varphi}(H)|^n.$$

https://doi.org/10.1017/S0017089507004053 Published online by Cambridge University Press

Now we prove that if $H_i \in \overline{\Omega}$ and $H = \cap H_i \in \overline{\Omega}$, then $\overline{\varphi}(H) = \cap\overline{\varphi}(H_i)$. By definition, it is clear that if $H_1 \leq H_2$, then

$$\overline{\varphi}(H_1) \leq \overline{\varphi}(H_2).$$

Therefore $\overline{\varphi}(H) \leq \cap\overline{\varphi}(H_i)$.

Let $R_i = H_i \cap S$. Note that $\cap R_i = H \cap S$ and hence $\overline{\varphi}(H) \cap M = (\cap R_i) \times K$. We also have $\overline{\varphi}(H_i) \cap M = R_i \times K$ and

$$(\cap\overline{\varphi}(H_i)) \cap M = (\cap R_i)K = \overline{\varphi}(H) \cap M.$$

Since $\overline{\varphi}(H) \leq \cap\overline{\varphi}(H_i)$ and both groups supplement M/K with the same intersection, counting orders, we get that $\overline{\varphi}(H) = \cap\overline{\varphi}(H_i)$. Note that $\overline{\Omega}$ contains all the maximal subgroups of types **b** and **c**. Consider now the sets

$$\Omega = \{H \in \overline{\Omega} \mid H \text{ is an intersection of maximals subgroups of type } \mathbf{b}\}$$

and

$$\Sigma = \{Y \leq X \mid YS = X \text{ with } Y \text{ an intersection of maximal subgroups}\}.$$

Let $H \in \Omega$, $H = \cap H_i$ and H_i maximal of type **b**. We have $\overline{\varphi}(H) = \cap\overline{\varphi}(H_i)$ and, by Proposition 2 (c), $C_G(S) \leq \overline{\varphi}(H_i)$ and so $C_G(S) \leq \overline{\varphi}(H)$. Therefore, we can define a map $\varphi : \Omega \to \Sigma$ with

$$\varphi(H) = (H \cap N)C_G(S)/C_G(S) = \overline{\varphi}(H)/C_G(S).$$

As $\overline{\varphi}(H) \cap S = H \cap S$, we have $|H \cap S| = |\varphi(H) \cap \mathrm{Inn}(S)|$.

Let $Y/C_G(S) \in \Sigma$. By Theorem 1, there exists $H \in \overline{\Omega}$ such that $\overline{\varphi}(H) = Y$. Assume that $Y = \cap Y_i$, where the subgroups Y_i are maximal in N. Then for each index i there exists $U_i \in \overline{\Omega}$ such that $\overline{\varphi}(U_i) = Y_i$. As $\overline{\varphi}(H) = Y \leq Y_i$ by Theorem 1, we can find some $k \in K$ with $H^k \leq U_i$. Hence we get $H \leq U_i^{k^{-1}}$ and $\overline{\varphi}(U_i^{k^{-1}}) = Y_i$.

This means that, changing notation, we can choose for each i a subgroup U_i maximal in G such that $H \leq U_i$ and $\overline{\varphi}(U_i) = Y_i$. By Proposition 2, U_i is of type **b** and so it belongs to Ω. We have

$$\overline{\varphi}(H) = Y = \cap Y_i = \cap\overline{\varphi}(U_i) = \overline{\varphi}(\cap U_i).$$

Using Theorem 1 we deduce that H and $\cap U_i$ are conjugated and since $H \leq \cap U_i \in \Omega$, we have $H = \cap U_i$ and $H \in \Omega$. Therefore $\varphi(H) = \overline{\varphi}(H)/C_G(S) = Y/C_G(S)$.

This means that φ is surjective. Clearly if $H_1, H_2 \in \Omega$ and $H_1 < H_2$, then $\varphi(H_1) < \varphi(H_2)$.

Let H be a supplement of $\mathrm{soc}(G)$ in G such that all maximal subgroups containing H are of type **b**. Suppose $\mu(G, H) \neq 0$. Then $H \in \Omega$. Recall that if $\mu(G, U) \neq 0$, then U is the intersection of maximal subgroups of G. It follows that if $H \leq U$ and $\mu(G, U) \neq 0$, then $U \in \Omega$.

Take $H_1, H_2 \in \Omega$ such that $H < H_1$ and $H < H_2$. Assume that $\varphi(H_1) = \varphi(H_2)$. Then $H \leq U = H_1 \cap H_2$ and so $U \in \Omega$. Applying $\overline{\varphi}$ we get $\overline{\varphi}(U) = \overline{\varphi}(H_1) \cap \overline{\varphi}(H_2)$ and as $\overline{\varphi}(H_1) = \overline{\varphi}(H_2)$, we have

$$\overline{\varphi}(U) = \overline{\varphi}(H_1) = \overline{\varphi}(H_2).$$

https://doi.org/10.1017/S0017089507004053 Published online by Cambridge University Press

We have already seen that $|G : U| = |N : \overline{\varphi}(U)|^n$ and so now we obtain $|G : U| = |G : H_1| = |G : H_2|$. Also as $U \leq H_1$, we deduce that $H_1 = U$. Analogously $U = H_2$. Hence $H_1 = H_2$.

For any $H \in \Omega$, φ is injective when restricted to $\{U \in \Omega \mid H < U\}$ and its image is $\{Y \in \Sigma \mid \varphi(H) < Y\}$.

Recall that the Möbius function is defined by $\mu_S(G) = 1$ and

$$\sum_{K \geq H} \mu_S(K) = 0$$

for $H < G$. Using the bijection above we finally get

$$\mu(G, H) = \mu(X, \varphi(H)).$$

□

THEOREM 4. *Let G be a monolithic finite group with nonabelian* soc(*G*)*. Let S be a simple component of* soc(*G*)*,* $N = N_G(S)$*,* $X = N/C_G(S)$ *and* $n = |G : N|$*. Suppose that all the maximal subgroups of G supplementing* soc(*G*) *are of type* **b**. *Then*

$$P_{G,\mathrm{soc}(G)}(t) = \sum_{H\mathrm{soc}(G)=G} \frac{\mu(G, H)}{|G : H|^t} = \sum_{YS=X} \frac{|X : Y|^{n-1}\mu(X, Y)}{(|X : Y|^n)^t} = P_{X,S}(nt - n + 1)$$

Moreover, denoting

$$P_{G,\mathrm{soc}(G)}(s) = \sum_{m\in\mathbb{N}} a_m m^{-s} \text{ and } P_{X,socX}(s) = \sum_{m\in\mathbb{N}} b_m m^{-s},$$

we have

$$a_{l^n} = l^{n-1} b_l.$$

Note that for any simple group S, any X with $S < X \leq \mathrm{Aut}(S)$ and any transitive permutation group P_n of degree n, the group $G = X \wr P_n$ satisfies the hypothesis of the theorem above.

Proof. Let $M = \mathrm{soc}(G) = S_1 \times \cdots \times S_n$, $S = S_1$ and $K = S_2 \times \cdots \times S_n$. Take a subgroup H of G such that $H\mathrm{soc}(G) = G$ and $\mu(G, H) \neq 0$.

By Proposition 3 it is $\mu(G, H) = \mu(X, \varphi(H))$ and $|G : H| = |X, \varphi(H)|^n$. Let $Y \leq X$ such that $YS = X$ and $\mu(X : Y) \neq 0$. We need to check that

$$|\{H \in \Omega \mid \varphi(H) = Y\}| = |X : Y|^{n-1},$$

where Ω is the same as in Proposition 3.

By Proposition 1, the subgroups $\{H \in \Omega \mid \varphi(H) = Y\}$ are conjugated under an element of K and so we have to check that $|K : N_K(H)| = |X : Y|^{n-1}$. Observe that $|N_K(H)|$ is the same for every H such that $\varphi(H) = Y$.

As in the proof of Theorem 1, G can be seen in a natural way as a subgroup of $X \wr P_n$, where P_n is the permutation group associated to the permutation action of G over the n components of soc(G); that is $P_n \simeq G/\cap N_G(S_i)$.

(To be more precise: first we choose a family $(1, g_2, \ldots, g_n)$ of representatives of the left cosets of N in G such that $S^{g_i} = S_i$. For $g \in G$ let $g_i g = c_{i,g} g_{i^\alpha}$, where α is the

https://doi.org/10.1017/S0017089507004053 Published online by Cambridge University Press

projection of g in P_n and $c^*_{i,g}$ is the projection of $c_{i,g} \in N$. It is easy to check that the map $g \mapsto (c^*_{1,g}, c^*_{2,g}, \ldots, c^*_{n,g})\alpha$ is a monomorphism.)

Note that N projects surjectively on the first component. Let $H \in \Omega$. As

$$(H \cap N)K \leq X \times (X \wr P_{n-1}) \text{ and } C_G(S_1) \leq (H \cap N)K,$$

$\varphi(H)$ is the projection in the first component. Choose $Y \leq X$ supplementing S and consider

$$H = (Y \wr P_n) \cap G.$$

It is easy to check that $\varphi(H) = Y$.

Let $t = (1, t_2, \ldots, t_n) \in K$ such that $t \in N_K(H)$. For any $(y_1, \ldots, y_n)\alpha \in H$, $[(y_1, \ldots, y_n)\alpha]^t \in H$; that is,

$$\left(y_1 t_{1^\alpha}, t_2^{-1} y_2 t_{2^\alpha}, \ldots\right)\alpha \in H.$$

In particular $y_1 t_{1^\alpha} \in Y$ and $t_{1^\alpha} \in Y$. As H acts transitively on $\{1, \ldots, n\}$, for any i there is some α with $t_{1^\alpha} = t_i$ and so we obtain $t \in H \cap K$. Obviously $H \cap K \in N_K(H)$ and this implies that $N_K(H) = H \cap K$. Moreover $H \cap M = (H \cap S_1) \times \cdots \times (H \cap S_n)$. Thus $|H \cap K| = |H \cap S|^{n-1}$.

By Proposition 3, $|H \cap S| = |Y \cap \mathrm{Inn}(S)|$. Therefore $|H \cap K| = |Y \cap \mathrm{Inn}(S)|^{n-1}$ and $|K : N_K(H)| = |K : (K \cap H)| = |S|^{n-1}/|Y \cap \mathrm{Inn}(S)|^{n-1} = |\mathrm{Inn}(S)Y : Y|^{n-1} = |X : Y|^{n-1}$.

Hence we get

$$P_{G,\mathrm{soc}(G)}(t) = \sum_{HN=G} \frac{\mu(G, H)}{|G : H|^t} = \sum_{YS=X} \frac{|X : Y|^{n-1}\mu(X, Y)}{(|X : Y|^n)^t}.$$

As

$$P_{X,S}(t) = \sum_{YS=X} \frac{\mu(X, Y)}{|X : Y|^t},$$

we deduce that

$$P_{G,\mathrm{soc}(G)}(t) = \sum_{YS=X} \frac{\mu(X, Y)}{|X : Y|^{nt-n+1}} = P_{X,S}(nt - n + 1).$$

From this we see that if we put

$$P_{G,\mathrm{soc}(G)}(s) = \sum_{m \in \mathbb{N}} a_m m^{-s}$$

and

$$P_{X,socX}(s) = \sum_{m \in \mathbb{N}} b_m m^{-s},$$

then $a_{l^n} = l^{n-1} b_l$. □

https://doi.org/10.1017/S0017089507004053 Published online by Cambridge University Press

THEOREM 5. *Let G be a finite monolithic group with nonabelian* soc(G), *S a simple component of* soc(G), *$N = N_G(S)$, $X = N/C_G(S)$ and $n = |G : N|$. Set*

$$P_{G,\mathrm{soc}(G)}(s) = \sum_{m\in\mathbb{N}} a_m m^{-s} \text{ and } P_{X,socX}(s) = \sum_{m\in\mathbb{N}} b_m m^{-s}.$$

Let m be an integer such that $|S|$ does not divide m. If $m = l^n$, then $a_{l^n} = l^{n-1} b_l$, and $a_m = 0$, otherwise.

Proof. Let m be an integer such that $m = |G : H|$, $|S| \nmid m$ and $\mu(G, H) \neq 0$. Assume that $H \leq U < G$ for any maximal U of type **a** or **c**. Then

$$|G : H| = |G : U||U : H|,$$

but $|S|$ divides $|G : U|$, so that $|S|$ divides m, which is a contradiction. Then H can be contained only in maximal subgroups of type **b** and so, by Proposition 3, $\mu(G, H) = \mu(X, \varphi(H))$ and $|G : H| = |X : \varphi(H)|^n$. Thus we get $m = l^n$. The same arguments as in the previous result prove that

$$|\{H \in \Omega \mid \varphi(H) = Y\}| = |X : Y|^{n-1},$$

and so we finally have

$$a_m = \sum_{\substack{H\mathrm{soc}(G)=G,\\ |G:H|=m}} \mu(G, H) = \sum_{\substack{YS=X,\\ |X:Y|=l}} |X : Y|^{n-1}\mu(X, Y) = l^{n-1} b_l.$$

□

ACKNOWLEDGEMENT. I thank Andrea Lucchini for the comments on these ideas.

REFERENCES

1. A. Ballester–Bolinches and L. M.Ezquerro, *Classes of finite groups* (Springer-Verlag, 2006).

2. N. Boston, A probabilistic generalization of the Riemann zeta functions, *Analytic number theory I* (1996), 155–162.

3. K. S. Brown, The coset poset and probabilistic zeta function of a finite group, *J. Algebra* **225** (2000), 989–1012.

4. E. Detomi and A. Lucchini, Crowns and factorization of the probabilistic zeta function of a finite group, *J. Algebra* **265** (2003), 651–668.

5. W. Gaschütz, Die Eulersche Funktion endlicher ausflösbarer Gruppen, *Illinois J. Math.* **3** (1959), 469–476.

6. F. Gross and L. G. Kovács, On normal subgroups which are direct products, *J. Algebra* **90** (1984), 133–168.

7. P. Hall, The Eulerian functions of a group, *Quart. J. Math. Oxford* **7** (1936), 134–151.

8. L. G. Kovács, Maximal subgroups in composite finite groups, *J. Algebra* **99** (1986), 114–131.

9. A. Mann, Positively finitely generated groups, *Forum Math* **8** (4) (1996), 429–459.

https://doi.org/10.1017/S0017089507004053 Published online by Cambridge University Press